SUMS INVOLVING FLOOR FUNCTION

By

Kritkhajohn Onphaeng

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree
Master of Science Program in Mathematics
Department of Mathematics
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ผลบวกที่เกี่ยวข้องกับฟังก์ชันพื้น

โดย
กฤตขจร อ่อนแพง

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
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The Graduate School, Silpakorn University has approved and accredited the Thesis title of "Sums involving floor function" submitted by Mr. Kritkhajohn Onphaeng as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics

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In this thesis, we study the properties of sums defined by Jacobsthal and generalized by Tverberg. We also introduce a new sum related to those sums and find their extreme values.

Department of Mathematics Graduate School, Silpakorn University

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ในวิทยานิพนธ์นี้เราศึกษาสมบัติต่างๆของผลบวกซึ่งนิยามโดย Jacobsthal และวางนัยทั่วไปโดย Tverberg เรายังได้ให้ผลบวกแบบใหม่ที่เกี่ยวข้องกับผลบวกแบบเดิมและหาค่าสุดขีดของผลบวกเหล่านั้น

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Chapter 1

Introduction

For each real number x, the largest integer which is less than or equal to x or the floor function of x is denoted by $\lfloor x \rfloor$. In addition, the fractional part of x, denoted by $\{x\}$, is defined by $\{x\} = x - \lfloor x \rfloor$. Problems involving floor function and fractional part have been of interest to mathematicians, especially number theorists and combinatorialists, for more than 100 years. For example, the famous Dirichlet divisor problem is to obtain an estimate for the sum $\sum_{n \le N} d(n)$, which can be written in the form involving floor function as

$$\sum_{n \le N} d(n) = \sum_{n \le N} \left\lfloor \frac{N}{n} \right\rfloor, \tag{1.1}$$

with an error term as small as possible. Here $d(n)$ is the number of positive divisors of n. In addition, the sum $\sum_{n \le N} d(n)$ counts the number of positive lattice points in the (x, y)-plane under the curve $xy = N$. So it is also connected to topics in arithmetic geometry.

Understanding floor function may lead to better estimate of (1.1) and other similar sums. For more details about Dirichlet's divisor problem, we refer the reader to [1, 13, 16, 20]. For other problems concerning with floor function or fractional part see, for example, in [2, 3, 4, 5, 6, 8, 7, 9, 11, 12, 14, 17, 18]. We are particularly interested in the sum introduced by Jacobsthal and generalized by Tverberg as given below.

Definition 1.1. *(Jacobsthal [10]) For each $a, b \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$, define $f_{a,b;m} : \mathbb{Z} \to \mathbb{Z}$ and $S_{a,b;m} : \mathbb{N} \cup \{0\} \to \mathbb{Z}$ by*

$$f_{a,b;m}(k) = \left\lfloor \frac{a+b+k}{m} \right\rfloor - \left\lfloor \frac{a+k}{m} \right\rfloor - \left\lfloor \frac{b+k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor, \text{ and} \tag{1.2}$$

$$S_{a,b;m}(K) = \sum_{k=0}^{K} f_{a,b;m}(k).$$

The above sum is also considered by Carlitz [3, 4] and Grimson [9], and is generalized by Tverberg [21] as follows.

Definition 1.2. *(Tverberg [21]) Let m and ℓ be positive integers and let C be a multiset of ℓ integers $a_1, a_2, \ldots, a_\ell$, i.e., $a_i = a_j$ is allowed for some $i \neq j$. Define $f_{C;m} : \mathbb{Z} \to \mathbb{Z}$ and $S_{C;m} : \mathbb{N} \cup \{0\} \to \mathbb{Z}$ by*

$$f_{C;m}(k) = \sum_{T \subseteq [1,\ell]} (-1)^{\ell - |T|} \left\lfloor \frac{k + \sum_{i \in T} a_i}{m} \right\rfloor, \text{ and}$$

$$S_{C;m}(K) = \sum_{k=0}^{K} f_{C;m}(k).$$

We sometimes write $f_{a_1,a_2,\ldots,a_\ell;m}(k)$ and $S_{a_1,a_2,\ldots,a_\ell;m}(K)$ instead of $f_{C;m}(k)$ and $S_{C;m}(K)$, respectively. The set $[1, \ell]$ appearing in the sum defining f is $\{1, 2, 3, \ldots, \ell\}$ and if $T = \emptyset$, then $\sum_{i \in T} a_i$ is defined to be zero.

Example 1.3. *If $C = \{a, b\}$, then $f_{C;m}(k)$ given in Definition 1.2 is the same as $f_{a,b;m}(k)$ given in (1.2), and if $C = \{a_1, a_2, a_3\}$, then $f_{C;m}(k)$ is*

$$\begin{aligned} f_{a_1,a_2,a_3;m}(k) = &\left\lfloor \frac{a_1+a_2+a_3+k}{m} \right\rfloor - \left\lfloor \frac{a_1+a_2+k}{m} \right\rfloor - \left\lfloor \frac{a_1+a_3+k}{m} \right\rfloor \\ &- \left\lfloor \frac{a_2+a_3+k}{m} \right\rfloor + \left\lfloor \frac{a_1+k}{m} \right\rfloor + \left\lfloor \frac{a_2+k}{m} \right\rfloor + \left\lfloor \frac{a_3+k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor. \end{aligned}$$

If $C = \{a_1, a_2, a_3, a_4\}$, *then* $f_{C;m}(k)$ *is*

$$\begin{aligned} f_{a_1,a_2,a_3,a_4;m}(k) = &\left\lfloor \frac{a_1+a_2+a_3+a_4+k}{m} \right\rfloor - \left\lfloor \frac{a_1+a_2+a_3+k}{m} \right\rfloor \\ &- \left\lfloor \frac{a_1+a_2+a_4+k}{m} \right\rfloor - \left\lfloor \frac{a_1+a_3+a_4+k}{m} \right\rfloor \\ &- \left\lfloor \frac{a_2+a_3+a_4+k}{m} \right\rfloor + \left\lfloor \frac{a_1+a_2+k}{m} \right\rfloor + \left\lfloor \frac{a_1+a_3+k}{m} \right\rfloor \\ &+ \left\lfloor \frac{a_1+a_4+k}{m} \right\rfloor + \left\lfloor \frac{a_2+a_3+k}{m} \right\rfloor + \left\lfloor \frac{a_2+a_4+k}{m} \right\rfloor \\ &+ \left\lfloor \frac{a_3+a_4+k}{m} \right\rfloor - \left\lfloor \frac{a_1+k}{m} \right\rfloor - \left\lfloor \frac{a_2+k}{m} \right\rfloor - \left\lfloor \frac{a_3+k}{m} \right\rfloor \\ &- \left\lfloor \frac{a_4+k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor. \end{aligned}$$

Jacobsthal obtained the lower and upper bounds of $S_{a,b;m}(K)$:

$$0 \le S_{a,b;m}(K) \le \left\lfloor \frac{m}{2} \right\rfloor \tag{1.3}$$

which are sharp bounds in the sense that one can not improve it to $S_{a,b;m}(K) > 0$ or $S_{a,b;m}(K) < \left\lfloor \frac{m}{2} \right\rfloor$. Tverberg [21] gave another proof of (1.3) and claimed (without proof) that

$$-2\left\lfloor \frac{m}{2} \right\rfloor \le S_{a_1,a_2,a_3;m}(K) \le \left\lfloor \frac{m}{3} \right\rfloor. \tag{1.4}$$

In this thesis, we give the proof of Tverberg's assertion and extends the result to the case of any positive integer $\ell \le 2$. We also obtain sharp upper and lower bounds for the sum $f_{a_1,a_2,\ldots,a_\ell;m}(k)$. In addition, we introduce the function g similar to f and obtain its bounds as well. Some of our results are published in Journal of Integer Sequences [15]. The function g and our main results are the following.

Definition 1.4. *Let* $g : \mathbb{R}^n \to \mathbb{Z}$ *be given by*

$$\begin{aligned} g(x_1, x_2, x_3, \ldots, x_n) = &\sum_{1\le i\le n} \lfloor x_i \rfloor - \sum_{1\le i_1<i_2\le n} \lfloor x_{i_1} + x_{i_2} \rfloor \\ &+ \sum_{1\le i_1<i_2<i_3\le n} \lfloor x_{i_1} + x_{i_2} + x_{i_3} \rfloor - \cdots \\ &+ (-1)^{n-1} \lfloor x_1 + x_2 + x_3 + \cdots + x_n \rfloor. \end{aligned}$$

In other words,

$$g(x_1, x_2, x_3, \ldots, x_n) = \sum_{\emptyset \neq T \subseteq [1,n]} (-1)^{|T|-1} \left\lfloor \sum_{i \in T} x_i \right\rfloor.$$

Theorem 1.5. *(Onphaeng and Pongsriiam [15]) For each $n \geq 2$, the function g given in Definition 1.4 has maximum value $2^{n-2} - 1$ and minimum value -2^{n-2}. The minimum occurs at least when $x_k = \frac{1}{2}$ for every $1 \leq k \leq n$. The maximum occurs at least when $x_k = \frac{1}{2} - \frac{1}{n^2}$ for every $1 \leq k \leq n$.*

Theorem 1.6. *(Onphaeng and Pongsriiam [15]) For each $\ell \geq 2$, $a_1, a_2, \ldots, a_\ell, k \in \mathbb{Z}$ and $m \geq 1$, we have*

$$-2^{\ell-2} \leq f_{a_1,a_2,\ldots,a_\ell;m}(k) \leq 2^{\ell-2}.$$

Moreover, $-2^{\ell-2}$ and $2^{\ell-2}$ are best possible in the sense that there are $a_1, a_2, \ldots, a_\ell, m, k$ which make the inequality becomes equality. More precisely the following statements hold.

(i) *If ℓ is odd, m is even, and $a_i = \frac{m}{2}$ for every $i = 1, 2, \ldots, \ell$, then $f_{a_1,a_2,\ldots,a_\ell;m}(0) = -2^{\ell-2}$ and $f_{a_1,a_2,\ldots,a_\ell;m}(\frac{m}{2}) = 2^{\ell-2}$.*

(ii) *If ℓ is even, m is even, and $a_i = \frac{m}{2}$ for every $i = 1, 2, \ldots, \ell$, then $f_{a_1,a_2,\ldots,a_\ell;m}(0) = 2^{\ell-2}$ and $f_{a_1,a_2,\ldots,a_\ell;m}(\frac{m}{2}) = -2^{\ell-2}$.*

Theorem 1.7. *(Onphaeng and Pongsriiam [15]) For each $\ell \geq 2$, $a_1, a_2, \ldots, a_\ell \in \mathbb{Z}$, $m \in \mathbb{N}$, and $K \in \mathbb{N} \cup \{0\}$, we have*

$$-2^{\ell-2} \left\lfloor \frac{m}{2} \right\rfloor \leq S_{a_1,a_2,\ldots,a_\ell;m}(K) \leq 2^{\ell-2} \left\lfloor \frac{m}{2} \right\rfloor. \tag{1.5}$$

Moreover, If ℓ is odd, then the lower bound $-2^{\ell-2} \left\lfloor \frac{m}{2} \right\rfloor$ is sharp and if ℓ is even, then the upper bound $2^{\ell-2} \left\lfloor \frac{m}{2} \right\rfloor$ is sharp in the sense that there are $a_1, a_2, \ldots, a_\ell, m, k$ which make the inequality becomes equality. More precisely, the following statements hold.

(i) *If ℓ is odd, m is even, and $a_i = \frac{m}{2}$ for every $i = 1, 2, \ldots, \ell$, then $S_{a_1,a_2,\ldots,a_\ell;m}(K) = -2^{\ell-2} \left\lfloor \frac{m}{2} \right\rfloor$.*

(ii) *If ℓ is even, m is even, and $a_i = \frac{m}{2}$ for every $i = 1, 2, \ldots, \ell$, then* $S_{a_1,a_2,\ldots,a_\ell;m}(K) = 2^{\ell-2} \left\lfloor \frac{m}{2} \right\rfloor$.

We remark that the extreme values of the functions g and $f_{a_1,a_2,\ldots,a_\ell;m}(k)$ are connected with Jacobsthal numbers J_n and Jacobsthal-Lucas numbers j_n defined, respectively, by the recurrence relations

$$J_0 = 0, \quad J_1 = 1, \quad J_n = J_{n-1} + 2J_{n-2} \quad \text{for } n \geq 2,$$

and

$$j_0 = 2, \quad j_1 = 1, \quad j_n = j_{n-1} + 2j_{n-2} \quad \text{for } n \geq 2.$$

The sequences $(J_n)_{n\geq 0}$ and $(j_n)_{n\geq 0}$ are, respectively, A001045 and A014551 in OEIS [19]. Recall that the Binet forms of Jacobsthal numbers J_n and Jacobsthal-Lucas numbers j_n are

$$J_n = \frac{2^n - (-1)^n}{3} \quad \text{and} \quad j_n = 2^n + (-1)^n \tag{1.6}$$

for every $n \geq 0$. Therefore we obtain the connection between Jacobsthal and Jacobsthal-Lucas numbers and sums introduced by Jacobsthal [10] and Tverberg [21] as follows.

Corollary 1.8. *(Onphaeng and Pongsriiam [15]) If n is odd, then the maximum and the minimum value of $g(x_1, x_2, x_3, \ldots, x_n)$ are j_{n-2} and $-1 - j_{n-2}$, respectively. If n is even, then the maximum and the minimum value of $g(x_1, x_2, x_3, \ldots, x_n)$ are $3J_{n-2}$ and $1 - j_{n-2}$, respectively.*

We organize this thesis as follows. In Chapter 2, we give preliminaries. In Chapter 3, we give the proof of (1.4). Finally, we give the proof of Theorems 1.5, 1.6, 1.7, and other related results in Chapter 4.

Chapter 2

Preliminaries

In this chapter, we recall some basic properties of floor function and binomial coefficients. Most of them can be found in any standard text in number theory and combinatorics but we give a proof for completeness.

2.1 Floor function

As introduced in the first chapter, for each $x \in \mathbb{R}$, we let $\lfloor x \rfloor$ be the largest integer less than or equal to x, and let $\{x\} = x - \lfloor x \rfloor$. Basic properties of $\lfloor x \rfloor$ and $\{x\}$ are as follows.

Lemma 2.1. *Let x be a real number. Then the following statements hold.*

(i) $\lfloor x \rfloor \in \mathbb{Z}$ *and* $0 \leq \{x\} < 1$.

(ii) $\lfloor \lfloor x \rfloor \rfloor = \lfloor x \rfloor$ *and* $\{\{x\}\} = \{x\}$.

(iii) *If n is a positive integer, then* $\left\lfloor \frac{\lfloor x \rfloor}{n} \right\rfloor = \left\lfloor \frac{x}{n} \right\rfloor$.

(iv) *If n is an integer, then* $\lfloor x + n \rfloor = \lfloor x \rfloor + n$.

Proof. The statement (i) follows immediately from the definition that $\lfloor x \rfloor \in \mathbb{Z}$ and $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$. Since $\lfloor x \rfloor \in \mathbb{Z}$, $\lfloor \lfloor x \rfloor \rfloor = \lfloor x \rfloor$. In addition, since $\{x\} \in [0, 1)$, we have $\{\{x\}\} = \{x\} - \lfloor \{x\} \rfloor = \{x\} - 0 = \{x\}$.So (ii) is proved.

Next we prove (iii). By (i), there exists an integer m such that $m = \left\lfloor \frac{x}{n} \right\rfloor$. By the definition of floor function, we have

$$m \leq \frac{x}{n} < m + 1.$$

Therefore

$$nm \leq x < n(m + 1).$$

Since $nm \in \mathbb{Z}$ and $nm \leq x$, we obtain $nm \leq \lfloor x \rfloor \leq x < n(m + 1)$. Then

$$m \leq \frac{\lfloor x \rfloor}{n} < m + 1.$$

By the definition of floor function, $\left\lfloor \frac{x}{n} \right\rfloor = m = \left\lfloor \frac{\lfloor x \rfloor}{n} \right\rfloor$.

Next we prove (iv). Let $m \in \mathbb{Z}$ be such that $m = \lfloor x \rfloor$. Then

$$m \leq x < m + 1$$

$$m + n \leq x + n < m + n + 1$$

By the definition of floor function, we see that

$$\lfloor x + n \rfloor = m + n = \lfloor x \rfloor + n.$$

□

Lemma 2.2. *Let x and y be real numbers. Then* $0 \leq \lfloor x + y \rfloor - \lfloor x \rfloor - \lfloor y \rfloor \leq 1$.

Proof. By the definition of fractional part and by Lemma 2.1 (iv), we have

$$\begin{aligned} \lfloor x + y \rfloor - \lfloor x \rfloor - \lfloor y \rfloor &= \lfloor \lfloor x \rfloor + \{x\} + \lfloor y \rfloor + \{y\} \rfloor - \lfloor x \rfloor - \lfloor y \rfloor \\ &= \lfloor \{x\} + \{y\} \rfloor + \lfloor x \rfloor - \lfloor x \rfloor + \lfloor y \rfloor - \lfloor y \rfloor \\ &= \lfloor \{x\} + \{y\} \rfloor. \end{aligned}$$

By Lemma 2.1 (i), $0 \leq \{x\} + \{y\} < 2$. So if $0 \leq \{x\} + \{y\} < 1$, then $\lfloor \{x\} + \{y\} \rfloor = 0$. If $1 \leq \{x\} + \{y\} < 2$, then $\lfloor \{x\} + \{y\} \rfloor = 1$. This implies the desired result.

□

Lemma 2.3. *(Hermite's Identity) Let x be a real number and m a positive integer. Then*

$$\sum_{k=0}^{m-1} \left\lfloor x + \frac{k}{m} \right\rfloor = \lfloor mx \rfloor. \tag{2.1}$$

Proof. Case 1: $x \in \mathbb{Z}$. By Lemma 2.1 (iv), the left hand side of (2.1) is

$$\begin{aligned}\sum_{k=0}^{m-1} \left(x + \left\lfloor \frac{k}{m} \right\rfloor \right) &= mx + \sum_{k=0}^{m-1} \left\lfloor \frac{k}{m} \right\rfloor \\ &= mx = \lfloor mx \rfloor.\end{aligned}$$

Case 2: $x \notin \mathbb{Z}$. Then $0 < \{x\} < 1$. We consider

$$\begin{aligned}\lfloor x \rfloor \leq \left\lfloor x + \frac{1}{m} \right\rfloor \leq \cdots \leq \left\lfloor x + \frac{m-1}{m} \right\rfloor &= \left\lfloor x + 1 - \frac{1}{m} \right\rfloor \\ &= \left\lfloor \lfloor x \rfloor + \{x\} + 1 - \frac{1}{m} \right\rfloor \\ &= \lfloor x \rfloor + 1 + \left\lfloor \{x\} - \frac{1}{m} \right\rfloor \\ &\leq \lfloor x \rfloor + 1.\end{aligned}$$

Then there exists $i \in \{1, 2, \ldots, m\}$ such that

$$\lfloor x \rfloor = \left\lfloor x + \frac{1}{m} \right\rfloor = \cdots = \left\lfloor x + \frac{i-1}{m} \right\rfloor$$

and

$$\left\lfloor x + \frac{i}{m} \right\rfloor = \left\lfloor x + \frac{i+1}{m} \right\rfloor = \cdots = \left\lfloor x + \frac{m-1}{m} \right\rfloor = \lfloor x \rfloor + 1. \tag{2.2}$$

Note that if $\lfloor x \rfloor = \lfloor x + \frac{1}{m} \rfloor = \cdots = \lfloor x + \frac{m-1}{m} \rfloor$, we take $i = m$ and (2.2) does not appear in the sum on the left hand side of (2.1). Hence

$$\sum_{k=0}^{m-1} \left\lfloor x + \frac{k}{m} \right\rfloor = i\lfloor x \rfloor + (m-i)(\lfloor x \rfloor + 1) = m\lfloor x \rfloor + m - i,$$

$$\{x\} + \frac{i-1}{m} < 1, \text{ and } \{x\} + \frac{i}{m} \geq 1.$$

The above two inequalities give us

$$m - i \leq m\{x\} < m - i + 1.$$

Then

$$m\lfloor x\rfloor + m - i \leq m\lfloor x\rfloor + m\{x\} = mx < m\lfloor x\rfloor + m - i + 1.$$

Therefore

$$\lfloor mx\rfloor = m\lfloor x\rfloor + m - i = \sum_{k=0}^{m-1}\left\lfloor x + \frac{k}{m}\right\rfloor.$$

□

2.2 Binomial coefficients

Recall that the binomial coefficients $\binom{n}{k}$ is defined for $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ by
$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!}, & \text{if } 0 \leq k \leq n; \\ 0, & \text{if } k < 0 \text{ or } k > n. \end{cases}$

The following are well-known identities which will be used in the proof of main results.

Lemma 2.4. *The following holds for all $n \in \mathbb{N}$ and $k \in \mathbb{Z}$.*

(i) $\binom{n}{k} = \binom{n}{n-k}$.

(ii) $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

Proof. If $k < 0$ or $k > n$, then both sides of (i) and of (ii) are zero. If $0 \leq k \leq n$, then this can be proved by straightforward algebraic manipulation. □

Theorem 2.5. *(Binomial Theorem) Let a and b be real numbers and n a nonnegative integer. Then*

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

Proof. This can be proved by induction on n together with Lemma 2.4 (ii). □

Lemma 2.6. *Let n be a positive integer. Then the following statements hold.*

(i) $\displaystyle\sum_{k=0}^{n}\binom{n}{k} = 2^n$.

(ii) $\displaystyle\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0.$

(iii) $\displaystyle\sum_{k=1}^{n}(-1)^{k-1}k\binom{n}{k} = 0$ *for* $n \geq 2$.

(iv) $\displaystyle\sum_{\substack{k=0\\k\equiv 0 \pmod 2}}^{n}\binom{n}{k} = 2^{n-1}.$

(v) $\displaystyle\sum_{\substack{k=0\\k\equiv 1 \pmod 2}}^{n}\binom{n}{k} = 2^{n-1}.$

Proof. By binomial theorem, we have

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k.$$

Substituting $a = b = 1$, we obtain (i). Similarly, (ii) follows from the substitution $a = 1$ and $b = -1$. For (iii), we consider the following as a function of x:

$$(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k. \tag{2.3}$$

Differentiating both sides of (2.3) with respect to x, we obtain

$$n(1+x)^{n-1} = \sum_{k=1}^{n}k\binom{n}{k}x^{k-1}. \tag{2.4}$$

Substituting $x = -1$ in (2.4), we obtain (iii). By adding (i) and (ii), we see that

$$\sum_{\substack{k=0\\k\equiv 0 \pmod 2}}^{n} 2\binom{n}{k} = \sum_{k=0}^{n}\left(1+(-1)^k\right)\binom{n}{k} = 2^n.$$

This implies (iv). Similarly, by subtracting (i) by (ii), we obtain (v). $\square$

Chapter 3

Proof of Tverberg's Assertion

3.1 Lemmas

As mentioned in the first chapter, Tverberg generalized the sums introduced by Jacobsthal, gave another proof of Jacobsthal's result, and claimed (without proof) that

$$-2\left\lfloor \frac{m}{2} \right\rfloor \le S_{a_1,a_2,a_3;m}(K) \le \left\lfloor \frac{m}{3} \right\rfloor.$$

In this section, we give the proof of his assertion. First, we prove the following lemma.

Lemma 3.1. *Let $a_1, a_2, a_3 \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$. The following statements hold.*

(i) *f is periodic with period m in each variable a_1, a_2, a_3, k. In other words, for any $q \in \mathbb{Z}$, $f_{a_1+qm,a_2,a_3;m}(k) = f_{a_1,a_2+qm,a_3;m}(k) = f_{a_1,a_2,a_3+qm;m}(k) = f_{a_1,a_2,a_3;m}(k+qm)$.*

(ii) *$f_{a_1,a_2,a_3;m}(k) = f_{a_2,a_1,a_3;m}(k) = \cdots = f_{a_3,a_2,a_1;m}(k)$. In other words, the permutation of a_1, a_2, a_3 does not change the value of $f_{a_1,a_2,a_3;m}(k)$.*

(iii) *$f_{0,a_2,a_3;m}(k) = f_{a_1,0,a_3;m}(k) = f_{a_1,a_2,0;m}(k) = 0$.*

Remark 3.2. *Lemma 3.1 can be generalized to the case of ℓ variables $a_1, a_2, \ldots, a_\ell$. Nevertheless, for the purpose of this section, we only need the case $\ell = 3$. The* *general case of* (i) *is used in the proof of our main results and will be proved in the next chapter (see Lemma 4.2* (ii)*). The general cases of* (ii) *and* (iii) *are not needed in this thesis.*

Proof. By Definition 1.2 and by Lemma 2.1 (iv), we obtain

$$\begin{aligned}
f_{a_1+qm,a_2,a_3;m}(k) &= \left\lfloor \frac{a_1+qm+a_2+a_3+k}{m} \right\rfloor - \left\lfloor \frac{a_1+qm+a_2+k}{m} \right\rfloor \\
&\quad - \left\lfloor \frac{a_1+qm+a_3+k}{m} \right\rfloor - \left\lfloor \frac{a_2+a_3+k}{m} \right\rfloor + \left\lfloor \frac{a_1+qm+k}{m} \right\rfloor \\
&\quad + \left\lfloor \frac{a_2+k}{m} \right\rfloor + \left\lfloor \frac{a_3+k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor \\
&= \left\lfloor \frac{a_1+a_2+a_3+k}{m} \right\rfloor + q - \left\lfloor \frac{a_1+a_2+k}{m} \right\rfloor - q \\
&\quad - \left\lfloor \frac{a_1+a_3+k}{m} \right\rfloor - q - \left\lfloor \frac{a_2+a_3+k}{m} \right\rfloor + \left\lfloor \frac{a_1+k}{m} \right\rfloor + q \\
&\quad + \left\lfloor \frac{a_2+k}{m} \right\rfloor + \left\lfloor \frac{a_3+k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor \\
&= \left\lfloor \frac{a_1+a_2+a_3+k}{m} \right\rfloor - \left\lfloor \frac{a_1+a_2+k}{m} \right\rfloor - \left\lfloor \frac{a_1+a_3+k}{m} \right\rfloor \\
&\quad - \left\lfloor \frac{a_2+a_3+k}{m} \right\rfloor + \left\lfloor \frac{a_1+k}{m} \right\rfloor + \left\lfloor \frac{a_2+k}{m} \right\rfloor + \left\lfloor \frac{a_3+k}{m} \right\rfloor \\
&\quad - \left\lfloor \frac{k}{m} \right\rfloor \\
&= f_{a_1,a_2,a_3;m}(k).
\end{aligned}$$

The equation $f_{a_1,a_2+qm,a_3;m}(k) = f_{a_1,a_2,a_3+qm;m}(k) = f_{a_1,a_2,a_3;m}(k+qm) = f_{a_1,a_2,a_3;m}(k)$ can be obtained in the same way as $f_{a_1+qm,a_2,a_3;m}(k) = f_{a_1,a_2,a_3;m}(k)$. This proves (i). The statement (ii) follows immediately from Definition 1.2. Next we prove (iii). By Definition 1.2, we have

$$\begin{aligned}
f_{0,a_2,a_3;m}(k) &= \left\lfloor \frac{0+a_2+a_3+k}{m} \right\rfloor - \left\lfloor \frac{0+a_2+k}{m} \right\rfloor - \left\lfloor \frac{0+a_3+k}{m} \right\rfloor \\
&\quad - \left\lfloor \frac{a_2+a_3+k}{m} \right\rfloor + \left\lfloor \frac{0+k}{m} \right\rfloor + \left\lfloor \frac{a_2+k}{m} \right\rfloor + \left\lfloor \frac{a_3+k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor \\
&= 0.
\end{aligned}$$

Similarly, $f_{a_1,0,a_3;m}(k) = f_{a_1,a_2,0;m}(k) = 0$. □

Lemma 3.3. *For each $\ell \geq 2$, $a_1, a_2, \ldots, a_\ell \in \mathbb{Z}$, $m \in \mathbb{N}$, and $K \in \mathbb{N} \cup \{0\}$, we have*

$$S_{a_1,a_2,\ldots,a_\ell;m}(m-1) = 0.$$

Proof. By Definition 1.2 and by Lemma 2.3, we obtain

$$\begin{aligned}
S_{a_1,a_2,\ldots,a_\ell;m}(m-1) &= \sum_{k=0}^{m-1} f_{a_1,a_2,\ldots,a_\ell;m}(k) \\
&= \sum_{k=0}^{m-1} \sum_{T\subseteq[1,\ell]} (-1)^{\ell-|T|} \left\lfloor \frac{k+\sum_{i\in T} a_i}{m} \right\rfloor \\
&= \sum_{T\subseteq[1,\ell]} (-1)^{\ell-|T|} \sum_{k=0}^{m-1} \left\lfloor \frac{k+\sum_{i\in T} a_i}{m} \right\rfloor \\
&= \sum_{T\subseteq[1,\ell]} (-1)^{\ell-|T|} \left\lfloor \sum_{i\in T} a_i \right\rfloor \\
&= (a_1 + a_2 + \cdots + a_\ell) \\
&\quad - \sum_{1\leq i_1<i_2<\ldots<i_{\ell-1}\leq\ell} (a_{i_1} + a_{i_2} + \cdots + a_{i_{\ell-1}}) \\
&\quad + \sum_{1\leq i_1<i_2<\ldots<i_{\ell-2}\leq\ell} (a_{i_1} + a_{i_2} + \cdots + a_{i_{\ell-2}}) + \cdots + \\
&\quad (-1)^{\ell-1} \sum_{1\leq i_1\leq\ell} (a_{i_1}).
\end{aligned}$$

Next, we consider

$$\sum_{1\leq i_1<i_2<\ldots<i_{\ell-k}\leq\ell} (a_{i_1} + a_{i_2} + \cdots + a_{i_{\ell-k}}) \tag{3.1}$$

where $k \in \{1, 2, \ldots, \ell-1\}$. The number of a_{i_r} appearing in the sum (3.1) is $\binom{\ell-1}{\ell-k-1}$ for each $r \in \{1, 2, \ldots, \ell\}$. By Lemma 2.4 (i) and Lemma 2.6 (ii), we obtain

$$
\begin{aligned}
S_{a_1,a_2,\ldots,a_\ell;m}(m-1) &= (a_1+a_2+\cdots+a_\ell) - \binom{\ell-1}{1}(a_1+a_2+\cdots+a_\ell) \\
&\quad + \binom{\ell-1}{2}(a_1+a_2+\cdots+a_\ell) + \cdots + \\
&\quad + (-1)^{\ell-1}\binom{\ell-1}{\ell-1}(a_1+a_2+\cdots+a_\ell) \\
&= (a_1+a_2+\cdots+a_\ell)\sum_{k=0}^{\ell-1}(-1)^k\binom{\ell-1}{k} = 0.
\end{aligned}
$$

□

Recall from (1.3) that

$$0 \le S_{a,b;m}(K) \le \left\lfloor \frac{m}{2} \right\rfloor.$$

We will apply the above inequality in the proof of the following theorem.

3.2 Tverberg's Assertion

Theorem 3.4. *Let $a_1, a_2, a_3 \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$. Then*

$$-2\left\lfloor \frac{m}{2} \right\rfloor \le S_{a_1,a_2,a_3;m}(K) \le \left\lfloor \frac{m}{3} \right\rfloor$$

Proof. First, we proof $-2\left\lfloor \frac{m}{2} \right\rfloor \le S_{a_1,a_2,a_3;m}(K)$. Recall that

$$
\begin{aligned}
f_{a_1,a_2,a_3;m}(k) &= \left\lfloor \frac{a_1+a_2+a_3+k}{m} \right\rfloor - \left\lfloor \frac{a_1+a_2+k}{m} \right\rfloor - \left\lfloor \frac{a_1+a_3+k}{m} \right\rfloor \\
&\quad - \left\lfloor \frac{a_2+a_3+k}{m} \right\rfloor + \left\lfloor \frac{a_1+k}{m} \right\rfloor + \left\lfloor \frac{a_2+k}{m} \right\rfloor + \left\lfloor \frac{a_3+k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor.
\end{aligned}
$$

By Definition 1.2, we have

$$f_{a_1+a_2,a_3;m}(k) = \left\lfloor \frac{a_1+a_2+a_3+k}{m} \right\rfloor - \left\lfloor \frac{a_1+a_2+k}{m} \right\rfloor - \left\lfloor \frac{a_3+k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor, \tag{3.2}$$

$$-f_{a_1,a_3;m}(k) = -\left\lfloor \frac{a_1+a_3+k}{m} \right\rfloor + \left\lfloor \frac{a_1+k}{m} \right\rfloor + \left\lfloor \frac{a_3+k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor, \tag{3.3}$$

$$-f_{a_2,a_3;m}(k) = -\left\lfloor \frac{a_2+a_3+k}{m} \right\rfloor + \left\lfloor \frac{a_2+k}{m} \right\rfloor + \left\lfloor \frac{a_3+k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor. \tag{3.4}$$

Summing (3.2), (3.3), and (3.4), we see that

$$f_{a_1,a_2,a_3;m}(k) = f_{a_1+a_2,a_3;m}(k) - f_{a_1,a_3;m}(k) - f_{a_2,a_3;m}(k). \tag{3.5}$$

By the definition of $S_{a_1,a_2,a_3;m}(K)$, (3.5), and (1.3), we obtain

$$\begin{aligned} S_{a_1,a_2,a_3;m}(K) &= \sum_{k=0}^{K} f_{a_1,a_2,a_3;m}(k) \\ &= \sum_{k=0}^{K} f_{a_1+a_2,a_3;m}(k) - \sum_{k=0}^{K} f_{a_1,a_3;m}(k) - \sum_{k=0}^{K} f_{a_2,a_3;m}(k) \\ &= S_{a_1+a_2,a_3;m}(K) - S_{a_1,a_3;m}(K) - S_{a_2,a_3;m}(K) \\ &\geq 0 - \left\lfloor \frac{m}{2} \right\rfloor - \left\lfloor \frac{m}{2} \right\rfloor = -2\left\lfloor \frac{m}{2} \right\rfloor. \end{aligned}$$

Next, we prove $S_{a_1,a_2,a_3;m}(K) \leq \lfloor \frac{m}{3} \rfloor$. By Lemma 3.1 (i) and Lemma 3.3, we can assume that $a_1, a_2, a_3, k, K \in [0, m-1]$. By Lemma 3.1 (ii) and Lemma 3.1 (iii), we can assume that $0 < a_1 \leq a_2 \leq a_3$.

Case 1: $a_1 + a_2 + a_3 \leq m$. Then $a_1 + a_2 + a_3 \geq a_2 + a_3 \geq a_1 + a_3 \geq \max\{a_1 + a_2, a_3\} \geq \min\{a_1 + a_2, a_3\} \geq a_2 \geq a_1$. If $k \in [0, m - a_1 - a_2 - a_3)$, then $f(k) = 0$. If $k \in [m-a_1-a_2-a_3, m-a_2-a_3)$, then $f(k) = 1$. If $k \in [m-a_2-a_3, m-a_1-a_3)$, then $f(k) = 0$. If $k \in [m-a_1-a_3, m-\max\{a_1+a_2, a_3\})$, then $f(k) = -1$. If $k \in [m - \max\{a_1 + a_2, a_3\}, m - \min\{a_1 + a_2, a_3\})$, then $f(k) = -2$ or $f(k) = 0$. If $k \in [m-\min\{a_1+a_2, a_3\}, m-a_2)$, then $f(k) = -1$. If $k \in [m - a_2, m - a_1)$, then $f(k) = 0$. If $k \in [m - a_1, m)$, then $f(k) = 1$. By Lemma 3.3, we obtain

$$\begin{aligned} S_{a_1,a_2,a_3;m}(K) &\leq S_{a_1,a_2,a_3;m}(m - a_2 - a_3) \\ &= m - a_2 - a_3 - (m - a_1 - a_2 - a_3) = a_1. \end{aligned}$$

By $a_1 \leq a_2 \leq a_3$ and $a_1+a_2+a_3 \leq m$, we have $a_1 \leq \lfloor \frac{m}{3} \rfloor$. Then $S_{a_1,a_2,a_3;m}(K) \leq \lfloor \frac{m}{3} \rfloor$.

Case 2: $m < a_1 + a_2 + a_3 < 2m$.

Case 2.1: $m < a_1 + a_2 + a_3 < 2m$ and $a_2 + a_3 < m$. Then $a_2 + a_3 \geq a_1 + a_3 \geq \max\{a_1 + a_2, a_3\} \geq \min\{a_1 + a_2, a_3\} \geq a_2 \geq a_1 \geq a_1 + a_2 + a_3 - m$. If

$k \in [0, m - a_2 - a_3)$, then $f(k) = 1$. If $k \in [m - a_2 - a_3, m - a_1 - a_3)$, then $f(k) = 0$. If $k \in [m - a_1 - a_3, m - \max\{a_1 + a_2, a_3\})$, then $f(k) = -1$. If $k \in [m - \max\{a_1 + a_2, a_3\}, m - \min\{a_1 + a_2, a_3\})$, then $f(k) = -2$ or $f(k) = 0$. If $k \in [m - \min\{a_1 + a_2, a_3\}, m - a_2)$, then $f(k) = -1$. If $k \in [m - a_2, m - a_1)$, then $f(k) = 0$. If $k \in [m - a_1, 2m - a_1 - a_2 - a_3)$, then $f(k) = 1$. If $k \in [2m - a_1 - a_2 - a_3, m)$, then $f(k) = 2$. By Lemma 3.3, we obtain

$$\begin{aligned} S_{a_1,a_2,a_3;m}(K) &\leq S_{a_1,a_2,a_3;m}(m - a_2 - a_3) \\ &= m - a_2 - a_3 - 0 = m - a_2 - a_3. \end{aligned}$$

By $m < a_1 + a_2 + a_3$, $a_2 + a_3 < m$ and $a_1 \leq a_2 \leq a_3$, we have $m - a_2 - a_3 \leq \left\lfloor \frac{m}{3} \right\rfloor$. Then $S_{a_1,a_2,a_3;m}(K) \leq \left\lfloor \frac{m}{3} \right\rfloor$.

Case 2.2: $m < a_1 + a_2 + a_3 < 2m$, $a_2 + a_3 \geq m$ and $a_1 + a_3 < m$.

Then $a_1 + a_3 \geq \max\{a_1 + a_2, a_3\} \geq \min\{a_1 + a_2, a_3\} \geq a_2 \geq a_1 + a_2 + a_3 - m \geq \max\{a_1, a_2 + a_3 - m\} \geq \min\{a_1, a_2 + a_3 - m\}$. If $k \in [0, m - a_1 - a_3)$, then $f(k) = 0$. If $k \in [m - a_1 - a_3, m - \max\{a_1 + a_2, a_3\})$, then $f(k) = -1$. If $k \in [m - \max\{a_1 + a_2, a_3\}, m - \min\{a_1 + a_2, a_3\})$, then $f(k) = -2$ or $f(k) = 0$. If $k \in [m - \min\{a_1 + a_2, a_3\}, m - a_2)$, then $f(k) = -1$. If $k \in [m - a_2, 2m - a_1 - a_2 - a_3)$, then $f(k) = 0$. If $k \in [2m - a_1 - a_2 - a_3, m - \max\{a_1, a_2 + a_3 - m\})$, then $f(k) = 1$. If $k \in [m - \max\{a_1, a_2 + a_3 - m\}, m - \min\{a_1, a_2 + a_3 - m\})$, then $f(k) = 2$ or $f(k) = 0$. If $k \in [m - \min\{a_1, a_2 + a_3 - m\}, m)$, then $f(k) = 1$. By Lemma 3.3, we have $S_{a_1,a_2,a_3;m}(K) \leq 0$.

Case 2.3: $m < a_1 + a_2 + a_3 < 2m$, $a_1 + a_3 \geq m$ and $a_1 + a_2 < m$.

Then $\max\{a_1 + a_2, a_3\} \geq \min\{a_1 + a_2, a_3\} \geq a_1 + a_2 + a_3 - m \geq a_2 \geq \max\{a_1, a_2 + a_3 - m\} \geq \min\{a_1, a_2 + a_3 - m\} \geq a_1 + a_3 - m$. If $k \in [0, m - \max\{a_1 + a_2, a_3\})$, then $f(k) = -1$. If $k \in [m - \max\{a_1 + a_2, a_3\}, m - \min\{a_1 + a_2, a_3\})$, then $f(k) = -2$ or $f(k) = 0$. If $k \in [m - \min\{a_1 + a_2, a_3\}, 2m - a_1 - a_2 - a_3)$, then $f(k) = -1$. If $k \in [2m - a_1 - a_2 - a_3, m - a_2)$, then $f(k) = 0$. If $k \in [m - a_2, m - \max\{a_1, a_2 + a_3 - m\})$, then $f(k) = 1$. If $k \in [m - \max\{a_1, a_2 + a_3 - m\}, m - \min\{a_1, a_2 + a_3 - m\})$, then $f(k) = 2$ or $f(k) = 0$. If $k \in [m - \min\{a_1, a_2 + a_3 - m\}, 2m - a_1 - a_3)$, then $f(k) = 1$. If

$k \in [2m-a_1-a_3, m)$, then $f(k) = 0$. By Lemma 3.3, we have $S_{a_1,a_2,a_3;m}(K) \leq 0$.

Case 2.4: $m < a_1 + a_2 + a_3 < 2m$ and $a_1 + a_2 \geq m$.

Then $a_1 + a_2 + a_3 - m \geq a_3 \geq a_2 \geq \max\{a_1, a_2 + a_3 - m\} \geq \min\{a_1, a_2 + a_3 - m\} \geq a_1 + a_3 - m \geq a_1 + a_2 - m$. If $k \in [0, 2m - a_1 - a_2 - a_3)$, then $f(k) = -2$. If $k \in [2m - a_1 - a_2 - a_3, m - a_3)$, then $f(k) = -1$. If $k \in [m - a_3, m - a_2)$, then $f(k) = 0$. If $k \in [m - a_2, m - \max\{a_1, a_2 + a_3 - m\})$, then $f(k) = 1$. If $k \in [m - \max\{a_1, a_2 + a_3 - m\}, m - \min\{a_1, a_2 + a_3 - m\})$, then $f(k) = 2$ or $f(k) = 0$. If $k \in [m - \min\{a_1, a_2 + a_3 - m\}, 2m - a_1 - a_3)$, then $f(k) = 1$. If $k \in [2m - a_1 - a_3, 2m - a_1 - a_2)$, then $f(k) = 0$. If $k \in [2m - a_1 - a_2, m)$, then $f(k) = -1$. By Lemma 3.3, we obtain

$$\begin{aligned}
S_{a_1,a_2,a_3}(K) &\leq S_{a_1,a_2,a_3}(2m - a_1 - a_2 - 1) \\
&= S_{a_1,a_2,a_3}(m-1) - \sum_{k=2m-a_1-a_2}^{m-1} f_{a_1,a_2,a_3}(k) \\
&= 0 - (-1)(m - (2m - a_1 - a_2)) \\
&= m - (2m - a_1 - a_2) = a_1 + a_2 - m.
\end{aligned}$$

By $a_1+a_2+a_3 < 2m$, $a_1+a_2 \geq m$ and $a_1 \leq a_2 \leq a_3$, we have $a_1+a_2-m \leq \left\lfloor \frac{m}{3} \right\rfloor$. Then $S_{a_1,a_2,a_3}(K) \leq \left\lfloor \frac{m}{3} \right\rfloor$.

Case 3: $2m \leq a_1 + a_2 + a_3$.

Then $a_3 \geq a_2 \geq \max\{a_1, a_2 + a_3 - m\} \geq \min\{a_1, a_2 + a_3 - m\} \geq a_1 + a_3 - m \geq a_1 + a_2 - m \geq a_1 + a_2 + a_3 - 2m$. If $k \in [0, m - a_3)$, then $f(k) = -1$. If $k \in [m - a_3, m - a_2)$, then $f(k) = 0$. If $k \in [m - a_2, m - \max\{a_1, a_2 + a_3 - m\})$, then $f(k) = 1$. If $k \in [m - \max\{a_1, a_2 + a_3 - m\}, m - \min\{a_1, a_2 + a_3 - m\})$, then $f(k) = 2$ or $f(k) = 0$. If $k \in [m - \min\{a_1, a_2 + a_3 - m\}, 2m - a_1 - a_3)$, then $f(k) = 1$. If $k \in [2m - a_1 - a_3, 2m - a_1 - a_2)$, then $f(k) = 0$. If $k \in [2m-a_1-a_2, 3m-a_1-a_2-a_3)$, then $f(k) = -1$. If $k \in [3m-a_1-a_2-a_3, m)$, then $f(k) = 0$. By Lemma 3.3, we obtain

$$\begin{aligned} S_{a_1,a_2,a_3}(K) &\leq S_{a_1,a_2,a_3}(2m - a_1 - a_3 - 1) \\ &= 3m - a_1 - a_2 - a_3 - (2m - a_1 - a_2) = m - a_3. \end{aligned}$$

By $2m \leq a_1 + a_2 + a_3$ and $a_1 \leq a_2 \leq a_3$, we have $m - a_3 \leq \left\lfloor \frac{m}{3} \right\rfloor$. Then $S_{a_1,a_2,a_3}(K) \leq \left\lfloor \frac{m}{3} \right\rfloor$. By Case 1, Case 2, and Case 3 $S_{a_1,a_2,a_3}(K) \leq \left\lfloor \frac{m}{3} \right\rfloor$. Next we show that $\left\lfloor \frac{m}{3} \right\rfloor$ is sharp: if $m \equiv 0 \pmod 3$ and $a_1 = a_2 = a_3 = \frac{m}{3}$. It easy to see that $f(0) = f(1) = \cdots = f\left(\frac{m}{3} - 1\right) = 1$. Then $S_{a_1,a_2,a_3}\left(\frac{m}{3} - 1\right) = \frac{m}{3}$.

□

Chapter 4

Proof of Our Main Results

4.1 Lemmas

Lemma 4.1. *For each* $\ell \geq 2$*, we have*

(i) $f_{a_1,a_2,\ldots,a_\ell;m}(0) = (-1)^{\ell-1} g\left(\frac{a_1}{m}, \frac{a_2}{m}, \cdots, \frac{a_\ell}{m}\right)$,

(ii) $f_{a_1,a_2,\ldots,a_\ell;m}(k) = (-1)^{\ell} g\left(\frac{a_1}{m}, \frac{a_2}{m}, \cdots, \frac{a_\ell}{m}, \frac{k}{m}\right) + (-1)^{\ell-1} g\left(\frac{a_1}{m}, \frac{a_2}{m}, \cdots, \frac{a_\ell}{m}\right)$.

Proof. This follows easily from the definitions of f and g but we give a proof for completeness. We have

$$\begin{aligned} f_{a_1,a_2,\ldots,a_\ell;m}(0) &= \sum_{T\subseteq[1,\ell]} (-1)^{\ell-|T|} \left\lfloor \sum_{i\in T} \left(\frac{a_i}{m}\right) \right\rfloor \\ &= \sum_{\emptyset\neq T\subseteq[1,\ell]} (-1)^{\ell-|T|} \left\lfloor \sum_{i\in T} \left(\frac{a_i}{m}\right) \right\rfloor \\ &= (-1)^{\ell-1} \sum_{\emptyset\neq T\subseteq[1,\ell]} (-1)^{1-|T|} \left\lfloor \sum_{i\in T} \left(\frac{a_i}{m}\right) \right\rfloor \\ &= (-1)^{\ell-1} g\left(\frac{a_1}{m}, \frac{a_2}{m}, \ldots, \frac{a_\ell}{m}\right). \end{aligned}$$

Next let $a_{\ell+1} = k$. Then we obtain

$$
\begin{aligned}
&(-1)^{\ell} g\left(\frac{a_1}{m}, \frac{a_2}{m}, \cdots, \frac{a_\ell}{m}, \frac{k}{m}\right) + (-1)^{\ell-1} g\left(\frac{a_1}{m}, \frac{a_2}{m}, \cdots, \frac{a_\ell}{m}\right) \\
=&(-1)^{\ell}\left(\sum_{\emptyset \neq T \subseteq [1,\ell+1]} (-1)^{|T|-1}\left\lfloor \sum_{i\in T}\left(\frac{a_i}{m}\right)\right\rfloor - \sum_{\emptyset \neq T \subseteq [1,\ell]} (-1)^{|T|-1}\left\lfloor \sum_{i\in T}\left(\frac{a_i}{m}\right)\right\rfloor\right) \\
=&(-1)^{\ell} \sum_{\substack{T \subseteq [1,\ell+1] \\ \ell+1 \in T}} (-1)^{|T|-1}\left\lfloor \sum_{i\in T}\left(\frac{a_i}{m}\right)\right\rfloor \\
=&(-1)^{\ell} \sum_{T \subseteq [1,\ell]} (-1)^{|T|}\left\lfloor \frac{k+\sum_{i\in T} a_i}{m}\right\rfloor \\
=&f_{a_1,a_2,\ldots,a_\ell;m}(k).
\end{aligned}
$$

□

Lemma 4.2. *The following statements hold.*

(i) *For each $i \in \{1, 2, \ldots, n\}$ and $q \in \mathbb{Z}$, we have*

$$g(x_1, x_2, \ldots, x_i + q, \ldots, x_n) = g(x_1, x_2, \ldots, x_n).$$

In particular, g has period 1 in each variable.

(ii) *For each $i \in \{1, 2, \ldots, \ell\}$ and $q \in \mathbb{Z}$, we have*

$$f_{a_1,a_2,\ldots,a_i+qm,\ldots,a_\ell;m}(k) = f_{a_1,a_2,\ldots,a_\ell;m}(k) = f_{a_1,a_2,\ldots,a_\ell;m}(k+qm).$$

In particular, f has period m in each variable $a_1, a_2, \ldots, a_\ell$ and k.

Proof. Since $\lfloor q+x\rfloor = q+\lfloor x\rfloor$ for every $q\in\mathbb{Z}$ and $x\in\mathbb{R}$, we see that

$$\begin{aligned}
g(x_1,x_2,\ldots,x_i+q,\ldots,x_n) &= \left(q+\sum_{i=1}^{n}\lfloor x_i\rfloor\right)\\
&\quad-\left(\binom{n-1}{1}q+\sum_{1\le i_1<i_2\le n}\lfloor x_{i_1}+x_{i_2}\rfloor\right)\\
&\quad+\left(\binom{n-1}{2}q+\sum_{1\le i_1<i_2<i_3\le n}\lfloor x_{i_1}+x_{i_2}+x_{i_3}\rfloor\right)\\
&\quad-\cdots+(-1)^{n-1}\left(\binom{n-1}{n-1}q+\lfloor x_1+x_2+\cdots+x_n\rfloor\right)\\
&= g(x_1,x_2,\ldots,x_n)+q\sum_{0\le k\le n-1}(-1)^k\binom{n-1}{k}\\
&= g(x_1,x_2,\ldots,x_n).
\end{aligned}$$

This proves (i). Next we prove (ii). By Lemma 4.1 (ii) and by (i), we obtain

$$\begin{aligned}
f_{a_1,a_2,\ldots,a_i+qm,\ldots,a_\ell;m}(k) &= (-1)^\ell g\left(\frac{a_1}{m},\frac{a_2}{m},\ldots,\frac{a_i}{m}+q,\ldots,\frac{a_\ell}{m},\frac{k}{m}\right)\\
&\quad+(-1)^{\ell-1}g\left(\frac{a_1}{m},\frac{a_2}{m},\ldots,\frac{a_i}{m}+q,\ldots,\frac{a_\ell}{m}\right)\\
&= (-1)^\ell g\left(\frac{a_1}{m},\frac{a_2}{m},\ldots,\frac{a_\ell}{m},\frac{k}{m}\right)\\
&\quad+(-1)^{\ell-1}g\left(\frac{a_1}{m},\frac{a_2}{m},\ldots,\frac{a_\ell}{m}\right)\\
&= f_{a_1,a_2,\ldots,a_\ell;m}(k).
\end{aligned}$$

Similarly, $f_{a_1,a_2,\ldots,a_\ell;m}(k+qm) = f_{a_1,a_2,\ldots,a_\ell;m}(k)$. This completes the proof. □

4.2 Proof of Main Results

Proof of Theorem 1.5. If $n=2$, then the result is the same as Lemma 2.2 that

$$-1\le\lfloor x\rfloor+\lfloor y\rfloor-\lfloor x+y\rfloor\le 0.$$

The inequality in Lemma 2.2 is sharp: if $x=y=\frac{1}{2}$ the left inequality in Lemma 2.2 becomes equality, and if $x=y=\frac{1}{4}$ the right inequality in Lemma 2.2 becomes equality. The result when $n\ge 3$ is obtained from the case $n=2$

and a careful selection of pairs. For illustration purpose, we first give a proof for the case $n = 3$ and $n = 4$. Recall that

$$g(x_1, x_2, x_3) = \lfloor x_1 \rfloor + \lfloor x_2 \rfloor + \lfloor x_3 \rfloor - \lfloor x_1 + x_2 \rfloor - \lfloor x_1 + x_3 \rfloor - \lfloor x_2 + x_3 \rfloor + \lfloor x_1 + x_2 + x_3 \rfloor.$$

We obtain by Lemma 2.2 that

$$0 \le \lfloor x_1 + x_2 + x_3 \rfloor - \lfloor x_1 + x_2 \rfloor - \lfloor x_3 \rfloor \le 1, \tag{4.1}$$

$$-1 \le -\lfloor x_2 + x_3 \rfloor + \lfloor x_2 \rfloor + \lfloor x_3 \rfloor \le 0, \tag{4.2}$$

$$-1 \le -\lfloor x_1 + x_3 \rfloor + \lfloor x_1 \rfloor + \lfloor x_3 \rfloor \le 0. \tag{4.3}$$

Summing (4.1), (4.2), and (4.3), the middle terms give $g(x_1, x_2, x_3)$. Then $-2 \le g(x_1, x_2, x_3) \le 1$. Next we consider

$$\begin{aligned} g(x_1, x_2, x_3, x_4) = {} & \lfloor x_1 \rfloor + \lfloor x_2 \rfloor + \lfloor x_3 \rfloor + \lfloor x_4 \rfloor - \lfloor x_1 + x_2 \rfloor - \lfloor x_1 + x_3 \rfloor \\ & - \lfloor x_1 + x_4 \rfloor - \lfloor x_2 + x_3 \rfloor - \lfloor x_2 + x_4 \rfloor - \lfloor x_3 + x_4 \rfloor \\ & + \lfloor x_1 + x_2 + x_3 \rfloor + \lfloor x_1 + x_2 + x_4 \rfloor + \lfloor x_1 + x_3 + x_4 \rfloor \\ & + \lfloor x_2 + x_3 + x_4 \rfloor - \lfloor x_1 + x_2 + x_3 + x_4 \rfloor. \end{aligned}$$

Again, we obtain by Lemma 2.2 the following inequalities:

$$-1 \le -\lfloor x_1 + x_2 + x_3 + x_4 \rfloor + \lfloor x_1 + x_2 + x_3 \rfloor + \lfloor x_4 \rfloor \le 0, \tag{4.4}$$

$$0 \le \lfloor x_1 + x_2 + x_4 \rfloor - \lfloor x_1 + x_2 \rfloor - \lfloor x_4 \rfloor \le 1, \tag{4.5}$$

$$0 \le \lfloor x_1 + x_3 + x_4 \rfloor - \lfloor x_1 + x_3 \rfloor - \lfloor x_4 \rfloor \le 1, \tag{4.6}$$

$$0 \le \lfloor x_2 + x_3 + x_4 \rfloor - \lfloor x_2 + x_3 \rfloor - \lfloor x_4 \rfloor \le 1, \tag{4.7}$$

$$-1 \le -\lfloor x_1 + x_4 \rfloor + \lfloor x_1 \rfloor + \lfloor x_4 \rfloor \le 0, \tag{4.8}$$

$$-1 \le -\lfloor x_2 + x_4 \rfloor + \lfloor x_2 \rfloor + \lfloor x_4 \rfloor \le 0, \tag{4.9}$$

$$-1 \le -\lfloor x_3 + x_4 \rfloor + \lfloor x_3 \rfloor + \lfloor x_4 \rfloor \le 0. \tag{4.10}$$

Summing (4.4) to (4.10), we see that $-4 \le g(x_1, x_2, x_3, x_4) \le 3$.

Next we prove the general case $n \ge 5$. The expression of the form $\lfloor x_{i_1} + x_{i_2} + \cdots + x_{i_k} \rfloor$ will be called a *k-bracket*. So for each $1 \le k \le n$, there are

$\binom{n}{k}$ k-brackets appearing in the sum defining $g(x_1, x_2, \ldots, x_n)$. We first pair up the n-bracket with an $(n-1)$-bracket and a 1-bracket as follows:

$$s_1 = (-1)^{n-1}\lfloor x_1+x_2+\cdots+x_n\rfloor+(-1)^{n-2}\lfloor x_1+x_2+\cdots+x_{n-1}\rfloor+(-1)^{n-2}\lfloor x_n\rfloor. \tag{4.11}$$

Notice that the sign of $\lfloor x_n\rfloor$ in (4.11) may or may not be the same as that appearing in the sum defining $g(x_1, x_2, \ldots, x_n)$ but it is the same as the sign of $\lfloor x_1 + x_2 + \cdots + x_{n-1}\rfloor$ so that we can apply Lemma 2.2 to obtain the bound for s_1. Next we pair up the remaining $(n-1)$-brackets with $(n-2)$-brackets and 1-brackets as follows:

$$(-1)^{n-2}\lfloor x_{i_1}+x_{i_2}+\cdots+x_{i_{n-1}}\rfloor+(-1)^{n-3}\lfloor x_{i_1}+x_{i_2}+\cdots+x_{i_{n-2}}\rfloor+(-1)^{n-3}\lfloor x_{i_{n-1}}\rfloor, \tag{4.12}$$

where $1 \le i_1 < i_2 < \ldots < i_{n-1} \le n$. We note again that the sign of $\lfloor x_{i_1} + x_{i_2} + \cdots + x_{i_{n-1}}\rfloor$ and $\lfloor x_{i_1} + x_{i_2} + \cdots + x_{i_{n-2}}\rfloor$ in (4.12) are the same as those appearing in the sum defining $g(x_1, x_2, \ldots, x_n)$ while the sign of $\lfloor x_{i_{n-1}}\rfloor$ in (4.12) may or may not be the same, but we can apply Lemma 2.2 to obtain the bound of (4.12). Since $\lfloor x_1 + x_2 + \cdots + x_{n-1}\rfloor$ appears in (4.11), the term $x_{i_{n-1}}$ appearing in the $(n-1)$-brackets in (4.12) is always x_n. So in fact (4.12) is

$$(-1)^{n-2}\lfloor x_{i_1}+x_{i_2}+\cdots+x_{i_{n-2}}+x_n\rfloor+(-1)^{n-3}\lfloor x_{i_1}+x_{i_2}+\cdots+x_{i_{n-2}}\rfloor+(-1)^{n-3}\lfloor x_n\rfloor. \tag{4.13}$$

Then we sum (4.13) over all possibles $1 \le i_1 < i_2 < \ldots < i_{n-2} < n$, and call it s_2. That is

$$\begin{aligned} s_2 = (-1)^{n-2} &\sum_{1\le i_1<i_2<\ldots<i_{n-2}<n} \lfloor x_{i_1} + x_{i_2} + \cdots + x_{i_{n-2}} + x_n\rfloor \\ &+ (-1)^{n-3} \sum_{1\le i_1<i_2<\ldots<i_{n-2}<n} \lfloor x_{i_1} + x_{i_2} + \cdots + x_{i_{n-2}}\rfloor \\ &+ (-1)^{n-3}\binom{n-1}{n-2}\lfloor x_n\rfloor. \end{aligned}$$

We continue doing this process as follows. For each $0 \le \ell \le n-1$, let c_ℓ be the sum of all $\lfloor x_{i_1} + x_{i_2} + \cdots + x_{i_{n-\ell}}\rfloor$ with $1 \le i_1 < i_2 < \ldots < i_{n-\ell} \le n$, a_ℓ

the sum of all such terms with $i_{n-\ell} = n$, and b_ℓ the sum of all such terms with $i_{n-\ell} < n$. Therefore $c_\ell = a_\ell + b_\ell$. As usual, the empty sum is defined to be zero, so $b_0 = 0$. The number of $(n-\ell)$-brackets appearing in the sum defining c_ℓ is $\binom{n}{n-\ell}$, the number of $(n-\ell)$-brackets appearing in the sum defining a_ℓ is $\binom{n-1}{n-\ell-1}$, and the number of $(n-\ell)$-brackets appearing in the sum defining b_ℓ is $\binom{n-1}{n-\ell}$. In addition, we have

$$\begin{aligned} s_1 &= (-1)^{n-1}a_0 + (-1)^{n-2}b_1 + (-1)^{n-2}\lfloor x_n \rfloor, \\ s_2 &= (-1)^{n-2}a_1 + (-1)^{n-3}b_2 + (-1)^{n-3}\binom{n-1}{n-2}\lfloor x_n \rfloor. \end{aligned}$$

In general, for each $1 \le \ell \le n-1$, we let

$$s_\ell = (-1)^{n-\ell}a_{\ell-1} + (-1)^{n-\ell-1}b_\ell + (-1)^{n-\ell-1}\binom{n-1}{n-\ell}\lfloor x_n \rfloor.$$

Then

$$\begin{aligned} \sum_{1\le\ell\le n-1} s_\ell = (-1)^{n-1}a_0 + \sum_{2\le\ell\le n-1}(-1)^{n-\ell}a_{\ell-1} + \sum_{1\le\ell\le n-2}(-1)^{n-\ell-1}b_\ell + b_{n-1} \\ + \lfloor x_n \rfloor \sum_{1\le\ell\le n-1}(-1)^{n-\ell-1}\binom{n-1}{n-\ell}. \end{aligned} \tag{4.14}$$

Recall from Lemma 2.6 (ii) that $\sum_{0\le\ell\le n}(-1)^\ell\binom{n}{\ell} = 0$ for all $n \ge 1$. Therefore the last sum on the right hand side of (4.14) is

$$\begin{aligned} -\sum_{1\le\ell\le n-1}(-1)^{n-\ell}\binom{n-1}{n-\ell} &= -\sum_{1\le\ell\le n-1}(-1)^{\ell}\binom{n-1}{\ell} \\ &= -\sum_{0\le\ell\le n-1}(-1)^{\ell}\binom{n-1}{\ell} + 1 = 1. \end{aligned}$$

Therefore the last term in (4.14) is $\lfloor x_n \rfloor$. Replacing ℓ by $\ell+1$ in the first sum on the right hand side of (4.14), we see that

$$\begin{aligned}
\sum_{1\le \ell\le n-1} s_\ell &= (-1)^{n-1}a_0 + \sum_{1\le \ell\le n-2}(-1)^{n-\ell-1}(a_\ell+b_\ell) + b_{n-1} + \lfloor x_n\rfloor \\
&= (-1)^{n-1}c_0 + \sum_{1\le \ell\le n-2}(-1)^{n-\ell-1}c_\ell + b_{n-1} + \lfloor x_n\rfloor \\
&= (-1)^{n-1}c_0 + \sum_{1\le \ell\le n-2}(-1)^{n-\ell-1}c_\ell + c_{n-1} \qquad (4.15) \\
&= \sum_{0\le \ell\le n-1}(-1)^{n-\ell-1}c_\ell \\
&= g(x_1, x_2, \ldots, x_n),
\end{aligned}$$

where (4.15) can be obtained from the definition of c_{n-1}, b_{n-1}, and a_{n-1} that

$$\begin{aligned}
c_{n-1} &= \lfloor x_1\rfloor + \lfloor x_2\rfloor + \cdots + \lfloor x_n\rfloor, \\
b_{n-1} &= \lfloor x_1\rfloor + \lfloor x_2\rfloor + \cdots + \lfloor x_{n-1}\rfloor, \\
a_{n-1} &= \lfloor x_n\rfloor, \quad \text{and} \\
c_{n-1} &= a_{n-1} + b_{n-1}.
\end{aligned}$$

We apply Lemma 2.2 and (4.1) to (4.11) to obtain

$$0 \le s_1 \le 1 \text{ if } n \text{ is odd, and } -1 \le s_1 \le 0 \text{ if } n \text{ is even.}$$

Similarly, applying Lemma 2.2 and (4.1) to (4.13), we see that such sum lies in $[0, 1]$ if n is even, and lies in $[-1, 0]$ if n is odd. Therefore

$$0 \le s_2 \le \binom{n-1}{n-2} \text{ if } n \text{ is even, and } -\binom{n-1}{n-2} \le s_2 \le 0 \text{ if } n \text{ is odd.}$$

In general, for each $1 \le \ell \le n-1$, we have

$$\begin{aligned}
0 \le s_\ell \le \binom{n-1}{n-\ell}&, \text{ if } n \text{ and } \ell \text{ have the same parity,} \\
-\binom{n-1}{n-\ell} \le s_\ell \le 0&, \text{ if } n \text{ and } \ell \text{ have a different parity.}
\end{aligned}$$

Since $g(x_1, x_2, \ldots, x_n) = \sum_{1\le \ell\le n-1} s_\ell$, we obtain, for odd n,

$$-\sum_{\substack{1\le \ell\le n-1 \\ \ell \text{ is even}}}\binom{n-1}{n-\ell} \le g(x_1, x_2, \ldots, x_n) \le \sum_{\substack{1\le \ell\le n-1 \\ \ell \text{ is odd}}}\binom{n-1}{n-\ell},$$

and for even n,

$$-\sum_{\substack{1\le \ell\le n-1\\ \ell \text{ is odd}}}\binom{n-1}{n-\ell}\le g(x_1,x_2,\ldots,x_n)\le \sum_{\substack{1\le \ell\le n-1\\ \ell \text{ is even}}}\binom{n-1}{n-\ell}.$$

Recall from Lemma 2.6 (iv) and Lemma 2.6 (v) that

$$\sum_{\substack{0\le k\le n\\ k \text{ is even}}}\binom{n}{k}=\sum_{\substack{0\le k\le n\\ k \text{ is odd}}}\binom{n}{k}=2^{n-1}.$$

Therefore if n is odd, then

$$\sum_{\substack{1\le \ell\le n-1\\ \ell \text{ is odd}}}\binom{n-1}{n-\ell}=\sum_{\substack{1\le \ell\le n-1\\ \ell \text{ is even}}}\binom{n-1}{\ell}=2^{n-2}-1, \text{ and}$$

$$\sum_{\substack{1\le \ell\le n-1\\ \ell \text{ is even}}}\binom{n-1}{n-\ell}=\sum_{\substack{1\le \ell\le n-1\\ \ell \text{ is odd}}}\binom{n-1}{\ell}=\sum_{\substack{0\le \ell\le n-1\\ \ell \text{ is odd}}}\binom{n-1}{\ell}=2^{n-2}.$$

Similarly, if n is even, then

$$\sum_{\substack{1\le \ell\le n-1\\ \ell \text{ is odd}}}\binom{n-1}{n-\ell}=2^{n-2} \text{ and } \sum_{\substack{1\le \ell\le n-1\\ \ell \text{ is even}}}\binom{n-1}{n-\ell}=2^{n-2}-1.$$

Hence $-2^{n-2}\le g(x_1,x_2,\ldots,x_n)\le 2^{n-2}-1$, as required. Next we show that the lower bound -2^{n-2} and the upper bound $2^{n-2}-1$ are actually the minimum and the maximum of $g(x_1,x_2,\ldots,x_n)$, respectively. Recall that the fractional part of a real number x, denoted by $\{x\}$, is defined by $\{x\}=x-\lfloor x\rfloor$. Let $x_k=\frac{1}{2}$ for every $k=1,2,\ldots,n$. Then

$$\begin{aligned} g(x_1,x_2,\ldots,x_n) &= \sum_{1\le k\le n}(-1)^{k-1}\left\lfloor\frac{k}{2}\right\rfloor\binom{n}{k}\\ &= \sum_{1\le k\le n}(-1)^{k-1}\left(\frac{k}{2}\right)\binom{n}{k}-\sum_{1\le k\le n}(-1)^{k-1}\left\{\frac{k}{2}\right\}\binom{n}{k}\\ &= \frac{1}{2}\sum_{1\le k\le n}(-1)^{k-1}k\binom{n}{k}-\frac{1}{2}\sum_{\substack{1\le k\le n\\ k \text{ is odd}}}\binom{n}{k}, \end{aligned} \tag{4.16}$$

where the last equality is obtained from the fact that $\left\{\frac{k}{2}\right\}=0$ if k is even and $\left\{\frac{k}{2}\right\}=\frac{1}{2}$ if k is odd. By Lemmas 2.6 (iii) and 2.6 (v), we obtain

$$g(x_1,x_2,\ldots,x_n)=0-\frac{1}{2}\left(2^{n-1}\right)=-2^{n-2}.$$

This shows that -2^{n-2} is the minimun value of g. Next let $x_k = \frac{1}{2} - \frac{1}{n^2}$ for every $k = 1, 2, \ldots, n$. Then

$$g(x_1, x_2, \ldots, x_n) = \sum_{1 \le k \le n} (-1)^{k-1} \left\lfloor \frac{k}{2} - \frac{k}{n^2} \right\rfloor \binom{n}{k}. \tag{4.17}$$

If $1 \le k \le n$ and k is even, then $\left\lfloor \frac{k}{2} - \frac{k}{n^2} \right\rfloor = \frac{k}{2} - 1 = \left\lfloor \frac{k-1}{2} \right\rfloor$. If $1 \le k \le n$ and k is odd, then $\left\lfloor \frac{k}{2} - \frac{k}{n^2} \right\rfloor = \left\lfloor \frac{k-1}{2} + \frac{1}{2} - \frac{k}{n^2} \right\rfloor = \left\lfloor \frac{k-1}{2} \right\rfloor$. Therefore (4.17) becomes

$$g(x_1, x_2, \ldots, x_n) = \sum_{1 \le k \le n} (-1)^{k-1} \left\lfloor \frac{k-1}{2} \right\rfloor \binom{n}{k}. \tag{4.18}$$

Now we can evaluate the sum (4.18) by using the same method as in (4.16). We write $\left\lfloor \frac{k-1}{2} \right\rfloor = \frac{k-1}{2} - \left\{ \frac{k-1}{2} \right\}$ and we know that $\left\{ \frac{k-1}{2} \right\} = 0$ if k is odd and $\left\{ \frac{k-1}{2} \right\} = \frac{1}{2}$ if k is even. Then (4.18) can be written as

$$g(x_1, x_2, \ldots, x_n) = \frac{1}{2} \sum_{1 \le k \le n} (-1)^{k-1} k \binom{n}{k} - \frac{1}{2} \sum_{1 \le k \le n} (-1)^{k-1} \binom{n}{k} + \frac{1}{2} \sum_{\substack{1 \le k \le n \\ k \text{ is even}}} \binom{n}{k}.$$

The first sum is zero by Lemma 2.6 (iii). The second sum is 1 by Lemma 2.6 (ii). By Lemma 2.6 (iv), we obtain

$$g(x_1, x_2, \ldots, x_n) = 0 - \frac{1}{2} + \frac{1}{2}\left(2^{n-1} - 1\right) = 2^{n-2} - 1.$$

□

Proof of Corollary 1.8. This follows immediately from (1.6) and Theorem 1.5.

□

Next we give the proof of Theorem 1.6. Although we can write $f_{a_1, a_2, \ldots, a_\ell; m}(k)$ in terms of $g(x_1, x_2, \ldots, x_n)$ as given in Lemma 4.1, we do not know the proof which applies Theorem 1.5 to obtain Theorem 1.6. Nevertheless, we can use the same idea in the proof of Theorem 1.5 to prove Theorem 1.6.

Proof of Theorem 1.6. By Lemma 4.2 (ii), we can assume that $a_i \in [0, m-1]$ for every $1 \le i \le \ell$. Therefore

$$\left\lfloor \frac{a_i}{m} \right\rfloor = 0 \text{ for every } i \in \{1, 2, \ldots, \ell\}. \tag{4.19}$$

If $\ell = 2$, then the result follows from (4.19) and Lemma 2.2, and we have

$$0 \leq \left\lfloor \frac{a_1 + a_2 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + k}{m} \right\rfloor \leq 1, \tag{4.20}$$

and

$$-1 \leq -\left\lfloor \frac{a_2 + k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor \leq 0. \tag{4.21}$$

Summing (4.20) and (4.21), we obtain $-1 \leq f_{a_1,a_2;m}(k) \leq 1$. The result when $\ell \geq 3$ is based on a careful selection of pairs and the case $\ell = 2$. For illustration purpose, we first give a proof for the case $\ell = 3$ and $\ell = 4$. Recall that

$$\begin{aligned} f_{a_1,a_2,a_3;m}(k) = &\left\lfloor \frac{a_1 + a_2 + a_3 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_2 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_3 + k}{m} \right\rfloor \\ &- \left\lfloor \frac{a_2 + a_3 + k}{m} \right\rfloor + \left\lfloor \frac{a_1 + k}{m} \right\rfloor + \left\lfloor \frac{a_2 + k}{m} \right\rfloor + \left\lfloor \frac{a_3 + k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor. \end{aligned}$$

We obtain by Lemma 2.2 and (4.19) that

$$0 \leq \left\lfloor \frac{a_1 + a_2 + a_3 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_2 + k}{m} \right\rfloor \leq 1, \tag{4.22}$$

$$-1 \leq -\left\lfloor \frac{a_1 + a_3 + k}{m} \right\rfloor + \left\lfloor \frac{a_1 + k}{m} \right\rfloor \leq 0, \tag{4.23}$$

$$-1 \leq -\left\lfloor \frac{a_2 + a_3 + k}{m} \right\rfloor + \left\lfloor \frac{a_2 + k}{m} \right\rfloor \leq 0, \tag{4.24}$$

$$0 \leq \left\lfloor \frac{a_3 + k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor \leq 1. \tag{4.25}$$

Summing (4.22), (4.23), (4.24), and (4.25), we see that the middle term is $f_{a_1,a_2,a_3,m}(k)$. Therefore $-2 \leq f_{a_1,a_2,a_3;m}(k) \leq 2$. Next we consider

$$\begin{aligned} f_{a_1,a_2,a_3,a_4;m}(k) = &\left\lfloor \frac{a_1 + a_2 + a_3 + a_4 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_2 + a_3 + k}{m} \right\rfloor \\ &- \left\lfloor \frac{a_1 + a_2 + a_4 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_3 + a_4 + k}{m} \right\rfloor \\ &- \left\lfloor \frac{a_2 + a_3 + a_4 + k}{m} \right\rfloor + \left\lfloor \frac{a_1 + a_2 + k}{m} \right\rfloor + \left\lfloor \frac{a_1 + a_3 + k}{m} \right\rfloor \\ &+ \left\lfloor \frac{a_1 + a_4 + k}{m} \right\rfloor + \left\lfloor \frac{a_2 + a_3 + k}{m} \right\rfloor + \left\lfloor \frac{a_2 + a_4 + k}{m} \right\rfloor \\ &+ \left\lfloor \frac{a_3 + a_4 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + k}{m} \right\rfloor - \left\lfloor \frac{a_2 + k}{m} \right\rfloor - \left\lfloor \frac{a_3 + k}{m} \right\rfloor \\ &- \left\lfloor \frac{a_4 + k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor. \end{aligned}$$

Again, we obtain by Lemma 2.2 and (4.19) the following inequalities:

$$0 \leq \left\lfloor \frac{a_1 + a_2 + a_3 + a_4 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_2 + a_3 + k}{m} \right\rfloor \leq 1, \tag{4.26}$$

$$-1 \leq -\left\lfloor \frac{a_1 + a_2 + a_4 + k}{m} \right\rfloor + \left\lfloor \frac{a_1 + a_2 + k}{m} \right\rfloor \leq 0, \tag{4.27}$$

$$-1 \leq -\left\lfloor \frac{a_1 + a_3 + a_4 + k}{m} \right\rfloor + \left\lfloor \frac{a_1 + a_3 + k}{m} \right\rfloor \leq 0, \tag{4.28}$$

$$-1 \leq -\left\lfloor \frac{a_2 + a_3 + a_4 + k}{m} \right\rfloor + \left\lfloor \frac{a_2 + a_3 + k}{m} \right\rfloor \leq 0, \tag{4.29}$$

$$0 \leq \left\lfloor \frac{a_1 + a_4 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + k}{m} \right\rfloor \leq 1, \tag{4.30}$$

$$0 \leq \left\lfloor \frac{a_2 + a_4 + k}{m} \right\rfloor - \left\lfloor \frac{a_2 + k}{m} \right\rfloor \leq 1, \tag{4.31}$$

$$0 \leq \left\lfloor \frac{a_3 + a_4 + k}{m} \right\rfloor - \left\lfloor \frac{a_3 + k}{m} \right\rfloor \leq 1, \tag{4.32}$$

$$-1 \leq -\left\lfloor \frac{a_4 + k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor \leq 0. \tag{4.33}$$

Summing (4.26) to (4.33), we see that $-4 \leq f_{a_1,a_2,a_3,a_4,m}(k) \leq 4$.

Next we prove the general case $\ell \geq 5$. The expression of the form $\left\lfloor \frac{a_{i_1}+a_{i_2}+\cdots+a_{i_r}+k}{m} \right\rfloor$ will be called an *r-bracket*. So for each $1 \leq r \leq \ell$, there are $\binom{\ell}{r}$ r-brackets appearing in the sum defining $f_{a_1,a_2,\ldots,a_\ell;m}(k)$. We follow closely the method used in the proof of Theorem 1.5. So we first pair up the ℓ-bracket with an $(\ell-1)$-bracket as follows:

$$s_1 = \left\lfloor \frac{a_1 + a_2 + \cdots + a_\ell + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_2 + \cdots + a_{\ell-1} + k}{m} \right\rfloor, \tag{4.34}$$

and we can apply Lemma 2.2 and (4.19) to obtain the bound for s_1. Next we pair up the remaining $(\ell-1)$-brackets with $(\ell-2)$-brackets as follows:

$$-\left\lfloor \frac{a_{i_1} + a_{i_2} + \cdots + a_{i_{\ell-1}} + k}{m} \right\rfloor + \left\lfloor \frac{a_{i_1} + a_{i_2} + \cdots + a_{i_{\ell-2}} + k}{m} \right\rfloor, \tag{4.35}$$

and we sum (4.35) over all $1 \leq i_1 < i_2 < \ldots < i_{\ell-1} \leq \ell$ and call it s_2. Since a_ℓ does not appear in the second term on the right hand side of (4.34), the term

$a_{i_{\ell-1}}$ appearing in (4.35) is always a_ℓ. So in fact

$$s_2 = -\sum_{1\le i_1<i_2<\ldots<i_{\ell-2}<\ell} \left\lfloor \frac{a_{i_1}+a_{i_2}+\cdots+a_{i_{\ell-2}}+a_\ell+k}{m} \right\rfloor + \sum_{1\le i_1<i_2<\ldots<i_{\ell-2}<\ell} \left\lfloor \frac{a_{i_1}+a_{i_2}+\cdots+a_{i_{\ell-2}}+k}{m} \right\rfloor.$$

We continue doing this process as follows. For each $1 \le r \le \ell$, let c_r be the sum of all $\left\lfloor \frac{a_{i_1}+a_{i_2}+\cdots+a_{i_r}+k}{m} \right\rfloor$ with $1 \le i_1 < i_2 < \cdots < i_r \le \ell$, a_r the sum of all such terms with $i_r = \ell$, and b_r the sum of all such terms with $i_r < \ell$. Therefore $c_r = a_r + b_r$, the number of summands of c_r is $\binom{\ell}{r}$, the number of summands of a_r is $\binom{\ell-1}{r-1}$, and the number of summands of b_r is $\binom{\ell-1}{r}$. As usual, the empty sum is defined to be zero, so $b_\ell = 0$. We have $s_1 = a_\ell - b_{\ell-1}$ and $s_2 = -a_{\ell-1} + b_{\ell-2}$. In general, for each $1 \le r \le \ell-1$, we let

$$s_r = (-1)^{r+1}a_{\ell-r+1} + (-1)^r b_{\ell-r} \text{ and } s_\ell = (-1)^{\ell+1}a_1 + (-1)^\ell \left\lfloor \frac{k}{m} \right\rfloor.$$

Then

$$0 \le s_r \le \binom{\ell-1}{\ell-r} \text{ if } r \text{ is odd, and } -\binom{\ell-1}{\ell-r} \le s_r \le 0 \text{ if } r \text{ is even,}$$

$$\begin{aligned}
\sum_{1\le r\le \ell} s_r &= a_\ell + \sum_{2\le r\le \ell-1} (-1)^{r+1} a_{\ell-r+1} + \sum_{1\le r\le \ell-2} (-1)^r b_{\ell-r} + (-1)^{\ell-1} b_1 + s_\ell \\
&= a_\ell + \sum_{1\le r\le \ell-2} (-1)^r (a_{\ell-r} + b_{\ell-r}) + (-1)^{\ell-1} b_1 + (-1)^{\ell+1} a_1 + \left\lfloor \frac{k}{m} \right\rfloor \\
&= c_\ell + \sum_{1\le r\le \ell-2} (-1)^r c_{\ell-r} + (-1)^{\ell-1} c_1 + \left\lfloor \frac{k}{m} \right\rfloor \\
&= \sum_{0\le r\le \ell-1} (-1)^r c_{\ell-r} + \left\lfloor \frac{k}{m} \right\rfloor \\
&= f_{a_1,a_2,\ldots,a_\ell;m}(k).
\end{aligned}$$

Therefore

$$-\sum_{\substack{1\le r\le \ell \\ r \text{ is even}}} \binom{\ell-1}{\ell-r} \le f_{a_1,a_2,\ldots,a_\ell;m}(k) \le \sum_{\substack{1\le r\le \ell \\ r \text{ is odd}}} \binom{\ell-1}{\ell-r}.$$

Replacing r by $r+1$, we see that

$$\sum_{\substack{1\le r\le \ell\\ r \text{ is odd}}} \binom{\ell-1}{\ell-r} = \sum_{\substack{0\le r\le \ell-1\\ r \text{ is even}}} \binom{\ell-1}{\ell-1-r}.$$

By Lemma 2.4 (i) and Lemma 2.6 (iv), we obtain

$$\sum_{\substack{0\le r\le \ell-1\\ r \text{ is even}}} \binom{\ell-1}{\ell-1-r} = \sum_{\substack{0\le r\le \ell-1\\ r \text{ is even}}} \binom{\ell-1}{r} = 2^{\ell-2}.$$

Similarly,

$$-\sum_{\substack{1\le r\le \ell\\ r \text{ is even}}} \binom{\ell-1}{\ell-r} = -2^{\ell-2}.$$

Therefore

$$-2^{\ell-2} \le f_{a_1,a_2,\ldots,a_\ell;m}(k) \le 2^{\ell-2}, \tag{4.36}$$

as required. If ℓ is odd, m is even, and $a_i = \frac{m}{2}$ for every $1 \le i \le \ell$, we obtain by Lemma 4.1 and Theorem 1.5 that $f_{a_1,a_2,\ldots,a_\ell;m}(0) = g\left(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}\right) = -2^{\ell-2}$and $f_{a_1,a_2,\ldots,a_\ell;m}(\frac{m}{2}) = (-1)^\ell g\left(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}\right) + (-1)^{\ell-1} g\left(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}\right) = 2^{\ell-2}$. If ℓ is even, m is even, and $a_i = \frac{m}{2}$ for every $1 \le i \le \ell$, we obtain similarly that $f_{a_1,a_2,\ldots,a_\ell;m}(0) = 2^{\ell-2}$ and $f_{a_1,a_2,\ldots,a_\ell;m}(\frac{m}{2}) = -2^{\ell-2}$. So $2^{\ell-2}$ and $-2^{\ell-2}$ in (4.36) cannot be improved. This completes The proof. □

Proof of Theorem 1.7. If $\ell = 2$, then the result is already proved by Jacobsthal [10]. See also another proof by Tverberg [21]. We recall from (1.3) that

$$0 \le S_{a,b;m}(K) \le \left\lfloor \frac{m}{2} \right\rfloor. \tag{4.37}$$

As before the result when $\ell \ge 3$ is based on the case $\ell = 2$ and a careful selection of pairs. The case $\ell = 3$ is already shown in the proof of Theorem 3.4. So we show more ideas by giving the proof for the case $\ell = 4$. We have the following equalities:

$$f_{a_1+a_2+a_3,a_4;m}(k) = \left\lfloor \frac{a_1+a_2+a_3+a_4+k}{m} \right\rfloor - \left\lfloor \frac{a_1+a_2+a_3+k}{m} \right\rfloor - \left\lfloor \frac{a_4+k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor, \tag{4.38}$$

$$-f_{a_1+a_2,a_4;m}(k) = -\left\lfloor \frac{a_1+a_2+a_4+k}{m} \right\rfloor + \left\lfloor \frac{a_1+a_2+k}{m} \right\rfloor + \left\lfloor \frac{a_4+k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor, \tag{4.39}$$

$$-f_{a_1+a_3,a_4;m}(k) = -\left\lfloor \frac{a_1+a_3+a_4+k}{m} \right\rfloor + \left\lfloor \frac{a_1+a_3+k}{m} \right\rfloor + \left\lfloor \frac{a_4+k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor, \tag{4.40}$$

$$-f_{a_2+a_3,a_4;m}(k) = -\left\lfloor \frac{a_2+a_3+a_4+k}{m} \right\rfloor + \left\lfloor \frac{a_2+a_3+k}{m} \right\rfloor + \left\lfloor \frac{a_4+k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor, \tag{4.41}$$

$$f_{a_1,a_4;m}(k) = \left\lfloor \frac{a_1+a_4+k}{m} \right\rfloor - \left\lfloor \frac{a_1+k}{m} \right\rfloor - \left\lfloor \frac{a_4+k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor, \tag{4.42}$$

$$f_{a_2,a_4;m}(k) = \left\lfloor \frac{a_2+a_4+k}{m} \right\rfloor - \left\lfloor \frac{a_2+k}{m} \right\rfloor - \left\lfloor \frac{a_4+k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor, \tag{4.43}$$

$$f_{a_3,a_4;m}(k) = \left\lfloor \frac{a_3+a_4+k}{m} \right\rfloor - \left\lfloor \frac{a_3+k}{m} \right\rfloor - \left\lfloor \frac{a_4+k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor. \tag{4.44}$$

Summing (4.38) to (4.44) and recalling the definition of $f_{a_1,a_2,a_3,a_4;m}(k)$, we see that

$$\begin{aligned} f_{a_1,a_2,a_3,a_4;m}(k) &= f_{a_1+a_2+a_3,a_4;m}(k) - f_{a_1+a_2,a_4;m}(k) - f_{a_1+a_3,a_4;m}(k) \\ &\quad - f_{a_2+a_3,a_4;m}(k) + f_{a_1,a_4;m}(k) + f_{a_2,a_4;m}(k) + f_{a_3,a_4;m}(k). \end{aligned} \tag{4.45}$$

Then we obtain from (4.45) and (4.37) that

$$\begin{aligned} S_{a_1,a_2,a_3,a_4;m}(K) &= S_{a_1+a_2+a_3,a_4;m}(K) - S_{a_1+a_2,a_4;m}(K) - S_{a_1+a_3,a_4;m}(K) \\ &\quad - S_{a_2+a_3,a_4;m}(K) + S_{a_1,a_4;m}(K) + S_{a_2,a_4;m}(K) + S_{a_3,a_4;m}(K) \\ &\le \left\lfloor \frac{m}{2} \right\rfloor - 0 - 0 - 0 + \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor = 4\left\lfloor \frac{m}{2} \right\rfloor. \end{aligned}$$

Similarly, $S_{a_1,a_2,a_3,a_4;m}(K) \ge -4\left\lfloor \frac{m}{2} \right\rfloor$. Next we prove the general case $\ell \ge 5$. The expression of the form $\left\lfloor \frac{a_{i_1}+a_{i_2}+\cdots+a_{i_r}+k}{m} \right\rfloor$ will be called an *r-bracket*. So for each $0 \le r \le \ell$, there are $\binom{\ell}{r}$ r-brackets appearing in the sum defining $f_{a_1,a_2,\ldots,a_\ell;m}(k)$. We first pair up the ℓ-bracket with an $(\ell-1)$-bracket, a 1-bracket and a 0-bracket as follows:

$$\begin{aligned} s_1(k) = &\left\lfloor \frac{a_1+a_2+\cdots+a_\ell+k}{m} \right\rfloor - \left\lfloor \frac{a_1+a_2+\cdots+a_{\ell-1}+k}{m} \right\rfloor - \left\lfloor \frac{a_\ell+k}{m} \right\rfloor \\ &+ \left\lfloor \frac{k}{m} \right\rfloor. \end{aligned} \tag{4.46}$$

So $s_1(k)$ is in fact $f_{a_1+a_2+\cdots+a_{\ell-1},a_\ell;m}(k)$ and we can apply (4.37) to obtain the inequality

$$0 \le S_{a_1+a_2+\cdots+a_{\ell-1},a_\ell;m}(K) = \sum_{k=0}^{K} s_1(k) \le \left\lfloor \frac{m}{2} \right\rfloor .$$

Next we pair up the remaining $(\ell-1)$-brackets with $(\ell-2)$-brackets, 1-brackets and 0-brackets as follows:

$$- \left\lfloor \frac{a_{i_1}+a_{i_2}+\cdots+a_{i_{\ell-1}}+k}{m} \right\rfloor + \left\lfloor \frac{a_{i_1}+a_{i_2}+\cdots+a_{i_{\ell-2}}+k}{m} \right\rfloor + \left\lfloor \frac{a_{i_{\ell-1}}+k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor , \tag{4.47}$$

and we sum (4.47) over all $1 \le i_1 < i_2 < \cdots < i_{\ell-1} \le \ell$ and call it $s_2(k)$. Since a_ℓ does not appear in the second term on the right hand side of (4.46), the term $a_{i_{\ell-1}}$ appearing in (4.47) is always a_ℓ. So in fact (4.47) is $-f_{a_{i_1}+a_{i_2}+\cdots+a_{i_{\ell-2}},a_\ell;m}(k)$ and

$$s_2(k) = - \sum_{1\le i_1<i_2<\ldots<i_{\ell-2}<\ell} f_{a_{i_1}+a_{i_2}+\cdots+a_{i_{\ell-2}},a_\ell;m}(k)$$

Furthermore,

$$\sum_{k=0}^{K} s_2(k) = - \sum_{1\le i_1<i_2<\ldots<i_{\ell-2}<\ell} S_{a_{i_1}+a_{i_2}+\cdots+a_{i_{\ell-2}},a_\ell;m}(K) \le 0,$$

where the last inequality is obtained from (4.37). We continue doing this process and follow closely the method used in the proof of Theorems 1.5 and 1.6. The well-known identities previously recalled will be applied without reference. For each $1 \le r \le \ell$, let $c_r(k)$ be the sum of all $\left\lfloor \frac{a_{i_1}+a_{i_2}+\cdots+a_{i_r}+k}{m} \right\rfloor$ with $1 \le i_1 < i_2 < \cdots < i_r \le \ell$, $a_r(k)$ the sum of all such terms with $i_r = \ell$, and $b_r(k)$ the sum of all such terms with $i_r < \ell$. Therefore $c_r(k) = a_r(k)+b_r(k)$, the number of r-brackets appearing in the sum defining $c_r(k)$ is $\binom{\ell}{r}$, the number of r-brackets appearing in the sum defining $a_r(k)$ is $\binom{\ell-1}{r-1}$, and the number of r-brackets appearing in the sum defining $b_r(k)$ is $\binom{\ell-1}{r}$. As usual, the empty sum is defined to be zero, so $b_\ell(k) = 0$. We have $s_1(k) = a_\ell(k) - b_{\ell-1}(k) - a_1(k) + \left\lfloor \frac{k}{m} \right\rfloor$ and $s_2(k) = -a_{\ell-1}(k) + b_{\ell-2}(k) + \binom{\ell-1}{\ell-2} a_1(k) - \binom{\ell-1}{\ell-2} \left\lfloor \frac{k}{m} \right\rfloor$. In general, for each

$1 \leq r \leq \ell - 1$, we let

$$\begin{aligned} s_r(k) &= (-1)^{r+1} a_{\ell-r+1}(k) + (-1)^r b_{\ell-r}(k) + (-1)^r \binom{\ell-1}{\ell-r} a_1(k) \\ &\quad + (-1)^{r+1} \binom{\ell-1}{\ell-r} \left\lfloor \frac{k}{m} \right\rfloor \\ &= (-1)^{r+1} \sum_{1 \leq i_1 < i_2 < \ldots < i_{\ell-r} < \ell} f_{a_{i_1}+a_{i_2}+\cdots+a_{i_{\ell-r}}, a_\ell; m}(k). \end{aligned}$$

Then

$$\sum_{k=0}^{K} s_r(k) = (-1)^{r+1} \sum_{1 \leq i_1 < i_2 < \ldots < i_{\ell-r} < \ell} S_{a_{i_1}+a_{i_2}+\cdots+a_{i_{\ell-r}}, a_\ell; m}(K).$$

So by (4.37), we see that

$$0 \leq \sum_{k=0}^{K} s_r(k) \leq \binom{\ell-1}{\ell-r} \left\lfloor \frac{m}{2} \right\rfloor \text{ if } r \text{ is odd,}$$

and

$$-\binom{\ell-1}{\ell-r} \left\lfloor \frac{m}{2} \right\rfloor \leq \sum_{k=0}^{K} s_r(k) \leq 0 \text{ if } r \text{ is even.}$$

Similar to the proof of Theorems 1.5 and 1.6, we obtain

$$\begin{aligned} \sum_{1 \leq r \leq \ell-1} s_r(k) &= a_\ell + \sum_{2 \leq r \leq \ell-1} (-1)^{r+1} a_{\ell-r+1} + \sum_{1 \leq r \leq \ell-2} (-1)^r b_{\ell-r} + (-1)^{\ell-1} b_1 \\ &\quad + (-1)^{\ell+1} a_1 + (-1)^{\ell} \left\lfloor \frac{k}{m} \right\rfloor \\ &= a_\ell + \sum_{1 \leq r \leq \ell-2} (-1)^r (a_{\ell-r} + b_{\ell-r}) + (-1)^{\ell-1} b_1 + (-1)^{\ell+1} a_1 \\ &\quad + (-1)^{\ell} \left\lfloor \frac{k}{m} \right\rfloor \\ &= c_\ell + \sum_{1 \leq r \leq \ell-2} (-1)^r c_{\ell-r} + (-1)^{\ell-1} c_1 + (-1)^{\ell} \left\lfloor \frac{k}{m} \right\rfloor \\ &= \sum_{0 \leq r \leq \ell-1} (-1)^r c_{\ell-r} + (-1)^{\ell} \left\lfloor \frac{k}{m} \right\rfloor \\ &= f_{a_1, a_2, \ldots, a_\ell; m}(k). \end{aligned}$$

Therefore

$$-\sum_{\substack{1 \leq r \leq \ell-1 \\ r \text{ is even}}} \binom{\ell-1}{\ell-r} \left\lfloor \frac{m}{2} \right\rfloor \leq \sum_{k=0}^{K} f_{a_1, a_2, \ldots, a_\ell; m}(k) \leq \sum_{\substack{1 \leq r \leq \ell-1 \\ r \text{ is odd}}} \binom{\ell-1}{\ell-r} \left\lfloor \frac{m}{2} \right\rfloor. \tag{4.48}$$

The middle term in (4.48) is $S_{a_1,a_2,\ldots,a_\ell;m}(K)$. The left and right most terms in (4.48) are, respectively, equal to $-2^{\ell-2}\left\lfloor\frac{m}{2}\right\rfloor$ and $2^{\ell-2}\left\lfloor\frac{m}{2}\right\rfloor$ which can be evaluated by the well-known identity previously recalled. This proves the first part of the theorem. Next we show that one of the upper bound or lower bound is sharp. Let $C=\{a_1,a_2,\ldots,a_\ell\}$. Suppose ℓ is odd, m is even, and $a_i=\frac{m}{2}$ for every $1\le i\le\ell$. Then we obtain by Lemma 4.1 (i) and Theorem 1.5 that $f_{C;m}(0)=g\left(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}\right)=-2^{\ell-2}$. Let $0<k<\frac{m}{2}$. By the definition of $f_{C;m}(k)$, we see that

$$
\begin{aligned}
f_{C;m}(k) &= \sum_{T\subseteq[1,\ell]}(-1)^{\ell-|T|}\left\lfloor\frac{k}{m}+\frac{|T|}{2}\right\rfloor\\
&= \sum_{r=0}^{\ell}(-1)^{\ell-r}\binom{\ell}{r}\left\lfloor\frac{k}{m}+\frac{r}{2}\right\rfloor
\end{aligned}
\tag{4.49}
$$

Since $0<k<\frac{m}{2}$, we have $\frac{r}{2}<\frac{k}{m}+\frac{r}{2}<\frac{r+1}{2}$. So if r is even, then $\left\lfloor\frac{k}{m}+\frac{r}{2}\right\rfloor=\frac{r}{2}=\left\lfloor\frac{r}{2}\right\rfloor$ and if r is odd, then $\left\lfloor\frac{k}{m}+\frac{r}{2}\right\rfloor=\frac{r-1}{2}=\left\lfloor\frac{r}{2}\right\rfloor$. In any case, $\left\lfloor\frac{k}{m}+\frac{r}{2}\right\rfloor=\frac{r}{2}=\left\lfloor\frac{0}{m}+\frac{r}{2}\right\rfloor$. This implies that $f_{C;m}(k)=f_{C;m}(0)$ for every $k=0,1,2,\ldots,\frac{m}{2}-1$. Then

$$
S_{C;m}\left(\frac{m}{2}-1\right)=\sum_{k=0}^{\frac{m}{2}-1}f_{C;m}(k)=\frac{m}{2}f_{C;m}(0)=-2^{\ell-2}\left\lfloor\frac{m}{2}\right\rfloor
$$

So $-2^{\ell-2}\left\lfloor\frac{m}{2}\right\rfloor$ in (1.5) cannot be improved when ℓ is odd. Next suppose ℓ is even, m is even, and $a_i=\frac{m}{2}$ for every $1\le i\le\ell$. Then we obtain similarly that $f_{C;m}(k)=f_{C;m}(0)=2^{\ell-2}$ for every $k=0,1,2,\ldots,\frac{m}{2}-1$. Then $S_{C;m}(\frac{m}{2}-1)=2^{\ell-2}\left\lfloor\frac{m}{2}\right\rfloor$. So $2^{\ell-2}\left\lfloor\frac{m}{2}\right\rfloor$ in (1.5) cannot be improved when ℓ is even. This completes the proof.

□

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