AN UPPER BOUND FOR THE DISTANCE BETWEEN A POINT AND A ROOT OF A POLYNOMIAL

By

MR. Teraporn THONGSIRI

A Thesis Submitted in Partial Fulfillment of the Requirements

for Master of Science (MATHEMATICS)

Department of MATHEMATICS

Graduate School, Silpakorn University

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ขอบเขตบนสำหรับระยะทางระหว่างจุดกับรากของพหุนาม

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นายธีรพร ทองศิริ

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Title	An upper bound for the distance between a point and a root of a polynomial
By	Teraporn THONGSIRI
Field of Study	(MATHEMATICS)
Advisor	Somjate Chaiya

Graduate School Silpakorn University in Partial Fulfillment of the Requirements for the Master of Science

.... Dean of graduate school

(Associate Professor Jurairat Nunthanid, Ph.D.)

Approved by

.... Chair person

(Assistant Professor Dr. Jittisak Rakbud)

.... Advisor

(Assistant Professor Dr. Somjate Chaiya)

.... External Examiner

(Assistant Professor Dr. Thiradet Jiarasuksakun)

56305204 : MAJOR (MATHEMATICS)

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Newton's method is one of the most popular root-finding algorithms for meromorphic functions. In 2002, Dierk Schleicher established an explicit upper bound for the number of iterations of Newton's method for complex polynomials with a prescribed precision. In his work, Schleicher needed an upper bound, namely f_d, for the distance between a starting point z_0 and the root α, where z_0 is in the immediate basin of α and d is the degree of the polynomial. In 2011, Somjate Chaiya gave an algorithm to improve the value of f_d. In this research, we establish a new explicit bound for f_d.

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ระเบียบวิธีของนิวตันเป็นวิธีการที่นิยมใช้ในการหารากของฟังก์ชันเมโรมอร์ฟิก ในปี ค.ศ. 2002 เดิร์ค ไชเชอร์ได้ให้ค่าขอบเขตบนของจำนวนการกระทำซ้ำของระเบียบวิธีการของนิวตัวสำหรับพหุนามเชิงซ้อนภายใต้ค่าความคลาดเคลื่อนที่กำหนดไว้ ในการหาค่าขอบเขตบนนั้น เดิร์ค ไชเชอร์ต้องใช้ค่าขอบเขตบนของระยะทางระหว่างจุด z_0 ซึ่งอยู่ในอิมมิเดียทเบสินของราก α กับราก α โดยให้ f_d แทนค่าขอบเขตบนของระยะทางนี้ ในปี ค.ศ. 2011 สมเจตน์ ชัยยะได้นำแสนอขั้นตอนวิธีที่ทำให้ได้ค่าขอบเขตบนของ f_d ที่ดีขึ้น ในงานวิจัยนี้เราได้ให้ค่าขอบเขตบนอันใหม่ของ f_d ที่อยู่ในรูปชัดแจ้ง

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Teraporn THONGSIRI

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Chapter 1

Introduction

Let P be a polynomial of degree d. The Newton map induced by P is the function $N_p(z) = z - \frac{P(z)}{P'(z)}$. Let $\mathbb{N}$ be the set of positive integers. For each $k \in \mathbb{N}$, let N_p^k denote the k-iteration of N_p, that is, $N_p^1 = N_p$, $N_p^2 = N_p \circ N_p$, and $N_p^k = N_p^{k-1} \circ N_p$. For a root α of P, we say that a set $U \subsetneq \mathbb{C}_\infty$ is the immediate basin of α if $U \subseteq \mathbb{C}$ is the largest connected open set containing α and $N_p^k(z) \to \alpha$, as $k \to \infty$, for all $z \in U$. Every immediate basin U is forward invariant, that is, $N_p(U) = U$, and is simply connected. (See [1], [2])

In 2002, Dierk Schleicher (See [3]) provided an upper bound for the number of iterations of Newton's method for complex polynomials of a fixed degree with a prescribed precision. More precisely, Schleicher proved that if all roots of P are inside the unit disc and $0 < \varepsilon < 1$, there is a constant $n(d, \varepsilon)$ such that for every root α of P, there is a point z with $|z| = 2$ such that $\left|N_p^n(z) - \alpha\right| < \varepsilon$ for all $n > n(d, \varepsilon)$. Schleicher showed that $n(d, \varepsilon)$ can be chosen so that

$$n(d, \varepsilon) \leq \frac{9\pi d^4 f_d^2}{\varepsilon^2 \log 2} + \frac{|\log \varepsilon| + \log 13}{\log 2} + 1 \tag{1.1}$$

with

$$f_d = \frac{d^2(d-1)}{2(2d-1)} \binom{2d}{d}. \tag{1.2}$$

To obtain this eslimate, Schleicher employed several rough estimates which cause the bound far from an efficient upper bound, The main point that causes the extreme inefficiency is the way Schleicher used to obtain f_d which arose when he estimated an upper bound for the distance of a point z to a root α. Schleicher showed that if z is in the immediate basin of α and $|N_p(z) - z| = \delta$, then the distance between z and α is at most δf_d.

In 2011, Somjate Chaiya (See [4]) gave an algorithm to improve the value of f_d. Even though, it is not an explicit formula, it can be easily computed. The following is the result established by Chaiya.

Theorem 1.1. *Let $P(z)$ be a polynomial of degree $d \geq 3$, and let y be a positive number larger than $4d-3$. If z_0 is in an immediate basin of a root α and $|N_p(z_0) - z_0| = \varepsilon$, then $|z_o - \alpha| \leq \varepsilon M(d,y)$, where $M(d,y) = \max\{y, A_d + \frac{y(d-1)}{y-1}\}$ and A_d can be derived from the following iterative algorithm.*

Let $b = \frac{y(y-d)}{y-1}$, and

$$A_2 = \frac{y(d-1)[2d(y-2d+3)-3y-1]}{(y-1)(y-4d+3)}. \tag{1.3}$$

For $k = 2, \ldots, d-1$, set $a_k = 1 + \sum_{j=2}^{k-1} \frac{A_k}{A_k + A_j}$.

If $2A_k < b$ then let

$$A_{k+1} = A_k \left(\frac{(a_k + d - k)A_k + b(k+1-a_k-d)}{A_k(a_k+1) - ba_k} \right). \tag{1.4}$$

Otherwise let

$$A_{k+1} = A_k \left(\frac{a_k + d - k}{a_k} \right). \tag{1.5}$$

Note that the value of $M(d,y)$ in the theorem depends only on the constant y and the degree d. In this thesis, we first want to study the value of $M(d,y)$ as a function of y so that we can determine the best possible bound for $M(d,y)$ under this algorithm. Furthermore we want to find an explicit upper bound for f_d.

In chapter 2, we present definitions and some properties about dynamics of rational functions.

In chapter 3, we will find an upper bound of $M(d,y)$ in the cases of $d=3$ and $d \geq 4$ when the value y satisfies $2A_2 \geq b$. In the case $d=3$, we choose $y = 12+\sqrt{95}$, then $M(d,y) \leq 12+\sqrt{95}$ which is the best bound for this case. In the case $d \geq 4$, we choose $y = 4d^2-7d+3$, so that $2A_2 \geq b$. Then we obtain

$$M(d,y) \leq \frac{2C(d-1)(4d^2-7d+3)}{(4d^2-7d+2)(4d^2-11d+6)}(4d^3-13d^2+13d-2)$$

when $C = (d-1)\prod_{k=3}^{d-1}\left(\frac{2d^2-(k-1)d-2}{(k+1)d-2}\right)$.

In chapter 4, instead of finding y, we find an upper bound of f_d in a different way. We obtain that the distance of a point z and a root α of a polynomial of degree $d \geq 4$ is less than $(M_d+1)d\varepsilon$, where

$$M_d = (7+\sqrt{17})\prod_{k=3}^{\lfloor d/2 \rfloor}\frac{2d^2-k^2+3k-2d-2}{(k-1)(2d-k+2)}\prod_{n=\lfloor d/2 \rfloor+1}^{d-1}\frac{9d^2-4kd-2d-3}{d^2+4kd-2d-3}.$$

In chapter 5, we will compare the upper bounds of the distance between a point $z \in \mathbb{C}$ and a root of α, derived in Chapter 3 and Chapter 4, to the upper bound f_d, given by D. Schleicher.

Chapter 2

Complex Dynamics of the Rational Functions

We denote $\mathbb{C}\cup\{\infty\}$ by $\mathbb{C}_\infty$ and call it **the extended complex plane**.

Theorem 2.1. *[5] The function $\sigma : \mathbb{C}_\infty \times \mathbb{C}_\infty \to \mathbb{R}$, which is defined by*

$$\sigma(z,w) = \begin{cases} \frac{2|z-w|}{\left(1+|z|^2\right)^{1/2}\left(1+|w|^2\right)^{1/2}}, & \text{if} \quad z, w \in \mathbb{C}; \\ \frac{2}{\left(1+|z|^2\right)^{1/2}}, & \text{if} \quad z \in \mathbb{C} \text{ and } w = \infty; \\ 0, & \text{if} \quad z = w = \infty, \end{cases}$$

is a metric on $\mathbb{C}_\infty$.

The metric σ is called the **chordal metric** on $\mathbb{C}_\infty$. Note that the chordal metric on $\mathbb{C}_\infty$ is bounded.

2.1 Rational Functions

Definition 2.2. We say that P is **a polynomial function** or **a polynomial map** if P is of the form

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_{n-1} z^{n-1} + a_n z^n$$

when $n \in \mathbb{N}\cup\{0\}$, $a_n \neq 0$ and $a_i \in \mathbb{C}$, for all $i = 0, 1, \ldots, n$. We call n **the degree of the polynomial of** P, denoted by $\deg P$. We call 0 the zero polynomial function.

Definition 2.3. Let P and Q be polynomial functions. A function R which is defined by

$$R(z) = \frac{P(z)}{Q(z)},$$

is called **a rational function**. If P is a zero function, R is then also a zero function. If Q is a zero function but P is not a zero function, R is a constant function ∞. We define $R(\infty)$ as the limit of $R(z)$ as $z \to \infty$. **The degree of** R is defined by $\deg R = \max\{\deg P, \deg Q\}$. If $R = 0$ or $R = \infty$, we define $\deg R = 0$.

Definition 2.4. Let $D \subseteq \mathbb{C}$. A function $f : D \to \mathbb{C}$ is **holomorphic** in D if $f'(x)$ exists for all $x \in D$.

Definition 2.5. Let $D \subseteq \mathbb{C}$. A function $f : D \to \mathbb{C}_\infty$ is **meromorphic** in D if each point of $z \in D$ has a neighborhood in which either f or $\dfrac{1}{f}$ is holomorphic.

Definition 2.6. Let D_1 and D_2 be subsets of $\mathbb{C}$. A function $f : D_1 \to D_2$ is **analytic** in D_1 if f is holomorphic or meromorphic in D_1

In fact, if R is a rational map with $\deg R = d$, then the number of the solutions of the equation $f(z) = w$ is exactly d (counting multiplicities).

Theorem 2.7. *[5] Let $D \subseteq \mathbb{C}$ and let R and S be the finite degree rational functions on the domain D. Then*

(i) $\deg(RS) = \deg(R)\deg(S)$,

(ii) $\deg(R^n) = (\deg(R))^n$.

Theorem 2.8. *[5] The rational functions of degree one are Möbius transformations.*

Definition 2.9. Let R and S be rational functions. We say that R and S are **conjugate** if there exists a Möbius transformation g such that $S = gRg^{-1}$

Theorem 2.10. *[5] Let R and S be rational functions. If R and S are conjugate with a Möbius transformation g, i.e. $S = gRg^{-1}$, then*

(i) $\deg(R) = \deg(S)$,

(ii) $S^n = gR^n g^{-1}$,

(iii) $g(z)$ is a fixed point of S if and only if z is a fixed point of R.

Theorem 2.11. *[5] Let R be rational function. Then R is a polynomial if and only if $R^{-1}\{\infty\} = \{\infty\}$. In general, a non-constant rational R is conjugate to a polynomial if and only if there exists $w \in \mathbb{C}_\infty$ such that $R^{-1}\{w\} = \{w\}$.*

Theorem 2.12. *[5] A non-constant rational function of degree d has precisely $d+1$ fixed points in $\mathbb{C}_\infty$.*

Next, let (X_1, d_1) and (X_2, d_2) be metric spaces.

Definition 2.13. A family $\mathcal{F}$ of maps from (X_1, d_1) into (X_2, d_2) is **equicontinuous** at x_0 if for every positive real number ε, there exists a positive real number δ such that

$$d_2\left(f\left(x_0\right), f\left(x\right)\right) < \varepsilon$$

for all $x \in X$, $d_1\left(x_0, x\right) < \delta$, and for all $f \in \mathcal{F}$.

Definition 2.14. A sequence $\{f_n\}$ of maps from (X, d) into (X_1, d_1) **converges locally uniformly** on X to some function f if for each point $x \in X$ has a neighborhood on which f_n converges uniformly to f.

Definition 2.15. A family $\mathcal{F}$ of maps from (X_1, d_1) into (X_2, d_2) is **normal** in X_1 if for every sequence of functions in $\mathcal{F}$ contains a subsequence which converges locally uniformly on X_1.

Theorem 2.16 (Arzela-Ascoli Theorem)**.** *[5] Let D be an open connected subset of the complex sphere, and let $\mathcal{F}$ be a family of continuous maps on D into the sphere. Then $\mathcal{F}$ is equicontinuous in D if and only if it is a normal family in D.*

Theorem 2.17 (Vitali's Theorem)**.** *[5] Let D be a sub-domain of the complex sphere, and a sequence of analytic functions $\{f_n\}$ be normal in D. If f_n converge pointwise to some function f on some non-empty open subset W of D, then there exists an analytic F on D such that f_n converge locally uniformly to F on D and $f = F$ on W.*

Let $\mathfrak{C}$ be the class of continuous functions of $\mathbb{C}_\infty$ into itself and let $\mathfrak{R}$ be the subclass of rational functions.

Theorem 2.18. *[5] Let f_n be the sequence of analytic functions in a domain D of $\mathbb{C}_\infty$. If f_n converges uniformly on D to f with respect to σ. Then f is analytic in D*

Theorem 2.19. *[5] The map $\deg : \mathfrak{R} \to \mathbb{N}_0$ is continuous. In particular, if the rational functions R_n converge uniformly on the complex sphere to a function R, then R is rational and for all sufficiently large n, $\deg R_n = \deg R$.*

2.2 Fatou Sets and Julia Sets

In this section, let X be a set and $g : X \to X$ be a map.

Definition 2.20. Let $E \subseteq X$. We say that E is

(i) **forward invariant** under g if $g(E) = E$;

(ii) **backward invariant** under g if $g^{-1}(E) = E$;

(iii) **completely invariant** under g if g is both forward and backward invariant.

Note that if g is surjective then $g(g^{-1}(E)) = E$. So if E is backward invariant under g, E is completely invariant under g.

Theorem 2.21. *[5] Let R be a rational map of degree at least two. If a finite set E is completely invariant under R, then E has at most two elements.*

Lemma 2.22. *[5] Let E be a subset of X and $g, h : X \to X$ be functions. Suppose that g is surjective and h is bijective. If E is completely invariant under g, then $h(E)$ is completely invariant under hgh^{-1}.*

Lemma 2.23. *[5] Let $g : X \to X$ be surjective. The intersection of a family of completely invariant sets under g is completely invariant under g.*

Let E_0 be a subset of X. By lemma 2.23, we have that

$$E = \bigcap \{F \subseteq X : F \text{ is completely invariant and } E_0 \subseteq F\}$$

is completely invariant. In the other words, E is the smallest completely invariant set that contains E_0. We say that E_0 generates E.

Next, we define the relation $\sim$ on X by $x \sim y$ if and only if there exist non-negative integers m and n such that

$$g^m(x) = g^n(y). \tag{2.1}$$

Theorem 2.24. *[5] The relation* $\sim$ *that is defined by the relation* (2.1) *is an equivalence relation.*

In Theorem 2.24. We call the equivalence class containing x **"**orbit of x**"**, denoted by $[x]$.

Theorem 2.25. *[5] Let x be a point in X. If g be surjective, then* $[x]$ *is the completely invariant set generated by* $\{x\}$.

By Theorem 2.25, we have that a set E is completely invariant if and only if E is a union of equivalence classes $[x]$.

Theorem 2.26. *[5] Let g be a continuous and open map of a topological space X onto itself. If E is completely invariant, then the complement* $(X - E)$*, the interior* $(\operatorname{Int} E)$*, the boundary* (∂E) *and the closure* $(\bar{E})$ *of E are completely invariant.*

Next, let R be a rational map. We consider the equivalence class $[z]$. By Theorem 2.25, the equivalence class $[z]$ is the smallest completely invariant set that contains z.

Definition 2.27. A point z is said to be **an exceptional point** for R if $[z]$ is finite, and the set of all exceptional points for R is denoted by $E(R)$.

Theorem 2.28. *[5] If* $\deg(R) \geq 2$*, then R has at most two exceptional points. Moreover,* *(i) if $E(R) = \{\zeta\}$, then R is conjugate to a polynomial with ζ corresponding to ∞;*

(ii) if $E(R) = \{\zeta_1, \zeta_2\}$ and $\zeta_1 \neq \zeta_2$, then R is conjugate to a map $z \to z^d$, for some integer d, where ζ_1 and ζ_2 correspond to 0 and ∞.

Definition 2.29. For each z, **the backward orbit of** z, denoted by $O^-(z)$, is the set

$$O^-(z) = \{w : \exists n \in \mathbb{N}_0, R^n(w) = z\} = \bigcup_{n \in \mathbb{N}_0} R^{-n}\{z\}.$$

We call the points in $O^-(z)$ the predecessors of z.

Remark 2.30. *[5] For each $z \in X$, $O^-(z) \subseteq [z]$.*

Theorem 2.31. *[5] The backward orbit $O^-(z)$ of z is finite if and only if z is an exceptional point.*

Theorem 2.32. *[5] Let (X_1, d_1) and (X_2, d_2) be metric spaces and let $\mathcal{F}$ be a family of maps of (X_1, d_1) into (X_2, d_2). Then there is a maximal open subset of X_1 on which $\mathcal{F}$ is equicontinuous. In particular, if f maps a metric space (X, d) into itself, then there is a maximal open subset of X on the the family of iterates $\{f^n\}$ is equicontinuous.*

Definition 2.33. Let R be a non-constant rational function. **The Fatou set of** R is the maximal open subset of $\mathbb{C}_\infty$ on which $\{R^n\}$ is equicontinuous, denoted by $\mathbf{F}(R)$. And **the Julia set of** R is the complement of the Fatou set of R in $\mathbb{C}_\infty$, denoted by $\mathbf{J}(R)$.

By the definition of the Fatou set and the Julia set, we have that the Fatou set is open and the Julia set is closed. Moreover, the Julia set is compact under the chordal metric.

Theorem 2.34. *[5] Let R be a non-constant rational function, let g be a Möbius map, and let $S = gRg^{-1}$. Then $\mathbf{F}(S) = g(\mathbf{F}(R))$ and $\mathbf{J}(S) = g(\mathbf{J}(S))$.*

Theorem 2.35. *[5] Let R be a non-constant rational function and $p \in \mathbb{N}$. Then $\mathbf{F}(R^p) = \mathbf{F}(R)$ and $\mathbf{J}(R^p) = \mathbf{J}(R)$.*

Next, let R be a rational function of degree greater than or equal to 2.

Theorem 2.36. *[5] Let R be a rational function. Then $\mathbf{F}(R)$ and $\mathbf{J}(R)$ are complete invariant under R.*

Theorem 2.37. *[5] Let P be a polynomial map such that $\deg P \geq 2$. Then $\infty \in \mathbf{F}(P)$ and the component of $\mathbf{F}(P)$ that contains ∞ is complete invariant under P.*

Corollary 2.38. *[5] If $\deg(R) \geq 2$, then the exceptional points of R lie in $\mathbf{F}(R)$.*

Theorem 2.39. *[5] Let f be continuous map of a topological space X onto itself, and suppose that X has only a finite number of components X_j. Then for some integer m, each X_j is completely invariant under f^m.*

Theorem 2.40. *[5] $\mathbf{J}(R)$ is infinite.*

Theorem 2.41. *[5] Let E be a closed, completely invariant subset of complex sphere. Then either:*

(i) E has at most two elements and $E \subseteq E(R) \subseteq \mathbf{F}(R)$; or

(ii) E is infinite and $\mathbf{J}(R) \subseteq E$.

By Theorem 2.41, we conclude that $\mathbf{J}(R)$ is the smallest closed and completely invariant set with at least three points.

Theorem 2.42. *[5] Either $\mathbf{J}(R) = \mathbb{C}_\infty$ or $\operatorname{Int} \mathbf{J}(R) = \phi$.*

Theorem 2.43. *[5] $\mathbf{J}(R)$ is a perfect set and uncountable.*

Theorem 2.44. *[5] Let W be a non-empty open set such that $W \cap \mathbf{J}(R) \neq \phi$. Then:*

(i) $\mathbb{C}_\infty - E(R) \subseteq \bigcup_{n=0}^{\infty} R^n(W)$; and

(ii) $\mathbf{J}(R) \subseteq R^n(W)$, for all sufficiently large integers n.

Definition 2.45. Let $\zeta \in \mathbb{C}_\infty$. We said to be ζ is **a periodic point of** R if there is an integer n such that ζ is a fixed point of R^n.

Theorem 2.46. *[5] $\mathbf{J}(R)$ is contained in the closure of the set of periodic points of R.*

Theorem 2.47. *[5] Let $z \in \mathbb{C}_\infty$.*

(i) if z is not exceptional, then $\mathbf{J}(R)$ is contained in the closure of $O^-(z)$,

(ii) if $z \in \mathbf{J}(R)$, then $\mathbf{J}(R)$ is the closure of $O^-(z)$.

Theorem 2.48. *[5] Let E be a compact subset of the complex sphere with the property that for all $z \in \mathbf{F}(R)$, the sequence $\{R^n(z) : n \in \mathbb{N}\}$ does not accumulate at any point of E. Then given for any open set U which contains $\mathbf{J}(R)$, $R^{-n}(E) \subseteq U$ for all sufficiently large n.*

Theorem 2.49. *[5] Let R and S be are rational maps such that the degree of R and S are at least two. If R and S commute, then $\mathbf{J}(R) = \mathbf{J}(S)$.*

2.3 Dynamics of Newton's Method

In 1669, Newton investigates a method to approximate a real root ζ of the equation

$$x^3 - 2x - 5 = 0. \tag{2.2}$$

He started with an approximation $x_0 = 2$, and wrote $x = 2 + y$. If x is a real root of the equation, then the original equation becomes

$$y^3 + 6y^2 + 10y - 1 = 0.$$

Neglecting the non-linear terms, he then got $y = \frac{1}{10}$ and so took $x_1 = 2.1$ as his next approximation to ζ. He then substituted $x = 2.1 + q$ into the equation (2.2) and obtained the equation

$$q^3 + \frac{63}{10}q^2 + \frac{1123}{100}q - \frac{61}{1000} = 0.$$

Again neglecting the non-linear terms, he then got $q = -\frac{61}{11230}$ and took $x_2 = x_1 + q = \frac{11761}{5615} = 2.0946...$ as his next approximation to ζ. By repeat the process, he got a better approximation for the actual root $\zeta = 2.09455148....$ His method was systematiclly discussed by Joseph Raphson in 1690. Raphson described the method in terms of the successive approximations x_n instead of the more complicated sequence of polynomials used by Newton. However, both Newton and Raphson used purely an algebraic method to derive the method and they restricted its use to polynomials.

In 1740, Thomas Simpson described Nemton's method as an iterative method for solving general nonlinear equation by using fluxional calculus. Simpson's method is now known as the Newton-Raphson method or Newton's method [5, 6, 7]. Newton's method is the iterative algorithm

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2.3}$$

for $n \leq 1$, where f is a differentiable real function. If we choose a real number x_0 well enough, the sequence x_n will converge to a real root ζ of the equation $f(x) = 0$.

In 1879, Cayley [8, 9] ignored the restriction of reality of the function f in Newton's method and used Newton's Method to find complex roots of complex functions. He called this method the Newton-Fourier method. So the problem concerning to the area of the initial points x_0 such that the sequence x_n will converge to a root of the equation $f(x) = 0$ falls into the scope of the study of the Fatou set of the function

$$F(z) = z - \frac{f(z)}{f'(z)} \tag{2.4}$$

when f is a meromorphic or non-constant entire function.

Let f be a meromorphic or non-constant entire function. We define $N_f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ by

$$N_f(z) = z - \frac{f(z)}{f'(z)} \tag{2.5}$$

for all $z \in \mathbb{C}_\infty$. The function N_f is called the **Newton map** induced by f.

We will denote N_f by N if there is no need to specify functions. Given a zero α of f, it is clear that α is a fixed point of N. From Lemma 2.52, it follows that $\alpha \in \mathbf{F}(N)$. If U_α is the component of $\mathbf{F}(N)$ containing α, then every point z in U_α will converge to α under the iteration of N, i.e. $\lim_{k\to\infty} N^k(z) = \alpha$. Moreover, every point $z \in \cup_{k=0}^{\infty} N^{-k}(U_\alpha)$ converges to α under the iteration of N, where $N^{-k}(U_\alpha)$ denotes the inverse image of U_α under N^k.

Now, let f denote a polynomial of degree $k \geq 2$, and let N be the Newton map induced by f. From the definition, we easily derive the following results:

Lemma 2.50. *We have* $\deg N = k$ *if all roots of* f *are simple, and* $\deg N < k$ *if the polynomial* f *has at least one multiple root.*

By Theorem 2.12, we have the following remark.

Remark 2.51. *A point* ζ *is a fixed point of* N *if and only if either* ζ *is a root of* f *or* $\zeta = \infty$. *If* $\deg N = k$, *then* N *has* $k+1$ *fixed points counting multiplicities in* $\bar{\mathbb{C}}$.

Lemma 2.52. *If* ζ *is a fixed point of* N, *then* ζ *is*

(i) a superattracting fixed point if ζ *is a simple root of* f;

(ii) an attracting fixed point if ζ *is a multiple root of* f; *or*

(iii) a repelling fixed point if $\zeta = \infty$.

The Julia set $\mathbf{J}(N_f)$ of a Newton map induced by a polynomial has a special structure. Shishikura [10] proved that $\mathbf{J}(N_f)$ is connected.

Theorem 2.53 (Shishikura's Theorem)**.** *[11] For every Newton map induced by a polynomial, the Julia set is connected.*

Definition 2.54. For every root ζ of a function f, **the basin of attraction of** ζ is the open set of points $z \in \mathbb{C}$ such that $N^k \to \zeta$ as $k \to \infty$. The **immediate basin of** ζ is the component of the Fatou set $\mathbf{F}(N)$ containing ζ.

Theorem 2.55. *(Mayer-Schleicher's Theorem) [12] Let f be a nonlinear entire function, and let ζ be a root of f. Then the immediate basin of ζ for the Newton map N_f is simply connected and unbounded.*

Corollary 2.56. *[12] Every immediate basin for the Newton map induced by a polynomial of degree at least two is simply connected and unbounded. Moreover, every component of a basin of attraction is simply connected.*

Chapter 3

The Upper Bound of $M(d, y)$

In 2011, Chaiya gave an algorithm to improve the value of f_d which was initially introdued by Schleicher. Even though, it is not an explicit formula, it can be easily computed. The following is the result established by Chaiya.

Theorem 3.1. *[4] Let $P(z)$ be a polynomial of degree $d \geq 3$, and let y be a positive number larger than $4d - 3$. If z_0 is in an immediate basin of a root α and $|N_p(z_0) - z_0| = \varepsilon$, then $|z_0 - \alpha| \leq M(d, y)\varepsilon$, where $M(d, y) = max\{y, A_d + \frac{y(d-1)}{y-1}\}$ and A_d can be derived from the following iterative algorithm.*
Let $b = \frac{y(y-d)}{y-1}$, and

$$A_2 = \frac{y(d-1)[2d(y-2d+3)-3y-1]}{(y-1)(y-4d-3)}.$$

For $k = 2, 3, \ldots, d-1$, set $a_k = 1 + \sum_{j=2}^{k-1} \frac{A_k}{A_k + A_j}$. If $2A_k < b$ then let

$$A_{k+1} = A_k \left(\frac{(a_k + d - k)A_k - (a_k + d - k - 1)b}{(a_k + 1)A_k - a_k b} \right).$$

Otherwise

$$A_{k+1} = A_k \left(\frac{a_k + d - k}{a_k} \right).$$

In the theorem, $M(d, y)\varepsilon$, an upper bound for the distance of a point z_0 to the a root α, depends on $y > 4d - 3$. In this chapter, we will find the value y to optimize the upper bound of $M(d, y)$ for some easy cases.

3.1 Some properties of A_k and a_k

Here, we study some properties of A_k and a_k appearing in Theorem 3.1

Lemma 3.2. *Let d be a positive integer greater than 3. For each $k = 2, 3, \ldots, d-1$, let A_k, a_k and A_d satisfy the algorithm in Theorem 3.1. We have that*

(i) $A_k > 0$ *for all* $k = 2, 3, \ldots, d$,

(ii) $a_k \geq 1$ *for all* $k = 2, 3, \ldots, d-1$,

(iii) $A_2 \leq A_3 \leq \ldots \leq A_d$,

(iv) $a_2 \leq a_3 \leq \ldots \leq a_{d-1}$,

(v) $a_k \leq k$ *for each* $k = 2, 3, \ldots, d-1$.

Proof. We will prove first that the statements (1) and (2) are true. Since $y > 4d-3$, we have that

$$\begin{aligned} A_2 &= \frac{y(d-1)[2d(y-2d+3)-3y-1]}{(y-1)(y-4d-3)} \\ &= \frac{y(d-1)[(2d-3)y-4d^2+6d-1]}{(y-1)(y-4d-3)} \\ &> \frac{y(d-1)[(2d-3)(4d-3)-4d^2+6d-1]}{(y-1)(y-4d-3)} \\ &= \frac{4y(d-1)^2(d-2)}{(y-1)(y-4d-3)} > 0 \end{aligned}$$

It clear that $a_2 = 1 \geq 1$.

Assume that there exists $k = 2, 3, \ldots, d-2$ such that $A_l > 0$ and $a_l > 1$ for all $l \leq k$. we will prove that $A_{k+1} > 0$ and $a_{k+1} > 0$. If $2A_k \geq b$, we have that

$A_{k+1} = A_k \left(\frac{a_k+d-k}{a_k}\right) = A_k \left(1 + \frac{d-k}{a_k}\right) > 0$. If $2A_k < b$, we get

$$\begin{aligned} A_{k+1} &= A_k \left(\frac{(a_k + d - k)A_k - (a_k + d - k - 1)b}{(a_k + 1)A_k - a_k b}\right). \\ &= A_k \left(1 + \frac{(d-k-1)(b - A_k)}{a_k \left(b - (1 + \frac{1}{a_k})A_k\right)}\right). \end{aligned}$$

Since $a_k \geq 1$, then $1 \leq 1 + \frac{1}{a_k} \leq 2$. It implies that $b - A_2 \geq b - (1 + \frac{1}{a_k})A_2 \geq b - 2A_2 > 0$. So $A_{k+1} > 0$. Since $A_l > 0$ for all $l = 2, 3, \ldots, k+1$,

$$a_k = 1 + \sum_{j=2}^{k-1} \frac{A_k}{A_k + A_j} \geq 1.$$

Next, we will prove that $A_d > 0$. If $2A_{d-1} \geq b$, we have that $A_d = A_{d-1}\left(\frac{a_{d-1}+1}{a_{d-1}}\right) = A_{d-1}\left(1 + \frac{1}{a_{d-1}}\right) > 0$. If $2A_{d-1} < b$, we get

$$A_d = A_{d-1}\left(\frac{(a_{d-1}+1)A_k - a_{d-1}b}{(a_{d-1}+1)A_k - a_{d-1}b}\right) = A_{d-1} > 0.$$

Hence, the statements (1) and (2) are true.

Next, we will prove that the statement (3) is true. If $2A_2 < b$, we have that

$$A_3 - A_2 = A_2 \left(\frac{(d-3)(b - A_2)}{b - 2A_2}\right) \geq 0.$$

If $2A_2 \geq b$, we have that

$$A_3 - A_2 = (d-2)A_2 \geq 0.$$

For each $k = 3, 4, 5, \ldots, d-1$. If $2A_k < b$, we have that

$$A_{k+1} - A_k = \frac{(d-k-1)(b - A_k)}{a_k \left(b - (1 + \frac{1}{a_k})A_k\right)} A_k \geq 0.$$

If $2A_k \geq b$, we have that

$$A_{k+1} - A_k = \frac{d-k}{a_k} A_k \geq 0.$$

From both cases, we conclude that $A_{k+1} \geq A_k$ for all $k = 2, 3, 4, \ldots, d-1$.

Next, we will prove that the statement (5) is true. Since (1) is true, we have that

$$a_k = 1 + \sum_{j=2}^{k-1} \frac{A_k}{A_k + A_j} \leq 1 + \sum_{j=2}^{k-1} 1 = k - 1 < k$$

for each $k = 2, 3, \ldots, d-1$.

Finally, we will prove that the statement (4) is true. For each $k = 2, 3, \ldots, d-1$, since (3) is true, $A_k \leq A_{k+1}$. By (1), we have that $A_k A_l \leq A_{k+1} A_l$ which yields that $A_k A_{k+1} + A_k A_l \leq A_k A_{k+1} + A_{k+1} A_l$. Hence,

$$\frac{A_k}{A_k + A_l} \leq \frac{A_{k+1}}{A_{k+1} + A_l}$$

for all $k = 2, 3, \ldots, d-1$ and for all $l = 2, 3, \ldots, k$. For each $k = 2, 3, \ldots, d-1$,

$$\begin{aligned} a_k &= 1 + \sum_{j=2}^{k-1} \frac{A_k}{A_k + A_j} \\ &\leq 1 + \sum_{j=2}^{k-1} \frac{A_{k+1}}{A_{k+1} + A_j} \\ &\leq 1 + \sum_{j=2}^{k} \frac{A_{k+1}}{A_{k+1} + A_j} \\ &= a_{k+1}. \end{aligned}$$

The proof is complete. □

3.2 An upper bound of $M(d, y)$ in the case of $d = 3$

In this section, we find the best upper bound of $M(d, y)$ in the case of $d = 3$. Note that $y \geq 4d - 3 = 9$. First we see that

$$\begin{aligned} 2A_2 \geq b &\Leftrightarrow \frac{4y(3y-19)}{(y-1)(y-9)} \geq \frac{y(y-3)}{y-1} \\ &\Leftrightarrow \frac{4(3y-19)}{(y-9)} \geq y - 3 \\ &\Leftrightarrow 12y - 76 \geq (y-3)(y-9) \\ &\Leftrightarrow 0 \geq y^2 - 24y + 49. \end{aligned}$$

This implies that if $9 < y \leq 12 + \sqrt{95}$, then $2A_2 \geq b$.

Next, we find an upper bound of $M(d, y)$ in the algorithm in Theorem 3.1 in the case of $d = 3$. If $9 < y \leq 12 + \sqrt{95}$, we have that $2A_2 \geq b$. This means that

$$A_3 = \left(\frac{a_2 + 3 - 2}{a_2}\right) A_2 = 2A_2 = \frac{4y(3y-19)}{(y-1)(y-9)}.$$

Since $A_3 + \frac{y}{y-1} = 2A_2 + \frac{y}{y-1} \geq b + \frac{y}{y-1} = y$, it follows that $M(3, y) = A_3 + \frac{y}{y-1} = \frac{y(13y-85)}{(y-1)(y-9)}$. We now consider the function $f(y) = \frac{y(13y-85)}{(y-1)(y-9)}$ on $(9, 12 + \sqrt{95})$. Since $f'(y) = -\frac{9(5y^2-26y+85)}{(y-1)^2(y-9)^2} = -\frac{45\left((y-\frac{13}{5})^2+\frac{256}{25}\right)}{(y-1)^2(y-9)^2} \leq 0$ for all $y \in (9, 12 + \sqrt{95})$, f is decreasing on $(9, 12 + \sqrt{95})$. We choose $y_0 = 12 + \sqrt{95}$, and then we have that $M(3, y_0) \approx 16.25804146$.

Next, we consider the case when $2A_2 < b$, which means $y > 12 + \sqrt{95}$. We have that

$$A_3 = \left(\frac{(a_2 + 3 - 2)A_2 - (a_2 + 3 - 2 - 1)b}{(a_2 + 1)A_2 - ba_2}\right) A_2 = \left(\frac{2A_2 - b}{2A_2 - b}\right) A_2 = A_2.$$

Since $A_3 + \frac{y}{y-1} = A_2 + \frac{y}{y-1} < 2A_2 + \frac{y}{y-1} < b + \frac{y}{y-1} = y$, we have that $M(3,y) = y$. So $\min\{M(3,y) : y > 12 + \sqrt{95}\} > 12 + \sqrt{95}$.

We conclude that if we choose $y = 12+\sqrt{95}$, then we get the least upper bound for the algorithm in Theorem 3.1 in the case $d = 3$ and it is 16.25804146. In the case $d = 3$, we conclude that we choose $y = 12 + \sqrt{95}$, then we get the best upper bound of $M(d,y)$, that is $12 + \sqrt{95}$.

3.3 An upper bound of $M(d,y)$ in the case of $2A_2 \geq b$ and $d \geq 4$

In this section, we consider only the case of $2A_2 \geq b$ and $d \geq 4$.

Lemma 3.3. *Let A_k, a_k and A_d be as in the algorithm in Theorem 3.1, for each $k = 2, 3, \ldots, d-1$. If $2A_2 \geq b$, then*

(i) $a_2 = 1$,

(ii) $a_3 = \frac{2d-1}{d}$,

(iii) $a_k \geq \frac{(k+1)d-2}{2d}$, *for* $k \geq 4$.

Proof. Suppose that $2A_2 \geq b$. By the definition in the algorithm, $a_2 = 1$, hence (1) follows. By Lemma 3.2 (3), we have that $A_2 \leq A_3 \leq \ldots \leq A_d$. Hence, we get $1 \leq \frac{A_3}{A_2} \leq \frac{A_4}{A_2} \leq \ldots \leq \frac{A_d}{A_2}$. Since $2A_2 \geq b$, we obtain

$$A_3 = \left(\frac{a_2 + d - 2}{a_2}\right) A_2 = (d-1)A_2.$$

So

$$d - 1 \leq \frac{A_4}{A_2} \leq \ldots \leq \frac{A_d}{A_2}.$$

We have that $a_3 = 1 + \frac{A_3}{A_3+A_2} = 1 + \frac{1}{1+\frac{1}{\frac{A_3}{A_2}}} = 1 + \frac{1}{1+\frac{1}{d-1}} = \frac{2d-1}{d}$. Hence (2) is true. For each $k = 4, 5, \ldots, d-1$, we have by Lemma 3.2 (3) that

$$\begin{aligned}
a_k &= 1 + \sum_{j=2}^{k-1} \frac{A_k}{A_k + A_j} \\
&= 1 + \frac{A_k}{A_k + A_2} + \sum_{j=3}^{k-2} \frac{A_k}{A_k + A_j} \\
&= 1 + \frac{1}{1 + \frac{1}{\frac{A_k}{A_2}}} + \sum_{j=3}^{k-2} \frac{A_k}{A_k + A_j} \\
&\geq 1 + \frac{1}{1 + \frac{1}{d-1}} + \sum_{j=3}^{k-2} \frac{A_k}{A_k + A_k} \\
&= \frac{2d-1}{d} + \frac{k-3}{2} \\
&= \frac{2(2d-1) + (k-3)d}{2d} \\
&= \frac{(k+1)d - 2}{2d}.
\end{aligned}$$

This proves (3). □

Lemma 3.4. *Let A_k, a_k and A_d be as in the algorithm in Theorem 3.1, for all $k = 2, 3, \ldots, d-1$. If $2A_2 \geq b$, then $A_d > b$.*

Proof. Suppose that $2A_2 \geq b$. By Lemma 3.2 (3), we have that $A_2 \leq A_3 \leq \ldots \leq A_d$. So, $b \leq 2A_2 \leq 2A_3 \leq \ldots \leq 2A_{d-1}$. For each $k = 3, 4, 5, \ldots, d$, we have

$$A_k = \left(\frac{a_{k-1} + d - (k-1)}{a_{k-1}} \right) A_{k-1}.$$

Let $B_k = \frac{a_k + d - k}{a_k}$, for each $k = 2, 3, 4, \ldots, d-1$. Then

$$A_d = B_{d-1} B_{d-2} \ldots B_4 B_3 B_2 A_2. \tag{3.1}$$

For each $k = 2, 3, 4, \ldots, d-1$, by Lemma 3.2 (5), we get

$$B_k = \frac{a_k + d - k}{a_k} = 1 + \frac{d-k}{a_k} \geq 1 + \frac{d-k}{k} = \frac{d}{k}.$$

Thus, the equation (3.1) and $d \geq 4$,

$$\begin{aligned}
A_d = B_{d-1}B_{d-2}\dots B_4B_3B_2A_2 &\geq \left(\frac{d}{d-1}\right)\left(\frac{d}{d-2}\right)\dots\left(\frac{d}{4}\right)\left(\frac{d}{3}\right)\left(\frac{d}{2}\right)A_2 \\
&\geq \left(\frac{d-1}{d-1}\right)\left(\frac{d-2}{d-2}\right)\dots\left(\frac{4}{4}\right)\left(\frac{d}{3}\right)\left(\frac{d}{2}\right)A_2 = \frac{d^2}{6}A_2 \\
&\geq \frac{4^2}{6}A_2 \geq 2A_2 > b.
\end{aligned}$$

The proof is now complete. $\square$

By Lemma 3.4, we have that if $2A_2 \geq b$, then $A_d + \frac{y(d-1)}{y-1} \geq y$. This means that $M(d,y) = A_d + \frac{y(d-1)}{y-1}$.

Lemma 3.5. *Let A_k, a_k and A_d be as in the algorithm in Theorem 3.1, for each $k = 2,3,\dots,d-1$. If $2A_2 \geq b$, then*

$$A_d \leq A_2(d-1)\prod_{k=3}^{d-1}\left(\frac{2d^2-(k-1)d-2}{(k+1)d-2}\right).$$

Proof. Suppose that $2A_2 \geq b$. We let $B_k = \frac{a_k+d-k}{a_k}$, for each $k = 2,3,4,\dots,d-1$. By Lemma 3.3, we get $B_2 = \frac{a_2+d-2}{a_2} = d-1$ and

$$\begin{aligned}
B_k &= \frac{a_k+d-k}{a_k} = 1 + \frac{d-k}{a_k} \\
&\leq 1 + \frac{d-k}{\frac{(k+1)d-2}{2d}} = \frac{(k+1)d-2+2d(d-k)}{(k+1)d-2} = \frac{2d^2-(k-1)d-2}{(k+1)d-2}
\end{aligned}$$

for each $d = 3,4,\dots,d-1$. From the equation (3.1), we have that

$$\begin{aligned}
A_d &= B_{d-1}B_{d-2}\dots B_4B_3B_2A_2 \\
&= A_2\left(B_2\prod_{k=3}^{d-1}B_k\right) \\
&\leq A_2(d-1)\prod_{k=3}^{d-1}\left(\frac{2d^2-(k-1)d-2}{(k+1)d-2}\right).
\end{aligned}$$

The proof is now complete. $\square$

Theorem 3.6. *Let A_k, a_k and A_d be as in the algorithm in Theorem 3.1, for each $k = 2, 3, \dots, d-1$. Then there exists $y > 4d-3$ such that*

$$M(d,y) \leq \frac{2C(d-1)(4d^2-7d+3)(4d^3-13d^2+13d-2)}{(4d^2-7d+2)(4d^2-11d+6)},$$

where $C = (d-1)\prod_{k=3}^{d-1}\left(\frac{2d^2-(k-1)d-2}{(k+1)d-2}\right)$.

Proof. We can see that $2A_2 \geq b$ if and only if

$$\frac{2y(d-1)[(2d-3)y-(4d^2-6d+1)]}{(y-1)(y-4d+3)} \geq \frac{y(y-d)}{y-1}$$

$$\frac{2(d-1)[(2d-3)y-(4d^2-6d+1)]}{(y-4d+3)} \geq (y-d)$$

$$2(d-1)(2d-3)y-2(d-1)(4d^2-6d+1) \geq (y-d)(y-4d+3)$$

$$y^2-(4d^2-5d+3)y+(8d^3-16d^2+11d-2) \leq 0.$$

So if $4d-3 < y \leq y^+$, when $y^+ = \frac{(4d^2-5d+3)+\sqrt{(4d^2-5d+3)^2-4(8d^3-16d^2+11d-2)}}{2}$, then $2A_2 \geq b$. Note that

$$\begin{aligned}(4d^2-5d+3)^2&-4(8d^3-16d^2+11d-2)\\ &= 16d^4-72d^3+113d^2-74d+16\\ &= (4d^2-9d+3)^2+8d^2-20d+8\\ &= (4d^2-9d+3)^2+4d(2d-5)+8\\ &\geq (4d^2-9d+3)^2+4(4)(2(4)-5)+8\\ &\geq (4d^2-9d+3)^2.\end{aligned}$$

Choose $y_0 = \frac{(4d^2-5d+3)+\sqrt{(4d^2-9d+3)^2}}{2} = 4d^2-7d+3$. It is clear that

$$4d-3 < 4d(d-\frac{7}{4})-3 < 4d^2-7d-3 < 4d^2-7d+3,$$

and that

$$4d^2 - 7d - 3 = \frac{(4d^2 - 5d + 3) + \sqrt{(4d^2 - 9d + 3)^2}}{2} < y^+.$$

Hence $4d - 3 < y_0 < y^+$. By choosing $y = y_0$, we get $2A_2 \geq b$.

By Lemma 3.5 and the result of the Lemma 3.4, we have

$$\begin{aligned}
M(d, y) &= A_d + \frac{y(d-1)}{y-1} \\
&\leq CA_2 + \frac{y(d-1)}{y-1} \\
&= \frac{Cy(d-1)[(2d-3)y - (4d^2 - 6d + 1)]}{(y-1)(y-4d+3)} + \frac{y(d-1)}{y-1} \\
&= \frac{(d-1)y}{(y-1)(y-4d+3)} \left[(2Cd - 3C + 1)y + (-4Cd^2 + 6Cd - C - 4d + 3)\right] \\
&\leq \frac{(d-1)y}{(y-1)(y-4d+3)} \left[(2Cd - 3C + C)y + (-4Cd^2 + 6Cd - C + 3C)\right] \\
&= \frac{(d-1)y}{(y-1)(y-4d+3)} \left[(2Cd - 2C)y + (-4Cd^2 + 6Cd + 2C)\right] \\
&= \frac{2C(d-1)y}{(y-1)(y-4d+3)} \left[(d-1)y + (-2d^2 + 3d + 1)\right] \\
&\leq \frac{2C(d-1)(4d^2 - 7d + 3)}{(4d^2 - 7d + 2)(4d^2 - 11d + 6)} \left[(d-1)(4d^2 - 7d + 3) + (-2d^2 + 3d + 1)\right] \\
&= \frac{2C(d-1)(4d^2 - 7d + 3)}{(4d^2 - 7d + 2)(4d^2 - 11d + 6)} (4d^3 - 13d^2 + 13d - 2).
\end{aligned}$$

The proof is now complete. □

In the proof of Theorem 3.6, we conclude that if we choose $y = 4d^2 - 7d + 3$, then $M(d, y)$ is less than $\frac{2C(d-1)(4d^2-7d+3)}{(4d^2-7d+2)(4d^2-11d+6)}(4d^3 - 13d^2 + 13d - 2)$ when $C = (d-1)\prod_{k=3}^{d-1}\left(\frac{2d^2-(k-1)d-2}{(k+1)d-2}\right)$.

Chapter 4

The Main Result

In Chapter 3, we found some upper bounds of $M(d, y)$ with some restriction. We never consider the case where $2A_2 < b$ in Chapter 3 because there were several cases that have to be considered. To avoid unpleasant and tedious work, we have to take a new look into the algorithm, and find a new way for deriving an upper bound of $|z_0 - \alpha|$.

4.1 Preliminary Results

Lemma 4.1. *[4] Let P be a polynomial. Let β be a complex number and $r > 0$. Suppose that $Re(\frac{(z-\beta)P'(z)}{P(z)}) \geq \frac{1}{2}$ whenever $|z - \beta| = r$ and $P(z) \neq 0$. Let U be an immediate basin of a root α of P. If $U \cap \bar{B}(\beta, r) \neq \varnothing$, then $\alpha \in B(\beta, r)$.*

Remark 4.2. *From Lemma 4.1, if β is a root of P and $Re(\frac{(z-\beta)P'(z)}{P(z)}) \geq \frac{1}{2}$ for all $|z - \beta| \geq r$, then the closed ball $\bar{B}(\beta, r)$ is contained in the immediate basin of β.*

Lemma 4.3. *Let P be a polynomial of degree d, and let z_0 be a point in the immediate basin of a root of P. Let $\alpha_1, \ldots, \alpha_d$ be all roots of P such that α_1 is the nearest root to z_0, and $|\alpha_1 - \alpha_k| \leq |\alpha_1 - \alpha_{k+1}|$ for all $k = 2, 3, \ldots, d-1$. If z_0 is* *not in the immediate basin of* α_j *for all* $j = 1, 2, \ldots, k$ *where* $2 \leq k \leq \frac{d}{2}$*, then*

$$|\alpha_1 - \alpha_{k+1}| < \frac{2d^2 - k^2 + 3k - 2d - 2}{(k-1)(2d-k+2)} A_k, \tag{4.1}$$

where A_k *is an upper bound of* $|\alpha_1 - \alpha_k|$

Proof. For $j = 2, 3, \ldots, d$, let $|\alpha_1 - \alpha_j| = r_j$. We prove by induction on k. First we show that it is true when $k = 2$. If $r_3 = r_2$, then the inequality (4.1) holds for $k = 2$. So suppose that $r_3 > r_2$. Let $A_2 \in (r_2, r_3)$. For $z \in \mathbb{C}$ with $|z - \alpha_1| = A_2$, we have

$$\begin{aligned} Re\left((z - \alpha_1)\frac{P'(z)}{P(z)}\right) &\geq 1 + \sum_{k=2}^{d} Re\left(\frac{z - \alpha_1}{z - \alpha_k}\right) \\ &\geq 1 + \frac{A_2}{A_2 + r_2} + \frac{A_2(d-2)}{A_2 - r_3} \geq 1 + \frac{1}{2} + \frac{A_2(d-2)}{A_2 - r_3}. \end{aligned}$$

Note that if $r_3 \geq (d-1)A_2$, then $1 + \frac{1}{2} + \frac{A_2(d-2)}{A_2 - r_3} \geq \frac{1}{2}$, and hence by Lemma 4.1, z_0 is in the immediate basin of either α_1 or α_2, which is not the case. Therefore $r_3 < (d-1)A_2$, and the inequality (4.1) holds for $k = 2$.

Next assume that for some $3 \leq n \leq \frac{d}{2}$ the statement holds for all $k \leq n - 1$. Here we let

$$A_{k+1} = \frac{2d^2 - k^2 + 3k - 2d - 2}{(k-1)(2d-k+2)} A_k$$

for each $k = 2, \ldots, n-1$. Notice that $A_2 < A_3 < \ldots < A_n$. It can be shown directly that $\frac{A_k}{A_k + A_{k+1}} < \frac{k-1}{d}$ when $k \geq 2$. So

$$\frac{A_n}{A_n + A_k} > \frac{A_{k+1}}{A_k + A_{k+1}} = 1 - \frac{A_k}{A_k + A_{k+1}} > 1 - \frac{k-1}{d}.$$

We may also assume that $r_{n+1} > A_n$, otherwise the inequality (4.1) is clearly true for $k = n$. For $z \in \mathbb{C}$ with $|z - \alpha_1| = A_n$, we have

$$
\begin{aligned}
Re\left((z-\alpha_1)\frac{P'(z)}{P(z)}\right) &\geq 1 + \sum_{k=2}^{d} Re\left(\frac{z-\alpha_1}{z-\alpha_k}\right) \\
&\geq 1 + \sum_{k=2}^{n} \frac{A_n}{A_n + r_k} + \frac{A_n(d-n)}{A_n - r_{n+1}} \\
&\geq 1 + \sum_{k=2}^{n-1} \frac{A_n}{A_n + A_k} + \frac{1}{2} + \frac{A_n(d-n)}{A_n - r_{n+1}} \\
&> 1 + \sum_{k=1}^{n-2} \left(1 - \frac{k}{d}\right) + \frac{1}{2} + \frac{A_n(d-n)}{A_n - r_{n+1}} \\
&= \frac{(n-1)(2d-n+2)}{2d} + \frac{1}{2} + \frac{A_n(d-n)}{A_n - r_{n+1}}.
\end{aligned}
$$

If $r_{n+1} \geq \frac{2d^2-n^2+3n-2d-2}{(n-1)(2d-n+2)}A_n$, then $\frac{(n-1)(2d-n+2)}{2d} + \frac{1}{2} + \frac{A_n(d-n)}{A_n - r_{n+1}} \geq \frac{1}{2}$, and hence by Lemma 4.1, z_0 is in the immediate basin of α_j for some $1 \leq j \leq n$. Therefore if z_0 is not in any immediate basin of α_j for all $j \in \{1, \ldots, n\}$, then the inequality (4.1) holds for $k = n$, as desired. The proof is now complete. $\square$

Lemma 4.4. *Let P be a polynomial of degree d, and let z_0 be a point in the immediate basin of a root of P. Let $\alpha_1, \ldots, \alpha_d$ be all roots of P such that α_1 is the nearest root to z_0, and that $|\alpha_1 - \alpha_k| \leq |\alpha_1 - \alpha_{k+1}|$ for all $k = 2, 3, \ldots, d-1$. If z_0 is not in the immediate basin of α_j for all $j = 1, 2, \ldots, k$ where $\frac{d}{2} < k \leq d-1$, then*

$$|\alpha_1 - \alpha_{k+1}| < \frac{9d^2 - 4kd - 2d - 3}{d^2 + 4kd - 2d - 3} A_k, \tag{4.2}$$

where A_k is an upper bound of $|\alpha_1 - \alpha_k|$

Proof. For $j = 2, 3, \ldots, d$, let $|\alpha_1 - \alpha_j| = r_j$. From Theorem 4.3, we let

$$A_{j+1} = \frac{2d^2 - j^2 + 3j - 2d - 2}{(j-1)(2d-j+2)} A_j$$

for each $2 \leq j \leq \frac{d}{2}$. For $2 \leq j \leq \frac{d}{2}$, if $A_k \geq A_j$, we obtain

$$\frac{A_k}{A_k + A_j} \geq \frac{A_{j+1}}{A_j + A_{j+1}} = 1 - \frac{A_j}{A_j + A_{j+1}} > 1 - \frac{j-1}{d}.$$

We may also assume that $r_{k+1} > A_k$, otherwise the inequality (4.2) clearly holds. For $z \in \mathbb{C}$ with $|z - \alpha_1| = A_k$, we have

$$\begin{aligned} Re\left((z-\alpha_1)\frac{P'(z)}{P(z)}\right) &\geq 1 + \sum_{j=2}^{d} Re\left(\frac{z-\alpha_1}{z-\alpha_j}\right) \\ &\geq 1 + \sum_{j=2}^{k} \frac{A_k}{A_k + r_j} + \frac{A_k(d-k)}{A_k - r_{k+1}} \\ &\geq 1 + \sum_{j=2}^{\lfloor d/2 \rfloor} \frac{A_k}{A_k + A_j} + \frac{1}{2}\left(k - \lfloor d/2 \rfloor\right) + \frac{A_k(d-k)}{A_k - r_{k+1}} \\ &> 1 + \sum_{j=1}^{\lfloor d/2 \rfloor - 1} \left(1 - \frac{j}{d}\right) + \frac{1}{2}\left(k - \lfloor d/2 \rfloor\right) + \frac{A_k(d-k)}{A_k - r_{k+1}} \\ &= \frac{N(d-N+1) + kd}{2d} + \frac{A_k(d-k)}{A_k - r_{k+1}}, \end{aligned}$$

where $N = \lfloor d/2 \rfloor$, the greatest integer that is less than or equal to $d/2$. If $r_{k+1} \geq \left(\frac{2d(d-k)}{N(d-N+1)+dk-d} + 1\right) A_k$, then $Re\left((z-\alpha_1)\frac{P'(z)}{P(z)}\right) > \frac{1}{2}$, and hence by Lemma 4.1, z_0 is in the immediate basin of α_j for some $1 \leq j \leq k$. Therefore if z_0 is not in any immediate basin of α_j for all $j \in \{1, \ldots, k\}$, then we must have

$$r_{k+1} < \left(\frac{2d(d-k)}{N(d-N+1) + dk - d} + 1\right) A_k.$$

Since $N \geq \frac{d-1}{2}$ and $N(d-N+1)$ increases as N increases on the interval $[1, (d+1)/2]$, we obtain, by substituting N by $(d-1)/2$,

$$r_{n+1} < \left(\frac{2d(d-k)}{N(d-N+1) + dk - d} + 1\right) A_k < \frac{9d^2 - 4kd - 2d - 3}{d^2 + 4kd - 2d - 3} A_k$$

as needed. The proof is now complete. $\square$

Lemma 4.5. *Let P be a polynomial of degree d, and let $z \in \mathbb{C}$ such that $|N_p(z) - z| = \varepsilon > 0$. If there at least $d-2$ roots of P are outside the open ball $B(z_0, y\varepsilon)$, where $y > d-2$, then there is a root α of P such that*

$$|z - \alpha| \leq \frac{2y\varepsilon}{y - d + 2}.$$

Proof. Let $\alpha_1, \alpha_2, \ldots, \alpha_d$ be all roots of P. Suppose that $|z - \alpha_k| \geq y\varepsilon$ for $k \geq 3$ and that $|z - \alpha_j| > \frac{2y\varepsilon}{y-d+2}$ for $j = 1, 2$, then

$$\begin{aligned} |N_p(z) - z| = \left|\frac{P'(z)}{P(z)}\right|^{-1} &\geq \left(\sum_{j=1}^{d} \frac{1}{|z - \alpha_j|}\right)^{-1} \\ &> \left(\frac{2(y-d+2)}{2y\varepsilon} + \frac{d-2}{y\varepsilon}\right)^{-1} = \varepsilon, \end{aligned}$$

a contradiction. Thus the lemma holds. □

From the above lemma, if we choose $y = d$, the following corollary follows.

Corollary 4.6. *Let P be a polynomial of degree d, and let $z \in \mathbb{C}$. There exists a root α of P such that $|z - \alpha| \leq d|N_p(z) - z|$.*

Indeed, this lemma is a well-known result about Newton's map of a polynomial (see [3]).

Lemma 4.7. *Let $d \geq 4$ The function*

$$\beta(c) = \frac{2d + 3c - 3 - \sqrt{4d^2 + 4d(c-3) + 9c^2 - 14c + 9}}{2(2d-1)} \tag{4.3}$$

is increasing on $c > 1$.

Proof. Since

$$\beta'(c) = \frac{1}{2(2d-1)}\left(3 - \frac{2d+9c-7}{\sqrt{9c^2+4cd+4d^2-14c-12d+9}}\right)$$

and

$$\left(9c^2+4cd+4d^2-14c-12d+9\right) - \left(\frac{2d+9c-7}{3}\right)^2 = \frac{16}{9}(2d-1)(d-2) > 0,$$

we have that

$$3 - \frac{2d+9c-7}{\sqrt{9c^2+4cd+4d^2-14c-12d+9}} > 0.$$

So $\beta'(c) > 0$ for all $c > 1$. We conclude that β is increasing on $c > 1$. □

Lemma 4.8. *Let $y > d-2$ and $d \geq 4$. The function*

$$T(c) = \frac{2cy}{(y-d+2)\beta(c)} \tag{4.4}$$

is increasing on $c > 1$.

Proof. Since

$$T'(c) = \frac{4y(2d-1)\left((2d-3)M - (4d^2+2dc-7c-12d+9)\right)}{(y-d+2)M\left(2d-3c-3-M\right)^2},$$

where $M = \sqrt{9c^2+4cd+4d^2-14c-12d+9}$, and

$$(2d-3)^2M^2 - (4d^2+2dc-7c-12d+9)^2 = 16c^2(2d-1)(d-2) > 0,$$

we have that $(2d-3)M - (4d^2+2dc-7c-12d+9) > 0$. So $T'(c) > 0$ for all $c > 1$. We conclude that β is increasing on $c > 1$. □

Theorem 4.9. *Let P be a polynomial of degree d, and let z_0 be a point in the immediate basin of a root α of P. Suppose that $|N(z_0) - z_0| = \varepsilon$. If $z_0 \neq \alpha$, then either $|z_0 - \alpha| < \frac{24+4\sqrt{17}}{5+\sqrt{17}}d\varepsilon$, or there are at least three roots of P lying inside the open ball $B(z_0, (6+\sqrt{17})d\varepsilon)$.*

Proof. Let $\alpha_1, \ldots, \alpha_d$ be all roots of P such that α_1 is the nearest root to z_0, and that $|\alpha_1 - \alpha_k| \leq |\alpha_1 - \alpha_{k+1}|$ for all $k = 2, 3, \ldots, d-1$. Let $y = (6 + \sqrt{17})d$. If $|z_0 - \alpha_k| < y\varepsilon$ for $k \in \{1, 2, 3\}$, then the theorem is true. So suppose that $|z_0 - \alpha_k| \geq y\varepsilon$ for all $k = 3, \ldots, d$. By Lemma 4.5, we have $|z_0 - \alpha_1| < y_0\varepsilon$, where

$$y_0 = \frac{2y}{y - d + 2} = \frac{(12 + 2\sqrt{17})d}{(5 + \sqrt{17})d + 2} < \frac{12 + 2\sqrt{17}}{5 + \sqrt{17}}.$$

Then for $k \geq 3$

$$|\alpha_1 - \alpha_k| \geq |z_0 - \alpha_k| - |z_0 - \alpha_1| > (y - y_0)\,\varepsilon = b\varepsilon, \tag{4.5}$$

where $b = \frac{(6+\sqrt{17})(5+\sqrt{17})d^2}{(5+\sqrt{17})d+2}$. For $k = 2, 3, \ldots, d$, let $|\alpha_1 - \alpha_k| = r_k$. Note that $r_k > b\varepsilon$ for all $k \geq 3$. Let $r_3 = cr_2$ for some c greater than or equal to 1, and let r be a positive number less than r_2. For $z \in \mathbb{C}$ with $|z - \alpha_1| = r$, we have

$$\begin{aligned} Re\left((z - \alpha_1)\frac{P'(z)}{P(z)}\right) &\geq 1 + \sum_{k=2}^{d} Re\left(\frac{z - \alpha_1}{z - \alpha_k}\right) \\ &\geq 1 + \frac{r}{r - r_2} + \frac{r(d-2)}{r - r_3} \\ &\geq 1 + \frac{r}{r - r_2} + \frac{r(d-2)}{r - cr_2} \end{aligned}$$

Note that if $r \leq \beta r_2$ then $1 + \frac{r}{r - r_2} + \frac{r(d-2)}{r - cr_2} \geq \frac{1}{2}$, where

$$\beta = \frac{2d + 3c - 3 - \sqrt{4d^2 + 4d(c-3) + 9c^2 - 14c + 9}}{2(2d-1)}.$$

As a consequence of Lemma 4.1, if $|z_0 - \alpha_1| \leq \beta r_2$, then z_0 is in the immediate basin of α_1 and the theorem holds because from above

$$|z_0 - \alpha_1| < y_0\varepsilon < \frac{12 + 2\sqrt{17}}{5 + \sqrt{17}}\varepsilon.$$

Next assume that z_0 is not in the immediate basin of α_1. So we must have

$$\beta r_2 < |z_0 - \alpha_1| < y_0\varepsilon,$$

which implies $r_2 < y_0\beta^{-1}\varepsilon$. Let $A_2 = y_0\beta^{-1}$. By Lemma 4.7, β is an increasing function with respect to c. Hence A_2 is a decreasing function with respect to c. Since $\beta^{-1} = 2d - 1$ when $c = 1$, it follows that $A_2 \leq (2d-1)y_0\varepsilon$. If z_0 is in the immediate basin of α_2, then $\alpha = \alpha_2$ and

$$|z_0 - \alpha_2| < |z_0 - \alpha_1| + |\alpha_1 - \alpha_2| < y_0\varepsilon + A_2\varepsilon \leq 2dy_0\varepsilon.$$

Hence the theorem is true for this case.

Finally suppose that z_0 is neither in the immediate basin of α_1 nor in the immediate basin of α_2. Since $r_3 > b\varepsilon > 2dy_0\varepsilon$, it follows that $r_3 > A_2\varepsilon$. Let δ be a sufficiently small positive number such that $r_3 > (1+\delta)r_2$. For $z \in \mathbb{C}$ satisfying $|z - \alpha_1| = (1+\delta)r_2$, we have

$$Re\left((z-\alpha_1)\frac{P'(z)}{P(z)}\right) \geq 1 + \frac{1}{2} + \frac{(1+\delta)r_2(d-2)}{(1+\delta)r_2 - r_3}.$$

If $r_3 \geq (d-1)(1+\delta)r_2$, then $1 + \frac{(1+\delta)r_2(d-2)}{(1+\delta)r_2 - r_3} \geq 0$ and hence $Re\left((z-\alpha_1)\frac{P'(z)}{P(z)}\right) \geq \frac{1}{2}$. Then by Lemma 4.1 z_0 is in the immediate basin of either α_1 or α_2, which is not the case. Hence $r_3 < (d-1)(1+\delta)r_2$. Since δ is arbitrary small, it follows that $r_3 \leq (d-1)r_2$. Since $r_3 = cr_2$, c must be less than or equal to $d-1$. By Lemma 4.8, we have that cA_2 is an increasing function with respect to c. By substituting c by $d-1$ into A_2, we derive that

$$r_3 = cr_2 < cA_2\varepsilon \leq \frac{2y_0(d-1)(2d-1)\varepsilon}{5d-6-\sqrt{17d^2-28d+32}}.$$

It can be easily shown that $\frac{2y_0(d-1)(2d-1)}{5d-6-\sqrt{17d^2-28d+32}} < b$. This gives a contradiction to the assumption that $r_3 > b\varepsilon$.

Therefore if $|z_0 - \alpha_k| \geq y\varepsilon$ for all $k = 3, \ldots, d$, then z_0 must be in the immediate basin of either α_1 or α_2, and in either case we get $|z_0 - \alpha| < 2dy_0\varepsilon = \frac{24+4\sqrt{17}}{5+\sqrt{17}} d\varepsilon$as desired. The proof is now complete. □

The following corollary is a consequence of Corollary 4.6 and Theorem 4.9.

Corollary 4.10. *Let P be a polynomial of degree $d \geq 3$, and let z_0 be a point in the immediate basin of a root of P. Let $\alpha_1, \ldots, \alpha_d$ be all roots of P such that α_1 is the nearest root to z_0, and that $|\alpha_1 - \alpha_k| \leq |\alpha_1 - \alpha_{k+1}|$ for all $k = 2, 3, \ldots, d-1$. If $|N(z_0) - z_0| = \varepsilon > 0$ and z_0 is not in the immediate basin of α_j for $j = 1, 2$, then $|\alpha_1 - \alpha_3| < (7 + \sqrt{17})d\varepsilon$.*

Proof. From Corollary 4.6, we have $|z_0 - \alpha_1| < d\varepsilon$. Combining with Theorem 4.9, we obtain

$$|\alpha_1 - \alpha_3| \leq |z_0 - \alpha_1| + |z_0 - \alpha_3| < \left(1 + (6 + \sqrt{17})\right) d\varepsilon = (7 + \sqrt{17})d\varepsilon.$$

we are done. □

4.2 Main Theorem

Now we are ready to prove our main theorem.

Main Theorem. *Let P be a polynomial of degree $d \geq 4$, and let z_0 be a point in the immediate basin of a root α of P. If $|N(z_0) - z_0| = \varepsilon > 0$, then $|z_0 - \alpha| < (M_d + 1)d\varepsilon$,* *where*

$$M_d = (7+\sqrt{17}) \prod_{k=3}^{\lfloor d/2 \rfloor} \frac{2d^2 - k^2 + 3k - 2d - 2}{(k-1)(2d-k+2)} \prod_{n=\lfloor d/2 \rfloor + 1}^{d-1} \frac{9d^2 - 4kd - 2d - 3}{d^2 + 4kd - 2d - 3}.$$

Proof. Let $\alpha_1, \ldots, \alpha_d$ be all roots of P such that α_1 is the nearest root to z_0, and that $|\alpha_1 - \alpha_k| \leq |\alpha_1 - \alpha_{k+1}|$ for all $k = 2, 3, \ldots, d-1$. If $\alpha \in \{\alpha_1, \alpha_2, \alpha_3\}$, then by Theorem 4.9 the result follows. Next suppose that $\alpha \notin \{\alpha_1, \alpha_2, \alpha_3\}$. Let $A_3 = (7+\sqrt{17})d\varepsilon$. Then, by Corollary 4.10, A_3 is an upper bound of $|\alpha_1 - \alpha_3|$. For $k \geq 3$, define A_{k+1} inductively by

$$A_{k+1} = \begin{cases} \frac{2d^2-k^2+3k-2d-2}{(k-1)(2d-k+2)} A_k & \text{if } \ k \leq \lfloor d/2 \rfloor, \\ \frac{9d^2-4kd-2d-3}{d^2+4kd-2d-3} A_k & \text{if } \ \lfloor d/2 \rfloor < k \leq d-1. \end{cases}$$

Notice that $A_3 < A_4 < \ldots < A_d = M_d d\varepsilon$. If $\alpha \neq \alpha_j$ for $j = 1, 2, \ldots, k$, then by Theorem 4.3 and Theorem 4.4 we have that A_{k+1} is an upper bound of $|\alpha_1 - \alpha_{k+1}|$. Hence in any case we must have $|\alpha - \alpha_1| < A_d$. Since by Corollary 4.6 $|\alpha_1 - z_0| \leq d\varepsilon$, we obtain that

$$|z_0 - \alpha| \leq |\alpha_1 - z_0| + |\alpha - \alpha_1| < M_d d\varepsilon + d\varepsilon,$$

as desired. The proof is now complete. $\square$

From Main Theorem, we conclude that if a point z is in the immediate basin of a root α, then the distance between z and α is less than $(M_d + 1)d\varepsilon$, where

$$M_d = (7+\sqrt{17}) \prod_{k=3}^{\lfloor d/2 \rfloor} \frac{2d^2 - k^2 + 3k - 2d - 2}{(k-1)(2d-k+2)} \prod_{n=\lfloor d/2 \rfloor + 1}^{d-1} \frac{9d^2 - 4kd - 2d - 3}{d^2 + 4kd - 2d - 3},$$

and $\varepsilon = |N(z) - z|$.

Chapter 5

Conclusions

In this chapter, we will present some computational results of the upper bounds for the distant between a point z and a root α that were derived in Chapter 3, namely $M(d,y)$, and in Chapter 4, namely M_d. Here $M(d,y)$ is computed, when y is selected, as appeared in Chapter 3, to be $4d^2 - 7d + 3$. The value f_d is the upper bound given by D. Schleicher, that is $f_d = \frac{d^2(d-1)}{2(2d-1)}\binom{2d}{d}$. We can see from the following tables that the bound $M(d,y)$ is better than f_d at least d times when $d \geq 11$. Furthermore, the bound M_d is better than f_d at least $2^{0.4d}d$ times if $d \geq 10$.

Table 1 : The values of $M(d,y)$, M_d and f_d.

d	f_d	$M(d,y)$	M_d
10	4.3758×10^{6}	4.6759×10^{5}	2.3145×10^{4}
20	1.3431×10^{13}	4.0121×10^{11}	8.6733×10^{8}
30	2.6159×10^{19}	3.5606×10^{17}	4.0491×10^{13}
40	4.2459×10^{25}	3.2766×10^{23}	2.0426×10^{18}
50	6.2420×10^{31}	3.0908×10^{29}	1.0721×10^{23}
100	2.2523×10^{62}	2.7817×10^{59}	5.4533×10^{46}
150	5.2563×10^{92}	2.8727×10^{89}	3.2504×10^{70}
200	1.0269×10^{123}	3.1467×10^{119}	2.0624×10^{94}
250	1.8205×10^{153}	3.5612×10^{149}	1.3535×10^{118}
300	3.0349×10^{183}	4.1149×10^{179}	9.0728×10^{141}
400	7.5122×10^{243}	5.7137×10^{239}	4.2411×10^{189}
500	1.6876×10^{304}	8.1990×10^{299}	2.0490×10^{237}
600	3.5656×10^{364}	1.2013×10^{360}	1.0107×10^{285}
700	7.2213×10^{424}	1.7855×10^{420}	5.0575×10^{332}
800	1.4179×10^{485}	2.6819×10^{480}	2.5571×10^{380}
900	2.7190×10^{545}	4.0606×10^{540}	1.3033×10^{428}
1000	5.1178×10^{605}	6.1871×10^{600}	6.6845×10^{475}

Table 2 : This table shows that $M(d, y)$ is better than f_d at least d times when $d \geq 11$.

d	f_d	$M(d, y)$	$\frac{f_d}{dM(d,y)}$
10	4.3758×10^{6}	4.6759×10^{5}	0.9358
11	2.0323×10^{7}	1.8360×10^{6}	1.0063
12	9.3117×10^{7}	7.1997×10^{6}	1.0778
20	1.3431×10^{13}	4.0121×10^{11}	1.6738
30	2.6159×10^{19}	3.5606×10^{17}	2.4489
40	4.2459×10^{25}	3.2766×10^{23}	3.2396
50	6.2420×10^{31}	3.0908×10^{29}	4.0390
100	2.2523×10^{62}	2.7817×10^{59}	8.0970
150	5.2563×10^{92}	2.8727×10^{89}	12.1981
200	1.0269×10^{123}	3.1467×10^{119}	16.3178
250	1.8205×10^{153}	3.5612×10^{149}	20.4477
300	3.0349×10^{183}	4.1149×10^{179}	24.5843
400	7.5122×10^{243}	5.7137×10^{239}	32.8699
500	1.6876×10^{304}	8.1990×10^{299}	41.1660
600	3.5656×10^{364}	1.2013×10^{360}	49.4690
700	7.2213×10^{424}	1.7855×10^{420}	57.7766
800	1.4179×10^{485}	2.6819×10^{480}	66.0876
900	2.7190×10^{545}	4.0606×10^{540}	74.4013
1000	5.1178×10^{605}	6.1871×10^{600}	82.7170

Table 3 : This table shows that M_d is better than f_d at least $2^{0.4d}d$ times when $d \geq 10$.

d	f_d	M_d	$\frac{f_d}{2^{0.4d}dM_d}$
10	4.3758×10^{6}	2.3145×10^{4}	1.1816×10^{0}
20	1.3431×10^{13}	8.6733×10^{8}	3.0246×10^{0}
30	2.6159×10^{19}	4.0491×10^{13}	5.2574×10^{0}
40	4.2459×10^{25}	2.0426×10^{18}	7.9294×10^{0}
50	6.2420×10^{31}	1.0721×10^{23}	1.1105×10^{1}
100	2.2523×10^{62}	5.4533×10^{46}	3.7564×10^{1}
150	5.2563×10^{92}	3.2504×10^{70}	9.3510×10^{1}
200	1.0269×10^{123}	2.0624×10^{94}	2.0594×10^{2}
250	1.8205×10^{153}	1.3535×10^{118}	4.2440×10^{2}
300	3.0349×10^{183}	9.0728×10^{141}	8.3883×10^{2}
400	7.5122×10^{243}	4.2411×10^{189}	3.0299×10^{3}
500	1.6876×10^{304}	2.0490×10^{237}	1.0251×10^{4}
600	3.5656×10^{364}	1.0107×10^{285}	3.3277×10^{4}
700	7.2213×10^{424}	5.0575×10^{332}	1.0500×10^{5}
800	1.4179×10^{485}	2.5571×10^{380}	3.2449×10^{5}
900	2.7190×10^{545}	1.3033×10^{428}	9.8701×10^{5}
1000	5.1178×10^{605}	6.6845×10^{475}	2.9650×10^{6}

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ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

1 พฤษภาคม 2561

การประชุมวิชาการคณิตศาสตร์บริสุทธิ์และประยุกต์ประจำปี 2561
30 พฤษภาคม – 1 มิถุนายน 2561

เรียน คุณธีรพร ทองศิริ

ในนามของคณะกรรมการจัดการประชุมวิชาการคณิตศาสตร์บริสุทธิ์และประยุกต์ประจำปี 2561 ขอแจ้งให้ท่านทราบว่า อนุกรรมการฝ่ายวิชาการของการประชุมฯ ได้พิจารณาบทคัดย่อของท่านในหัวข้อ

An upper bound for the distance between a point to a root of polynomials

เรียบร้อยแล้วและมีความยินดีขอเชิญท่านเข้าร่วมประชุมและเสนอผลงานในหัวข้อดังกล่าว และบทคัดย่อของท่านจะได้รับการตีพิมพ์ลงใน Proceedings of Annual Pure and Applied Mathematics Conference 2018 ทั้งนี้ กำหนดการนำเสนอจะแจ้งให้ท่านทราบต่อไป

นอกจากนี้ ท่านสามารถรับทราบข้อมูลต่าง ๆ ของการประชุมครั้งนี้ได้ทางเว็บไซต์

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ขอแสดงความนับถือ

(ผู้ช่วยศาสตราจารย์ ดร.คำรณ เมฆฉาย)
ประธานจัดการประชุมวิชาการ
คณิตศาสตร์บริสุทธิ์และประยุกต์ประจำปี 2561

ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์

คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

ขอมอบเกียรติบัตรเพื่อแสดงว่า

ธีรพร ทองศรี

ได้นำเสนอผลงานและเข้าร่วม

การประชุมวิชาการคณิตศาสตร์บริสุทธิ์และประยุกต์ประจำปี 2561 (*APAM 2018*)

ณ คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

ระหว่างวันที่ 30 พฤษภาคม – 1 มิถุนายน 2561

(ผู้ช่วยศาสตราจารย์ ดร.คำรณ เมฆฉาย)
ประธานคณะกรรมการจัดการประชุมฯ

(ศาสตราจารย์ ดร.กฤษณะ เนียมมณี)
หัวหน้าภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์

An Upper Bound for the Distance between a Point to a Root of Polynomials

Teraporn Thongsiri *†1

[1]Department of Mathematics, Faculty of Science, Silpakorn University

[1]teraporn.thongsiri@gmail.com

Abstract

Newton's method is one of the most popular root-finding algorithms for meromorphic functions. In 2002, Dierk Schleicher established an explicit upper bound for the number of iterations of Newton's method for complex polynomials with a prescribed precision. In his work, Schleicher needed an upper bound, namely f_d, for the distance between a starting point z_0 to the root α, where z_0 is in the immediate basin of α and d is the degree of the polynomial. In 2011, Somjate Chaiya gave an algorithm to improve the value of f_d. In this research, we establish a new explicit bound for f_d.

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*Corresponding author
†The author is supported by DPST.

Biography

Name	Mr. Teraporn Thongsiri
Date of Birth	January 15, 1991
Address	15/6 Moo 9 Phong Sawai, Muang, Ratchaburi, Thailand
Institution Attended	
2013	Bachelor of Science in Mathematics, Silpakorn University
2018	Master of Science in Mathematics, Silpakorn University