# FEASIBILITY OF DISTANCE-REGULAR GRAPHS

By
MISS Supalak SUMALROJ

A Thesis Submitted in partial Fulfillment of Requirements
for Doctor of Philosophy (MATHEMATICS)
Department of MATHEMATICS
Graduate School, Silpakorn University
Academic Year 2017

FEASIBILITY OF DISTANCE-REGULAR GRAPHS

By

MISS Supalak SUMALROJ

A Thesis Submitted in Partial Fulfillment of the Requirements

for Doctor of Philosophy (MATHEMATICS)

Department of MATHEMATICS

Graduate School, Silpakorn University

Academic Year 2017

| | |
|---|---|
| Title | Feasibility of distance-regular graphs |
| By | Supalak SUMALROJ |
| Field of Study | (MATHEMATICS) |
| Advisor | Chalermpong Worawannotai |

Science Silpakorn University in Partial Fulfillment of the Requirements for the Doctor of Philosophy

........................................................................ Dean of Graduate School

(Assoc. Prof. Panjai Tantatsanawong, Ph.D.)

Approved by

........................................................................ Chair person

(Assistant Professor Jittisak Rakbud, Ph.D.)

........................................................................ Advisor

(Chalermpong Worawannotai, Ph.D.)

........................................................................ External Examiner

(Assistant Professor Wongsakorn Charoenpanitseri, Ph.D.)

56305803 : MAJOR (MATHEMATICS)
KEY WORDS : DISTANCE-REGULAR GRAPH, LOCAL GRAPH, NONEXISTENCE, PARTIAL LINEAR SPACE

SUPALAK SUMALROJ : FEASIBILITY OF DISTANCE-REGULAR GRAPHS. THESIS ADVISOR : CHALERMPONG WORAWANNOTAI, Ph.D.

The problem of deciding whether a distance-regular graph with a given intersection array exists is a widely studied topic in distance-regular graphs. In 1989 Brouwer, Cohen and Neumaier have compiled a list of intersection arrays that passed known feasibility conditions, but the existence of corresponding distance-regular graphs were unknown for many of those arrays. Since then the arrays from the list are studied and the existence and nonexistence of distance-regular graphs associated to many arrays from the list are proved but more than half are still unknown.

In this thesis, we study three intersection arrays from the list, $\{22, 16, 5; 1, 2, 20\}$, $\{27, 20, 10; 1, 2, 18\}$, and $\{36, 28, 4; 1, 2, 24\}$. We prove that distance-regular graphs with these intersection arrays do not exist. To prove these, we assume that such graphs exist and derive some combinatorial properties of their local graphs to get contradictions.

# Acknowledgements

This thesis has been completed by the involvement of people about whom I would like to mention here.

I would like to express my deep gratitude to my thesis advisor, Dr. Chalermpong Worawannotai, for insightful suggestions on my work. He encouraged and advised me through the thesis process.

I also would like to thank to my thesis committees, Assistant Professor Dr. Jittisak Rakbud and Assistant Professor Dr. Wongsakorn Charoenpanitseri for their comments and suggestions.

Moreover, I would like to thank all the teachers who have instructed and taught me for valuable knowledge.

In addition, I would like to thank the Development and Promotion of Science and Technology Talents Project (DPST) for financial support throughout my undergraduate and graduate study.

Finally, I would like to thank my family, my friends and those whose names are not mentioned here but have greatly inspired and encouraged me throughout the period of this research.

Supalak SUMALROJ

# Table of contents

# List of tables

# List of figures

# Chapter 1

# Introduction

The problem of deciding whether a distance-regular graph with a given intersection array exists is a widely studied topic in distance-regular graphs. In 1989 Brouwer, Cohen and Neumaier [5] have compiled a list of intersection arrays that passed known feasibility conditions, but the existence of corresponding distance-regular graphs were unknown for many of those arrays. Since then the arrays from the list are studied and the existence and nonexistence of distance-regular graphs associated to many arrays from the list are proved [11, Section 17] but more than half are still unknown.

In this chapter we intend to recall some definitions and notations used in this thesis. Most of them follows Biggs [2], Bondy and Murty [3], and Brouwer, Cohen and Neumaier [5].

A *graph* is an ordered pair $\Gamma = (V(\Gamma), E(\Gamma))$ where $V(\Gamma)$ is a nonempty set of elements called *vertices* and $E(\Gamma)$ is a set of unordered pairs of (not necessary distinct) vertices called *edges*. For any edge $e = \{x, y\} \in E(\Gamma)$, we say that $x$ and $y$ are *adjacent* and we write $e = xy$. The vertices $x$ and $y$ are called the *end vertices* of an edge $e$. We say that the vertices $x$ and $y$ are *incident* with an edge $e$. A graph $\Gamma$ is said to be *finite* whenever both $V(\Gamma)$ and $E(\Gamma)$ are finite. The *order* of a graph $\Gamma$ is the number of vertices of $\Gamma$. An edge is called a *loop* whenever it has identical end vertices. Two or more edges that join the same end vertices are call *parallel edges*. A *simple graph* is a graph having no loops or parallel edges. All graphs we consider are finite and simple.

A graph $\Gamma'$ is a *subgraph* of a graph $\Gamma$ whenever $V(\Gamma') \subseteq V(\Gamma)$ and $E(\Gamma') \subseteq E(\Gamma)$. For a nonempty subset $S$ of $V(\Gamma)$, the *subgraph* of $\Gamma$ *induced* by $S$, is a graph with vertex set $S$ and edge set $\{xy \in E(\Gamma) | x, y \in S\}$. For a subset $S$

of $V(\Gamma)$, the *neighborhood* of $S$ in $\Gamma$, denoted by $N_\Gamma(S)$, is the set of all vertices in $\Gamma - S$ that are adjacent to at least one vertex of $S$. A *neighborhood* of a vertex $x$ in $\Gamma$, denoted by $N_\Gamma(x)$, is the set $\{y \in V(\Gamma) | xy \in E(\Gamma)\}$. The *degree* of $x$ in $\Gamma$ is $|N_\Gamma(x)|$. For any graph $\Gamma$, we identify $\Gamma$ with its vertex set $V(\Gamma)$. We denote the subgraph of $\Gamma$ induced by a subset $S$ of $V(\Gamma)$ by $S$ itself. For a vertex $x$ in $\Gamma$, the subgraph of $\Gamma$ induced by the neighbors of $x$ is called the *local graph* of $\Gamma$ with respect to $x$.

A *walk* in a graph is a finite sequence $x_0e_1x_1e_2\ldots e_{n-1}x_{n-1}e_nx_n$ of vertices and edges such that for $1 \leq i \leq n$, the edge $e_i$ has end vertices $x_{i-1}$ and $x_i$. A *path* is a walk with distinct vertices. A walk $C = x_0e_1x_1e_2...e_{n-1}x_{n-1}e_nx_0$ is called a *cycle* whenever the edges $e_1, e_2, \ldots, e_n$ and the vertices $x_0, x_1, \ldots, x_{n-1}$ of $C$ are distinct and $C$ has at least 3 edges. A cycle $C$ has *length* $n$, denoted by $C_n$, if the number of edges of $C$ is $n$. We may write a cycle $x_0e_1x_1e_2...e_{n-1}x_{n-1}e_nx_0$ by $x_0x_1\ldots x_{n-1}$. Two vertices $x$ and $y$ are *connected* whenever there exists a path from $x$ to $y$. We say that a graph $\Gamma$ is *connected* whenever every pair of its vertices are connected; otherwise $\Gamma$ is *disconnected*. For vertices $x$ and $y$ in $\Gamma$, the *distance* between $x$ and $y$, denoted by $d(x,y)$ is the length of a shortest path between $x$ and $y$ in $\Gamma$. The *diameter* of $\Gamma$, denoted by $diam(\Gamma)$, is the greatest distance between any pair of vertices in $\Gamma$. A *complete graph* is a simple graph in which any two distinct vertices are adjacent. A complete graph with $n$ vertices is denoted by $K_n$. A *clique* of a graph $\Gamma$ is a maximal complete subgraph of $\Gamma$. A *coclique* of a graph $\Gamma$ is a nonempty induced subgraph of $\Gamma$ with an empty set of edges.

A *regular graph* is a graph such that each vertex has the same degree. For an integer $k \geq 0$, a graph is *k-regular* whenever every vertex has degree $k$; in other words, a graph has *valency* $k$. Let $\Gamma$ denote a connected graph with diameter $d$. For a vertex $x \in V(\Gamma)$ and $0 \leq i \leq d$ let $\Gamma_i(x)$ denote the set of vertices at distance $i$ from $x$. The graph $\Gamma$ is called *distance-regular* whenever for all $0 \leq i \leq d$ and any two vertices $x$ and $y$ and distance $d(x,y) = i$, the numbers $b_i = |\Gamma_{i+1}(x) \cap \Gamma_1(y)|$, $c_i = |\Gamma_{i-1}(x) \cap \Gamma_1(y)|$ and $a_i = |\Gamma_i(x) \cap \Gamma_1(y)|$ depend only on $i$ where $\Gamma_{-1}(x)$ and $\Gamma_{d+1}(x)$ are unspecified. The numbers $b_i, c_i$ and $a_i$ are called the *intersection numbers* of $\Gamma$. For $0 \leq i \leq d$ define $k_i = |\Gamma_i(x)|$. In particular, $\Gamma$ is a regular graph with degree $k = b_0, b_d = c_0 = 0, c_1 = 1$ and $c_i + a_i + b_i = k$ for all $0 \leq i \leq d$. The sequence $\{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\}$ is called the *intersection array* of $\Gamma$. The *distribution diagram* for a distance-regular graph with intersection array $\{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\}$ is shown in Figure 1.1.

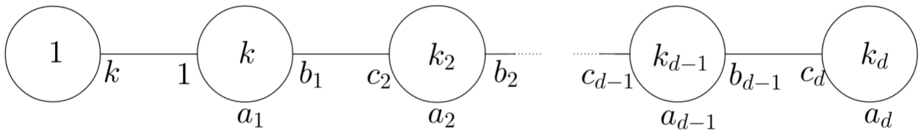


Figure 1.1: Distribution diagram for a distance-regular graph with intersection array $\{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\}$.

**Example 1.1.** The Heawood graph is a distance-regular graph on 14 vertices and diameter 3 with intersection array $\{3, 2, 2; 1, 1, 3\}$. The distribution diagram is shown in Figure 1.3. For a fixed vertex $x$ we display the sets $\Gamma_i(x)$ for $0 \leq i \leq 3$ in Figure 1.4.

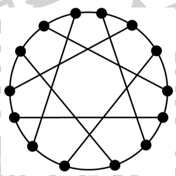

Figure 1.2: The Heawood graph.

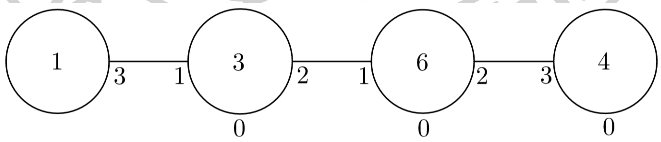


Figure 1.3: Distribution diagram for the Heawood graph.

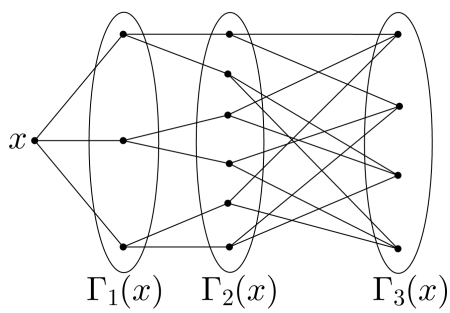


Figure 1.4: Illustration for the sets $\Gamma_i(x)$ of the Heawood graph.

A graph $\Gamma$ is called *strongly regular* with parameters $(|V(\Gamma)|, k, \lambda, \mu)$ whenever $\Gamma$ is $k$-regular, each two adjacent vertices have $\lambda$ common neighbors, and each two nonadjacent vertices have $\mu$ common neighbors. The connected strongly regular graphs are precisely the distance-regular graphs with diameter two and $k = b_0, \lambda = a_1$ and $\mu = c_2$.

For $0 \leq i \leq d$, let $A_i$ denote the $|V(\Gamma)| \times |V(\Gamma)|$ matrix whose rows and columns are indexed by the vertices of $\Gamma$ and

$$(A_i)_{xy} = \begin{cases} 1 & \text{if} \quad d(x, y) = i, \\ 0 & \text{if} \quad d(x, y) \neq i, \end{cases}$$

where $x, y \in V(\Gamma)$. We call $A_i$ the *i-th distance matrix* of $\Gamma$. In particular, we call $A = A_1$ the *adjacency matrix* of $\Gamma$. By construction the matrix $A_i$ is real and symmetric for $0 \leq i \leq d$.

The *eigenvalues* of $\Gamma$ are the eigenvalues of its adjacency matrix. Since an adjacency matrix is real and symmetric, its eigenvalues are real numbers. The *multiplicity* of an eigenvalue $\theta$ is the multiplicity of the root $\theta$ of the characteristic equation $det(\alpha I - A) = 0$. The *spectrum* of a graph is the set of numbers which are eigenvalues together with their multiplicities. If the distinct eigenvalues of a graph are $\theta_0 > \theta_1 > \cdots > \theta_d$ and their multiplicities are $m_0, m_1, \ldots, m_d$, respectively, then we write the spectrum of the graph as $\theta_0^{m_0} \theta_1^{m_1} \cdots \theta_d^{m_d}$.

An *incidence geometry* $(P, L)$ consists of a set $P$ whose elements are called points and a set $L$ whose elements are called lines together with an incidence relation between points and lines, that is, a subset of $P \times L$. A *partial linear space* is an incidence geometry such that every pair of distinct points lie on at most one common line and every line has at least two points.

**Example 1.2.** The Fano plane is a partial linear space with 7 points and 7 lines and each line has 3 points.

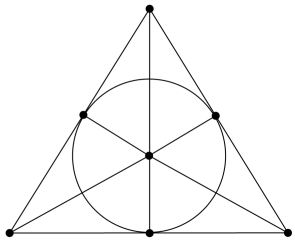

Figure 1.5: The Fano plane.

There are many results concerning existence and nonexistence of distance-regular graphs, as example:

In [9] Coolsaet and Degraer proved that there exists a unique distance-regular graph with intersection array $\{6, 5, 2; 1, 1, 3\}$ on 57 vertices. This graph is known as the Perkel graph.

In [5, Theorem 11.2.1 (13)] a distance-regular graph with intersection array $\{31, 30, 17; 1, 2, 15\}$ on 1024 vertices was constructed from studying the Kasami codes.

Brouwer and Pasechnik [6] proved that there exists a distance-regular graph with intersection array $\{26, 24, 19; 1, 3, 8\}$ on 729 vertices by constructing the subgraph of a dual polar graph.

Coolsaet and Jurišić [10] established the nonexistence of a distance-regular graph with intersection array $\{74, 54, 15; 1, 9, 60\}$ and of distance-regular graphs with intersection arrays $\{4r^3 + 8r^2 + 6r + 1, 2r(r + 1)(2r + 1), 2r^2 + 2r + 1; 1, 2r(r+1), (2r+1)(2r^2+2r+1)\}$ whrer $r$ is a positive integer by using equality in the Krein conditions.

There are many results that established the nonexistence of distance-regular graphs by studying the local graphs, as example:

Coolsaet [7] proved that a distance-regular graph with intersection array $\{21, 16, 8; 1, 4, 14\}$ does not exist by partitioning a local graph of a hypothetical distance-regular graph and constructing a partial linear space on the partition.

In [8] Coolsaet proved the nonexistence of a distance-regular graph with intersection array $\{13, 10, 7; 1, 2, 7\}$ by showing that its local graph is a disjoint union of triangles, hexagons and/or heptagons.

Coolsaet and Jurišić [10] proved the nonexistence of a distance-regular graph with intersection array $\{19, 12, 5; 1, 4, 15\}$ by showing that its local graph is a strongly regular graph with parameters $(19, 6, 1, 2)$ and it is known that a strongly regular graph with these parameters does not exist.

In [12] Gavrilyuk proved that distance-regular graphs with intersection arrays $\{55, 36, 11; 1, 4, 45\}$ and $\{56, 36, 9; 1, 3, 48\}$ do not exist by considering the coclique of local graphs.

Jurišić and Koolen [13] proved that there is no distance-regular graph of diameter four with intersection array $\{r(2s + 2rs - r^2), (r^2 - 1)(2s - r + 1), rs, 1; 1, rs, (r^2 - 1)(2s - r + 1), r(2s + 2rs - r^2)\}$ where $r$ and $s$ are odd integers by showing that its local graph is a strongly regular graph and considering its property.

In this thesis we show the nonexistence of distance-regular graphs with intersection arrays $\{22, 16, 5; 1, 2, 20\}$, $\{27, 20, 10; 1, 2, 18\}$, and $\{36, 28, 4; 1, 2, 24\}$. We obtain the results by studying local graphs of hypothetical distance-regular graphs and their combinatorial properties.

# Chapter 2
# Distance-Regular Graphs

In this chapter we provide some background and known results about distance-regular graphs. From now on we assume that $\Gamma$ is a distance-regular graph with degree $k$ and diameter $d$.

The following results are necessary conditions for the intersection arrays of distance-regular graphs.

**Lemma 2.1.** (See [5, pp. 127].) *For* $0 \leq i \leq d-1$,

*(i)* $k_0 = 1$,

*(ii)* $k_1 = k$,

*(iii)* $k_{i+1}c_{i+1} = k_i b_i$,

*(iv)* $|V(\Gamma)| = 1 + k_1 + k_2 + \cdots + k_d$.

*Proof.* $(i), (ii), (iv)$ follow from the definition of a distance-regular graph. $(iii)$ holds because each of $k_{i+1}c_{i+1}$ and $k_i b_i$ is equal to the number of edges between $\Gamma_i(x)$ and $\Gamma_{i+1}(x)$. □

**Proposition 2.2.** (See [5, Proposition 4.1.6].) *The following conditions hold:*

*(i)* $k = b_0 > b_1 \geq b_2 \geq \cdots \geq b_{d-1} > b_d = 0$,

*(ii)* $1 = c_1 \leq c_2 \leq \cdots \leq c_d \leq k$,

*(iii) if* $i + j \leq d$, *then* $c_i \leq b_j$,

*(iv) there exists an* $i$ *such that* $k_0 \leq k_1 \leq \cdots \leq k_i$ *and* $k_{i+1} \geq k_{i+2} \geq \cdots \geq k_d$,

*(v) all multiplicities are integers.*

*Proof.* ($i$) For $1 \leq i \leq d$, let $x, y, z \in V(\Gamma)$ such that $d(x, y) = i$ and $z \in \Gamma_1(x) \cap \Gamma_{i-1}(y)$. Let $w \in \Gamma_1(y) \cap \Gamma_{i+1}(x)$. Then $d(w, z) = i$ and thus $w \in \Gamma_1(y) \cap \Gamma_i(z)$. Thus $\Gamma_1(y) \cap \Gamma_{i+1}(x) \subseteq \Gamma_1(y) \cap \Gamma_i(z)$. So $b_i = |\Gamma_1(y) \cap \Gamma_{i+1}(x)| \leq |\Gamma_1(y) \cap \Gamma_i(z)| = b_{i-1}$.

($ii$) For $1 \leq i \leq d$, let $x, y, z \in V(\Gamma)$ such that $d(x, y) = i$ and $z \in \Gamma_1(x) \cap \Gamma_{i-1}(y)$. Let $w \in \Gamma_1(y) \cap \Gamma_{i-2}(z)$. Then $d(w, x) = i - 1$ and so $w \in \Gamma_1(y) \cap \Gamma_{i-1}(x)$. Thus $\Gamma_1(y) \cap \Gamma_{i-2}(z) \subseteq \Gamma_1(y) \cap \Gamma_{i-1}(x)$. Therefore $c_{i-1} = |\Gamma_1(y) \cap \Gamma_{i-2}(z)| \leq |\Gamma_1(y) \cap \Gamma_{i-1}(x)| = c_i$.

($iii$) Suppose that $i + j \leq d$. Let $x, y, z \in V(\Gamma)$ such that $d(x, y) = i + j$ and $z \in \Gamma_i(x) \cap \Gamma_j(y)$. Let $w \in \Gamma_1(z) \cap \Gamma_{i-1}(x)$. Then $d(w, y) = j + 1$ and hence $w \in \Gamma_1(z) \cap \Gamma_{j+1}(y)$. So $\Gamma_1(z) \cap \Gamma_{i-1}(x) \subseteq \Gamma_1(z) \cap \Gamma_{j+1}(y)$. Therefore $c_i = |\Gamma_1(z) \cap \Gamma_{i-1}(x)| \leq |\Gamma_1(z) \cap \Gamma_{j+1}(y)| = b_j$.

($iv$) By (i),(ii) and Lemma 2.1 (iii), we have $k_j/k_{j+1} = c_{j+1}/b_j \leq c_{j+2}/b_{j+1} = k_{j+1}/k_{j+2}$. Then there exists a $i$ such that $k_0 \leq k_1 \leq \cdots \leq k_i$ and $k_{i+1} \geq k_{i+2} \geq \cdots \geq k_d$

($v$) It follows from the definition of multiplicity. □

The following results give formulas of eigenvalues and their multiplicities of a distance-regular graph.

**Lemma 2.3.** (See [5, pp. 127].) *For* $0 \leq i \leq d$,

$$AA_i = c_{i+1}A_{i+1} + a_iA_i + b_{i-1}A_{i-1}$$

*where* $A$ *is the adjacency matrix of* $\Gamma$, $A_{-1} = A_{d+1} = 0$ *and* $b_{-1}$ *and* $c_{d+1}$ *are unspecified.*

*Proof.* For $x, y \in V(\Gamma)$, we have

$$\begin{aligned}(AA_i)_{xy} &= \sum_{z \in V(\Gamma)} A_{xz}(A_i)_{zy} \\ &= \sum_{z \in \Gamma_1(x) \cap \Gamma_i(y)} 1 \\ &= |\Gamma_1(x) \cap \Gamma_i(y)|\end{aligned}$$

$$= \begin{cases} b_{i-1} & \text{if} \quad d(x,y) = i-1, \\ a_i & \text{if} \quad d(x,y) = i, \\ c_{i+1} & \text{if} \quad d(x,y) = i+1, \\ 0 & \text{otherwise} \end{cases}$$

$$= (c_{i+1}A_{i+1} + a_iA_i + b_{i-1}A_{i-1})_{xy}.$$

The result follows. □

Let $\lambda$ denote an indeterminate. Define polynomials $\{v_i\}_{i=0}^{d+1}$ by $v_0(\lambda) = 1$, $v_1(\lambda) = \lambda$, and for $1 \leq i \leq d$, $\lambda v_i(\lambda) = c_{i+1}v_{i+1}(\lambda) + a_iv_i(\lambda) + b_{i-1}v_{i-1}(\lambda)$ where $v_{-1}(\theta) = v_{d+1}(\theta) = 0$, and $b_{-1}$ and $c_{d+1}$ are unspecified.

**Lemma 2.4.** (See [5, pp. 127, 128] and [14, Lemma 3.8].) *The following conditions hold:*

*(i)* *deg* $v_i = i$ $(0 \leq i \leq d+1)$,

*(ii)* *the coefficient of* $\lambda^i$ *in* $v_i$ *is* $(c_1c_2\cdots c_i)^{-1}$ $(0 \leq i \leq d+1)$,

*(iii)* $v_i(A) = A_i$ $(0 \leq i \leq d)$,

*(iv)* $v_{d+1}(A) = 0$,

*(v)* *the distinct eigenvalues of* $\Gamma$ *are precisely the zeros of* $v_{d+1}$.

Define a $(d+1) \times (d+1)$ matrix $B$ as follows:

$$B = \begin{bmatrix} a_0 & b_0 & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & \\ & c_2 & a_2 & \ddots & \\ & & \ddots & \ddots & b_{d-1} \\ \mathbf{0} & & & c_d & a_d \end{bmatrix}.$$

Observe that $v(\lambda)B = \lambda v(\lambda)$ where $v(\lambda) = (v_0(\lambda), v_1(\lambda), ..., v_d(\lambda))$. So $v(\lambda)$ is an eigenvector of $B$ corresponding to eigenvalue $\lambda$. The minimum polynomial of $B$

has degree $d+1$ and satisfies $a_d v_d(\lambda) + b_{d-1} v_{d-1}(\lambda) = \lambda v_d(\lambda)$ that is $\lambda v_d(\lambda) - a_d v_d(\lambda) - b_{d-1} v_{d-1}(\lambda) = 0$. By Lemma 2.3 and Lemma 2.4, the adjacency matrix $A$ has the same minimal polynomial as $B$. Moreover the minimal polynomial of $B$ is the characteristic polynomial of $B$.

**Proposition 2.5.** (See [2, Proposition 21.2].) *$\Gamma$ has $d+1$ distinct eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_d$ which are the eigenvalues of the matrix $B$.*

**Theorem 2.6.** *(Biggs' formula)* (See [2, Theorem 21.4].) *Let $\theta$ denote an eigenvalue of $\Gamma$. Then the multiplicity $m(\theta)$ of $\theta$ satisfies*

$$m(\theta) = \frac{|V(\Gamma)|}{\sum_{i=0}^{d} \frac{(v_i(\theta))^2}{k_i}}.$$

The following proposition gives an upper bound of the size of a clique of a distance-regular graph in terms of its smallest and largest eigenvalues.

**Proposition 2.7.** (See [5, Proposition 4.4.6].) *Let $\Gamma$ denote a distance-regular graph of diameter $d \geq 2$ with eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_d$. Then the size of a clique $K$ in $\Gamma$ is bounded by*

$$|K| \leq 1 - k/\theta_d.$$

# Chapter 3

# The nonexistence of a distance-regular graph with intersection array {27, 20, 10; 1, 2, 18}

In this chapter we investigate a distance-regular graph with intersection array $\{27, 20, 10; 1, 2, 18\}$. If a distance-regular graph with such array exists, then by Lemma 2.1, the number of vertices is 448 and the valency is 27. By Proposition 2.5 and Theorem 2.6, the spectrum of the graph is $27^1 9^{96} (-1)^{216} (-5)^{135}$ and the distribution diagram is shown in Figure 3.1.

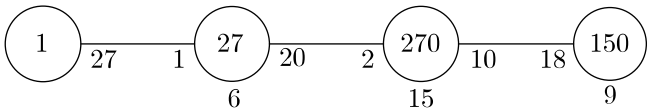


Figure 3.1: Distribution diagram for a distance-regular graph with intersection array {27,20,10;1,2,18}.

In addition, the distance-three graph is a strongly regular graph with parameters $(243, 150, 50, 50)$ which corresponds to a partial geometry $pg(15, 9, 5)$; according to Brouwer [4], it is unknown whether such a strongly regular graph and a partial geometry exist.

In this chapter we prove the nonexistence of a distance-regular graph with intersection array $\{27, 20, 10; 1, 2, 18\}$. In particular we assume that such a graph exists and derive some combinatorial properties of its local graph to display the contradiction.

The following results are combinatorial properties of a distance-regular graph.

**Lemma 3.1.** *Let* $\Gamma$ *denote a distance-regular graph and fix a vertex* $\infty$ *of* $\Gamma$*. Then each vertex in* $\Gamma_1(\infty)$ *is on at least* $\lceil \frac{1}{2}(a_1^2 + 1 - |\Gamma_1(\infty)|) \rceil$ *triangles.*

*Proof.* Let $u$ denote a vertex of $\Gamma_1(\infty)$. Let $u_1, u_2, ..., u_{a_1}$ denote the distinct neighbors of $u$ in $\Gamma_1(\infty)$. Let $N$ denote the number of triangles of $\Gamma_1(\infty)$ that contain $u$. Observe that $N$ is also the number of edges $u_iu_j$ where $1 \leq i < j \leq a_1$. Thus the number of vertices of $\Gamma_1(\infty)$ with distance at most 2 from $u$ is $1+a_1+(a_1-1)a_1-2N$. Therefore a vertex $u$ is on at least $\lceil \frac{1}{2}(a_1^2 + 1 - |\Gamma_1(\infty)|) \rceil$ triangles in $\Gamma_1(\infty)$. □

**Lemma 3.2.** *Let $\Gamma$ denote a distance-regular graph with $c_2 = 2$. Fix a vertex $\infty$ of $\Gamma$. Let $\Delta = \Gamma_1(\infty)$ denote the subgraph of $\Gamma$ induced by all vertices of $\Gamma$ adjacent to $\infty$. If $\Delta$ contains a cycle $C$ of length 4, then the subgraph induced by $C$ is a complete graph $K_4$.*

*Proof.* Suppose that $\Delta$ contains a cycle $C$ of length 4. Suppose there exist vertices $u$ and $v$ of $C$ that are not adjacent in $\Delta$. Then the distance between $u$ and $v$ is 2 and there exist two distinct paths from $u$ to $v$ of length 2 in $C$ and a path $u\infty v$ in $\Gamma$ which contradicts the fact that $c_2 = 2$. Thus any two distinct vertices of $C$ are adjacent. Therefore the subgraph induced by $C$ is a complete graph $K_4$. □

From now on we assume that $\Gamma$ is a distance-regular graph with intersection array $\{27, 20, 10; 1, 2, 18\}$. Then $\Gamma$ has eigenvalues $27, 9, -1$ and $-5$. Fix a vertex $\infty$ of $\Gamma$. Let $\Delta = \Gamma_1(\infty)$ denote the subgraph of $\Gamma$ induced by all vertices of $\Gamma$ adjacent to $\infty$. Then $\Delta$ is a 6-regular graph on 27 vertices. The following results give some properties of the local graph $\Delta$. By Lemma 3.2 we have that two nonadjacent vertices in $\Delta$ have at most one common neighbor in $\Delta$.

**Corollary 3.3.** *$\Delta$ does not contain a complete subgraph $K_i$ for all $i \geq 6$.*

*Proof.* By Proposition 2.7, the size of a clique in $\Gamma$ is at most 6. Thus the size of a clique in $\Delta$ is at most 5. □

**Lemma 3.4.** *Each vertex in $\Delta$ is on at least six subgraphs $K_3$'s of $\Delta$.*

*Proof.* Let $u$ denote a vertex of $\Delta$ and $u_1, u_2, u_3, u_4, u_5, u_6$ denote the distinct neighbors of $u$ in $\Delta$. From Lemma 3.1 the number of edges $u_iu_j$ for $1 \leq i < j \leq 6$ is at least 5. By the pigeonhole principle, there exists one vertex of $\{u_i | 1 \leq i \leq 6\}$

which is incident with at least 2 edges of the $N$ edges $u_iu_j$ so we may assume that $u_1$ is adjacent to $u_2$ and $u_3$. By Lemma 3.2 applied to the cycle $uu_2u_1u_3$, the vertices $u_2$ and $u_3$ are adjacent. Thus $u$ is on at least six subgraphs $K_3$'s of $\Delta$. □

By Lemma 3.2, Corollary 3.3 and Lemma 3.4, there are 3 possibilities for the subgraph of $\Delta$ induced by a vertex $w$ and its neighbors as shown in Figure 3.2.

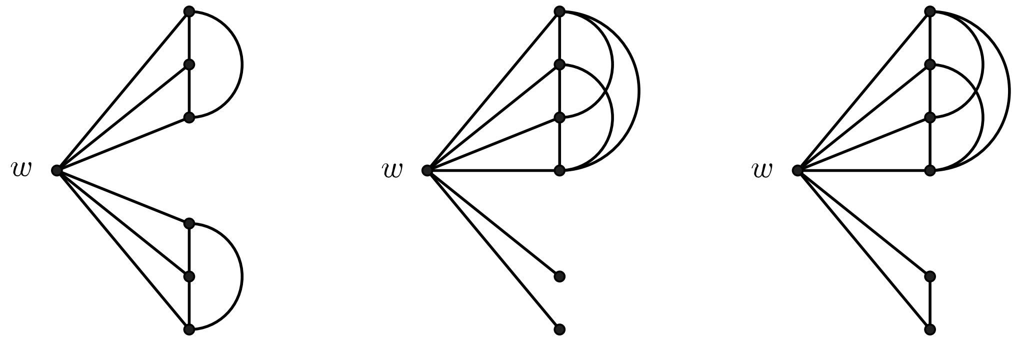


Figure 3.2: The 3 possibilities for the subgraph of $\Delta$ induced by a vertex $w$ and its neighbors.

**Lemma 3.5.** $\Delta$ *contains a complete subgraph* $K_5$.

*Proof.* Suppose that there is no a complete subgraph $K_5$ in $\Delta$. Then we have the first possibility of Figure 3.2, that is, each vertex is on two subgraphs $K_4$'s of $\Delta$. Thus the total number of subgraphs $K_4$'s in $\Delta$ is $27 \times 2/4$, a contradiction. □

Observe that we always have the second or the third possibility of Figure 3.2. The number of subgraphs $K_5$'s in $\Delta$ is $27/5$, but that is not an integer, a contradiction. Therefore such graph $\Gamma$ does not exist and we have the following theorem.

**Theorem 3.6.** *A distance-regular graph with intersection array* $\{27, 20, 10; 1, 2, 18\}$ *does not exist.*

# Chapter 4

# The nonexistence of a distance-regular graph with intersection array {36, 28, 4; 1, 2, 24}

In this chapter we consider the intersection array $\{36, 28, 4; 1, 2, 24\}$ [5, pp. 428]. If a distance-regular graph with such array exists, then by Lemma 2.1, the number of vertices is $625 = 5^4$ and the valency is 36. By Proposition 2.5 and Theorem 2.6, the spectrum of the graph is $36^1 11^{84} 6^{120} (-4)^{420}$ and the distribution diagram is shown in Figure 4.1.

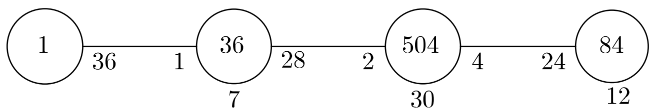


Figure 4.1: Distribution diagram for a distance-regular graph with intersection array {36,28,4;1,2,24}.

In addition, the distance-two graph is a strongly regular graph with parameters $(625, 504, 403, 420)$ which corresponds to an orthogonal array $OA(25, 21)$; according to Brouwer [4], such a strongly regular graph exists.

In this chapter we apply the result in Chapter 3 to prove the nonexistence of a distance-regular graph with intersection array $\{36, 28, 4; 1, 2, 24\}$. In particular we assume that such a graph exists and show that its local graph is a disjoint union of complete graphs $K_8$'s to get a contradiction.

From now on we assume that $\Gamma$ is a distance-regular graph with intersection array {36,28,4;1,2,24}. Then $\Gamma$ has eigenvalues $36, 11, 6$ and $-4$. Fix a vertex $\infty$ of $\Gamma$. Let $\Delta = \Gamma_1(\infty)$ denote the subgraph of $\Gamma$ induced by all vertices of $\Gamma$ adjacent to $\infty$. Then $\Delta$ is a 7-regular graph on 36 vertices.

**Corollary 4.1.** *Each vertex in $\Delta$ is on at least seven complete subgraphs $K_3$'s of $\Delta$.*

*Proof.* The result follows from Lemma 3.1. □

By Lemma 3.2 and Corollary 4.1, there are 6 possibilities for the subgraph of $\Delta$ induced by a vertex $u$ and its neighbors as shown in Figure 4.2.

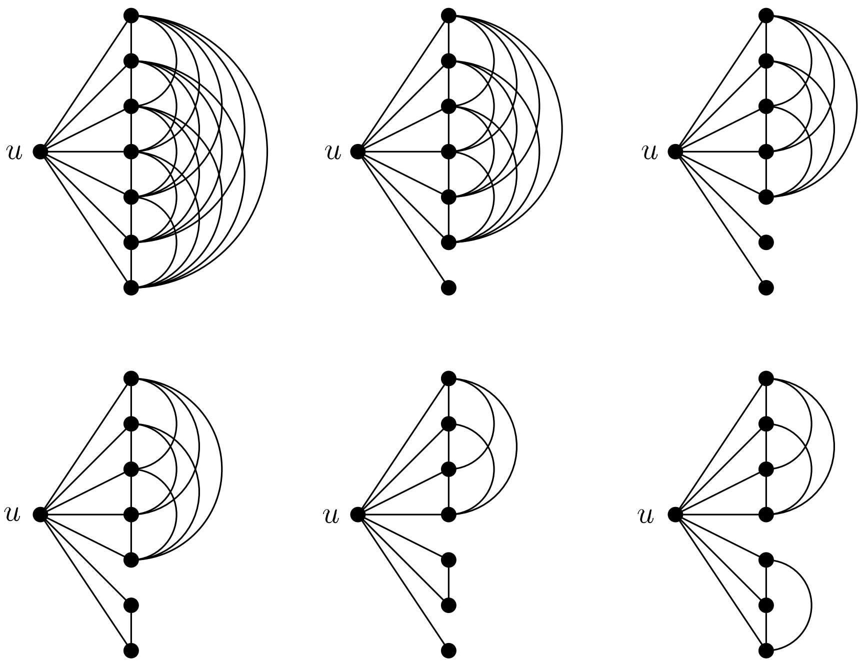


Figure 4.2: The 6 possibilities for the subgraph of $\Delta$ induced by a vertex $u$ and its neighbors.

Observe that each vertex of $\Delta$ is on a complete subgraph $K_5$. The following three lemmas show that a vertex of $\Delta$ and its neighbors induce a complete subgraph $K_8$.

**Lemma 4.2.** *Each vertex of $\Delta$ is on a complete subgraph $K_6$.*

*Proof.* Let $u$ denote a vertex of $\Delta$. Let $K$ denote a complete subgraph $K_5$ that contains $u$. Let $K = \{u, v, w, x, y\}$. For $t \in K$, let $N_{\Delta - K}(t) = \{t_1, t_2, t_3\}$. We first show that there exist two distinct vertices in $K$ which have a common neighbor in $\Delta - K$. Suppose not. Then $u_i, v_i, w_i, x_i$ and $y_i$ are distinct for all $1 \leq i \leq 3$. Let $u_{i1}, u_{i2}, u_{i3}$ and $u_{i4}$ denote neighbors of $u_i$ such that $u_i, u_{i1}, u_{i2}, u_{i3}$ and $u_{i4}$ induce a complete subgraph $K_5$ for $1 \leq i \leq 3$. Then $u_j \notin \{u_{i1}, u_{i2}, u_{i3}, u_{i4}\}$ for $i \neq j$ by Lemma 3.2. The vertices $u_{ij}$ and $u_{lm}$ are distinct for all $1 \leq i < l \leq 3$

and $1 \leq j, m \leq 4$ by Lemma 3.2. Since $\Delta$ has 36 vertices, we let $\Delta - (K \cup \{u_i, v_i, w_i, x_i, y_i, u_{ij} | 1 \leq i \leq 3, 1 \leq j \leq 4\}) = \{s_1, s_2, s_3, s_4\}$. By Lemma 3.2, the vertex $v_1$ is adjacent to at most one vertex in $\{u_{ij} | 1 \leq j \leq 4\}$ for each $1 \leq i \leq 3$ and $v_1$ is not adjacent to $u_j, w_j, x_j$ and $y_j$ for all $1 \leq j \leq 3$. Without loss of generality, we may assume that $v_1$ is adjacent to $s_1$. By Lemma 3.2, the vertices $v_2$ and $v_3$ are not adjacent to $s_1$. By similar arguments, we may assume that $v_2$ is adjacent to $s_2$, and $v_3$ is adjacent to $s_3$. Then $v_1$ is not adjacent to $s_2$ or $s_3$ by Lemma 3.2. By Corollary 4.1 applied to $v$, we may assume that $v_1$ and $v_2$ are adjacent. Since $v_1$ has degree 7 and $v_1$ is not adjacent to $s_2$ or $s_3$, we may assume that $v_1$ is adjacent to $u_{11}$ and $u_{21}$. By Corollary 4.1 applied to $v_1$, the subgraph induced by a vertex $v_1$ and its neighbors except $v$ and $v_2$ contains a complete subgraph $K_5$ and we may assume that $u_{11}$ is on such a subgraph. It follows that $u_{11}$ has degree at least 8, a contradiction. Thus there exist two distinct vertices in $K$ which have a common neighbor in $\Delta - K$. Without loss of generality, we may assume that $u$ and $v$ have a common neighbor in $\Delta - K$, say $z$. Then $z$ is adjacent to $w, x$ and $y$ by Lemma 3.2. The subgraph induced by $u, v, w, x, y$ and $z$ is a complete graph $K_6$. □

**Lemma 4.3.** *Each vertex of $\Delta$ is on a complete subgraph $K_7$.*

*Proof.* Let $u$ denote a vertex of $\Delta$. Let $F$ denote a complete subgraph $K_6$ that contains $u$. Let $F = \{u, v, w, x, y, z\}$. For $t \in F$, let $N_{\Delta - F}(t) = \{t_1, t_2\}$. We first show that there exist two distinct vertices in $F$ which have a common neighbor in $\Delta - F$. Suppose not. Then $u_i, v_i, w_i, x_i, y_i$ and $z_i$ are distinct for all $1 \leq i \leq 2$. By Lemma 3.2, the vertex $u_i$ is not adjacent to $v_j, w_j, x_j, y_j$ and $z_j$ for all $1 \leq i, j \leq 2$. Let $u_{i1}, u_{i2}, u_{i3}, u_{i4}$ and $u_{i5}$ denote neighbors of $u_i$ such that $u_i, u_{i1}, u_{i2}, u_{i3}, u_{i4}$ and $u_{i5}$ induce a complete subgraph $K_6$ for $1 \leq i \leq 2$. Then $u_j \notin \{u_{i1}, u_{i2}, u_{i3}, u_{i4}, u_{i5}\}$ for $i \neq j$ by Lemma 3.2. The vertices $u_{ij}$ and $u_{lm}$ are distinct for all $1 \leq i < l \leq 2$ and $1 \leq j, m \leq 5$ by Lemma 3.2. Since $\Delta$ has 36 vertices, we let $\Delta - (F \cup \{u_i, v_i, w_i, x_i, y_i, z_i, u_{ij} | 1 \leq i \leq 2, 1 \leq j \leq 5\}) = \{s_1, s_2, ..., s_8\}$. Consider a vertex $v_1$. By Lemma 3.2, the vertex $v_1$ is adjacent to at most one vertex in $\{u_{ij} | 1 \leq j \leq 5\}$ for each $1 \leq i \leq 2$ and the vertex $v_1$ is not adjacent to $u_j, w_j, x_j, y_j$ and $z_j$ for all

$1 \leq j \leq 2$. Without loss of generality, we may assume that $v_1$ is adjacent to $s_i$ for all $1 \leq i \leq 3$. By Lemma 3.2, the vertex $v_2$ is not adjacent to $s_i$ for all $1 \leq i \leq 3$. By similar arguments, we may assume that $v_2$ is adjacent to $s_i$ for all $4 \leq i \leq 6$. Consider a vertex $w_1$. By Lemma 3.2, the vertex $w_1$ is adjacent to at most one vertex in $\{u_{ij} | 1 \leq j \leq 5\}$ for each $1 \leq i \leq 2$, at most one vertex in $\{s_1, s_2, s_3\}$, at most one vertex in $\{s_4, s_5, s_6\}$ and $w_1$ is not adjacent to $u_j, v_j, x_j, y_j$ and $z_j$ for all $1 \leq j \leq 2$. We may assume that $w_1$ is adjacent to $s_7$. By Lemma 3.2, the vertex $w_2$ is not adjacent to $s_7$. By similar arguments, we assume that $w_2$ is adjacent to $s_8$. Consider the vertices $x_1, x_2, y_1, y_2, z_1$ and $z_2$. By similar arguments, we may assume that $x_1, y_1$ and $z_1$ are adjacent to $s_7$ and $x_2, y_2$ and $z_2$ are adjacent to $s_8$. Then $s_7$ is on at most six complete subgraphs $K_3$'s of $\Delta$ which contradicts Corollary 4.1. Thus there exist two distinct vertices in $F$ which have a common neighbor in $\Delta - F$. Without loss of generality, we may assume that $u$ and $v$ have a common neighbor in $\Delta - F$, say $t$. Then $t$ is adjacent to $w, x, y$ and $z$ by Lemma 3.2. The subgraph induced by $t, u, v, w, x, y$ and $z$ is a complete graph $K_7$. □

**Lemma 4.4.** *Each vertex of $\Delta$ is on a complete subgraph $K_8$.*

*Proof.* Let $u$ denote a vertex of $\Delta$. Let $G$ denote a complete subgraph $K_7$ that contains $u$. Let $G = \{t, u, v, w, x, y, z\}$. For $s \in G$, let $N_{\Delta - G}(s) = \{s_1\}$. We first show that there exist two distinct vertices in $G$ which have a common neighbor in $\Delta - G$. Suppose not. Then $t_1, u_1, v_1, w_1, x_1, y_1$ and $z_1$ are distinct. By Lemma 3.2, the vertex $t_1$ is not adjacent to $u_1, v_1, w_1, x_1, y_1$ and $z_1$. Let $N_{\Delta - G}(t_1) = \{t_{1i} | 1 \leq i \leq 6\}$. By Corollary 4.1, Lemma 4.2 and Lemma 4.3 applied to $t_1$, the vertices $t_{1i}$ and $t_{1j}$ are adjacent for all $1 \leq i < j \leq 6$. By Lemma 3.2, the vertex $u_1$ is adjacent to at most one vertex in $\{t_{1i} | 1 \leq i \leq 6\}$. By Corollary 4.1, Lemma 4.2 and Lemma 4.3 applied to $u_1$, there exist six vertices $u_{11}, u_{12}, u_{13}, u_{14}, u_{15}$ and $u_{16}$ of $\Delta - (G \cup \{t_1, u_1, v_1, w_1, x_1, y_1, z_1, t_{1i} | 1 \leq i \leq 6\})$ such that $u_{1i}$ is adjacent $u_1$ and $u_{1j}$ for all $1 \leq i < j \leq 6$. By similar arguments, any pair of vertices among $t_1, u_1, v_1, w_1, x_1, y_1$ and $z_1$ are not adjacent and do not have common neighbors. Therefore, $|\Delta| \geq 7 + 7 + 7 \times 6 = 56$ vertices, a contradiction. Thus there exist two distinct vertices in $G$ which have a common neighbor in $\Delta - G$. Without loss of generality, we may assume that $u$ and $v$ have a common neighbor in $\Delta - G$, say $s$. Then $s$ is adjacent to $t, w, x, y$ and $z$ by Lemma 3.2. Observe that the subgraph induced by $s, t, u, v, w, x, y$ and $z$ is a complete graph $K_8$. □

By Lemma 4.4 and since $\Delta$ is 7-regular, each component of $\Delta$ is a complete graph $K_8$. Since $|\Delta| = 36$ and $8 \nmid 36$, such graph $\Gamma$ does not exist and we have the following theorem.

**Theorem 4.5.** *A distance-regular graph with intersection array* $\{36, 28, 4; 1, 2, 24\}$ *does not exist.*

# Chapter 5

# The nonexistence of a distance-regular graph with intersection array {22, 16, 5; 1, 2, 20}

In this chapter we prove that a distance-regular graph with intersection array $\{22, 16, 5; 1, 2, 20\}$ does not exist. Our construction is inspired by [7] where the author cleverly partitioned a local graph of a hypothetical distance-regular graph with intersection array $\{21, 16, 8; 1, 4, 14\}$ and constructed a partial linear space on the partition. If a distance-regular graph with such array exists, then by Lemma 2.1, the number of vertices is $243 = 3^5$, which is relatively small, and the valency is 22. By Proposition 2.5 and Theorem 2.6, the spectrum of the graph is $22^1 7^{66} (-2)^{132} (-5)^{44}$ and the distribution diagram is shown in Figure 5.1.

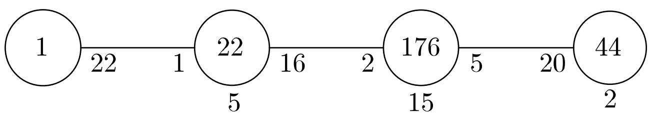


Figure 5.1: Distribution diagram for a distance-regular graph with intersection array {22,16,5;1,2,20}.

In addition, the distance-two graph is strongly regular with parameters $(243, 176, 130, 120)$; according to Brouwer [4], it is unknown whether such a strongly regular graph exists. Incidentally, there is a very interesting strongly regular graph on 243 vertices, valency 22, and $\mu = 2$: the Berlekamp-Van Lint-Seidel graph that corresponds to the ternary Golay code [1].

We apply the result in Chapter 3 to show that a distance-regular graph with intersection array $\{22, 16, 5; 1, 2, 20\}$ does not exist. From now on we assume that $\Gamma$ is a distance-regular graph with intersection array $\{22, 16, 5; 1, 2, 20\}$. Eigenvalues of $\Gamma$ are $22, 7, -2$ and $-5$. Fix a vertex $\infty$ of $\Gamma$. Let $\Delta = \Gamma_1(\infty)$ denote the subgraph of $\Gamma$ induced by all vertices of $\Gamma$ adjacent to $\infty$. Then $\Delta$ is a 5-regular graph on 22 vertices. The following results give some combinatorial properties of the local graph $\Delta$ of $\Gamma$.

**Corollary 5.1.** *$\Delta$ does not contain a complete subgraph $K_i$ for all $i \geq 5$.*

*Proof.* By Proposition 2.7, the size of a clique in $\Gamma$ is at most 5. Thus the size of a clique in $\Delta$ is at most 4. □

**Lemma 5.2.** *Each vertex in $\Delta$ is on at least two subgraphs $K_3$'s of $\Delta$.*

*Proof.* Observe that $a_1 = 5$. By Lemma 3.1, the result follows. □

By Lemma 3.2, Corollary 5.1 and Lemma 5.2, there are 3 possibilities for the subgraph of $\Delta$ induced by a vertex $u$ and its neighbors as shown in Figure 5.2.

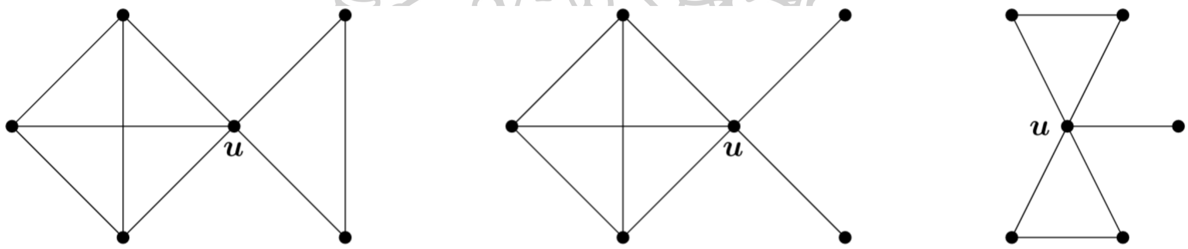


Figure 5.2: The 3 possibilities for the subgraph of $\Delta$ induced by a vertex $u$ and its neighbors.

**Lemma 5.3.** *$\Delta$ contains a complete subgraph $K_4$.*

*Proof.* Suppose not. Then the subgraph of $\Delta$ induced by a vertex in $\Delta$ and its neighbors must be isomorphic to the graph on the right in Figure 5.2. Thus each vertex in $\Delta$ is on exactly two $K_3$'s so $|\{(u, K_3)|K_3 \subseteq \Delta, u \in K_3\}| = 22 \times 2 = 44$. Since the number of vertices of $K_3$ is three, $3|44$, a contradiction. Thus $\Delta$ contains a complete subgraph $K_4$. □

Now we partition the vertex set of the local graph $\Delta$. For the rest of this chapter, fix a complete subgraph $K$ on four vertices of $\Delta$. Let $S = \Delta_1(K) = \{y \in \Delta - K | y \text{ is adjacent to some vertices in } K\}$ be the neighborhood of $K$ in $\Delta$ and define $R = \Delta - K - S$.

**Lemma 5.4.** *$K$ has size 4, $S$ has size 8, and $R$ has size 10.*

*Proof.* Clearly, $|K| = 4$. Let $u_1, u_2, u_3$ and $u_4$ denote the vertices in $K$. Since $\Delta$ is 5-regular, for each $1 \leq i \leq 4$ there exist two vertices in $S$ which are adjacent to $u_i$. If $u_i$ and $u_j$ have a common neighbor $s$ in $S$ for some $1 \leq i < j \leq 4$, then by Lemma 3.2, the vertex $s$ is adjacent to $u_l$ for all $1 \leq l \leq 4$ and hence $\{s, u_1, u_2, u_3, u_4\}$ induces a $K_5$ in $\Delta$ which contradicts Corollary 5.1. Thus $u_i$ and $u_j$ have no common neighbors in $S$ for all $1 \leq i < j \leq 4$. Therefore $|S| = 8$, and hence $|R| = |\Delta| - |K| - |S| = 22 - 4 - 8 = 10$. □

Let $u_1, u_2, u_3$ and $u_4$ denote the vertices of $K$. For $1 \leq i \leq 4$ let $s_{2i-1}$ and $s_{2i}$ denote the vertices of $S$ which are adjacent to $u_i$.

**Lemma 5.5.** *The only possible edges in $S$ are $s_{2i-1}s_{2i}$ for $1 \leq i \leq 4$. Moreover, the vertices $s_{2i-1}$ and $s_{2i}$ have no common neighbors in $R$.*

*Proof.* The result follows from Lemma 3.2. □

To further investigate the structure of $R$ we define an incidence geometry $G = (R, S)$ where elements of $R$ are regarded as points and elements of $S$ are regarded as lines, and a point $r \in R$ is on a line $s \in S$ if and only if the vertices $r$ and $s$ are adjacent in $\Gamma$.

**Lemma 5.6.** *$G$ is a partial linear space. Moreover each line in $G$ is incident with at least 3 points.*

*Proof.* Suppose two distinct points $r$ and $r'$ of $R$ are incident with two distinct lines $s$ and $s'$. Then the vertices $s, r, s'$ and $r'$ form a cycle in $\Delta$. By Lemma 3.2, the vertices $s$ and $s'$ are adjacent. Thus by Lemma 5.5 the vertices $s$ and $s'$ are adjacent to a common vertex $u$ in $K$. Now $u, s, r$ and $s'$ form a cycle in $\Delta$. By Lemma 3.2, the vertices $u$ and $r$ are adjacent, a contradiction. Thus every pair of distinct points lie on at most one common line.

By Lemma 5.5 and since $\Delta$ is 5-regular, it follows that each vertex of $S$ is adjacent to at least 3 vertices of $R$, that is, each line in $S$ is incident with at least 3 points in $R$. Therefore $G$ is a partial linear space. □

**Lemma 5.7.** *One of the following two conditions holds:*

*1). The number of edges in $S$ is 3. The number of edges in $R$ is 12. The number of edges between $S$ and $R$ is 26.*

*2). The number of edges in $S$ is 4. The number of edges in $R$ is 13. The number of edges between $S$ and $R$ is 24.*

*Proof.* First we will show that the subgraph induced by $S$ contains at least 3 edges.

Without loss of generality, we may assume that $s_7$ and $s_8$ are not adjacent. Then $s_7$ and $s_8$ are lines of size 4 in $G$. By Lemma 5.5, the lines $s_7$ and $s_8$ have no common points.

Suppose that $s_1$ is a line of size 4 in $G$. Then $s_1$ and $s_2$ are not adjacent and hence $s_2$ is also a line of size 4 in $G$. By Lemma 5.5, the lines $s_1$ and $s_2$ have no common points. Since every pair of distinct points lie on at most one common line and $|R| = 10$, the line $s_1$ is incident with one point of $s_7$, one point of $s_8$ and other two points not on $s_7$ or $s_8$. Similarly, the line $s_2$ is incident with one point of $s_7$, one point of $s_8$ and two points not on $s_1, s_7$ or $s_8$. Thus $G$ has more than 10 points, a contradiction. Therefore $s_1$ is a line of size 3 in $G$. Similarly, $s_i$ is a line of size 3 in $G$ for all $2 \leq i \leq 6$.

Thus $s_{2i-1}$ is adjacent to $s_{2i}$ for all $1 \leq i \leq 3$ and hence the subgraph induced by $S$ contains at least 3 edges.

If $S$ contains exactly 4 edges, then the number of edges between $S$ and $R$ is $3 \times 8 = 24$ and the number of edges in $R$ is $(5 \times 10 - 24)/2 = 13$. If $S$ contains exactly 3 edges, then the number of edges between $S$ and $R$ is $(3 \times 6) + (4 \times 2) = 26$ and the number of edges in $R$ is $(5 \times 10 - 26)/2 = 12$. □

**Lemma 5.8.** *Each vertex in $R$ has degree at least 2 in $R$. Moreover there are at least 4 vertices in $R$ with degree 2 in $R$.*

*Proof.* If a vertex $r$ in $R$ is adjacent to 5 vertices in $S$, then $r$ is adjacent to $s_{2i-1}$ and $s_{2i}$ for some $1 \leq i \leq 4$. The vertices $r, s_{2i-1}, u_i$ and $s_{2i}$ form a cycle in $\Delta$. By Lemma 3.2, the vertices $u_i$ and $r$ are adjacent, a contradiction. Thus each vertex in $R$ is adjacent to at most 4 vertices in $S$.

Suppose that there exists a vertex $r_1$ in $R$ such that the number of edges from $r_1$ to S is 4. By Lemma 3.2, we may assume that $r_1$ is adjacent to $s_1, s_3, s_5$ and $s_7$. By Lemma 5.2 applied to $r_1$, there exist $i, j \in \{1, 3, 5, 7\}$, $i \neq j$, such that $s_i$ and $s_j$ are adjacent which contradicts Lemma 5.5. Thus there are no vertices in $R$ which are adjacent to 4 vertices in $S$. That is each vertex in $R$ has degree at least 2 in $R$.

If there are at most 3 vertices in $R$ with degree 2 in $R$, then the number of edges between $R$ and $S$ is less than or equal to $(3 \times 3) + (7 \times 2) = 23$ which contradicts Lemma 5.7. Thus there are at least 4 vertices in $R$ with degree 2 in $R$. □

By Lemma 5.7 and Lemma 5.8, there are 8 possibilities for the degree sequence of $R$ as shown in Table 5.3.

| The number of vertices in the induced subgraph $R$ with degree $i$ | | | | $\|E(R)\|$ |
|---|---|---|---|---|
| $i = 2$ | $i = 3$ | $i = 4$ | $i = 5$ | |
| 4 | 6 | 0 | 0 | 13 |
| 5 | 4 | 1 | 0 | 13 |
| 6 | 3 | 0 | 1 | 13 |
| 6 | 2 | 2 | 0 | 13 |
| 6 | 4 | 0 | 0 | 12 |
| 7 | 2 | 1 | 0 | 12 |
| 8 | 0 | 2 | 0 | 12 |
| 8 | 1 | 0 | 1 | 12 |

Figure 5.3: The 8 possibilities for the degree sequence of $R$.

By Lemma 5.7, either $|E(R)| = 12$ or $|E(R)| = 13$. We now rule out both possibilities. We start with the latter.

**Lemma 5.9.** $|E(R)| \neq 13$.

*Proof.* Suppose that $|E(R)| = 13$. By Lemma 5.7, the subgraph induced by $S$ contains 4 edges and the number of edges between $S$ and $R$ is 24. Thus each vertex in $S$ is adjacent to 3 vertices in $R$. By Lemma 3.2 and Lemma 5.2, there are 8 distinct edges $e_1, e_2, \ldots, e_8$ in $R$ such that $s_i$ is adjacent to both ends of $e_i$ for $1 \leq i \leq 8$. Let $T = \{e_1, e_2, \ldots, e_8\}$.

Suppose that there exists a vertex $r \in R$ which has degree 5 in $R$. Let $r_1, r_2, r_3, r_4$ and $r_5$ denote the distinct neighbors of $r$ in $R$. Then for each $i \in \{1,2,3,4,5\}$, $rr_i \notin T$. Since $R$ has 13 edges, $E(R) - \{rr_1, rr_2, rr_3, rr_4, rr_5\} = T$. By Lemma 5.2 applied to $r$, we may assume that $r_1$ and $r_2$ are adjacent. Thus $e_i = r_1r_2$ for some $1 \leq i \leq 8$. So the vertices $s_i, r_1, r$ and $r_2$ form a cycle in $\Delta$ and hence $r$ is adjacent to $s_i$, a contradiction. Therefore each vertex in $R$ has degree at most 4 in $R$. By Lemma 5.8, each vertex in $R$ is adjacent to 1, 2 or 3 vertices in $S$.

Now suppose that $r$ is a vertex in $R$ with degree 3 in $R$. Let $N_R(r) = \{r_1, r_2, r_3\}$. Without loss of generality, we may assume that $N_S(r) = \{s_1, s_3\}$.

<u>Case 1</u> : $s_i$ and $r_j$ are not adjacent for all $i \in \{1,3\}$ and $j \in \{1,2,3\}$.

Then $r_j$ and $r_k$ are adjacent for all $1 \leq j < k \leq 3$ by Lemma 5.2 applied to $r$. By Lemma 3.2, the edges $rr_1, rr_2, rr_3, r_1r_2, r_1r_3, r_2r_3 \notin T$. Since $R$ contains 13 edges, $8 = |T| \leq |E(R) - \{rr_1, rr_2, rr_3, r_1r_2, r_1r_3, r_2r_3\}| = 7$, a contradiction. Thus Case 1 cannot occur.

<u>Case 2</u> : $s_1$ is adjacent to exactly one vertex in $\{r_1, r_2, r_3\}$.

Without loss of generality, we may assume that $s_1$ is adjacent to $r_3$. Then $s_1$ is not adjacent to $r_1$ and $r_2$. Since $s_1$ is adjacent to 3 vertices in $R$, there exists a vertex $r_4 \in R - \{r, r_1, r_2, r_3\}$ such that $r_4$ is adjacent to $s_1$. By Lemma 3.2, the vertex $s_2$ is not adjacent to $r_i$ for $1 \leq i \leq 4$. Since $s_2$ is adjacent to 3 vertices in $R$, there exist $r_5, r_6, r_7 \in R - \{r, r_1, r_2, r_3, r_4\}$ such that $r_5, r_6, r_7$ are adjacent to $s_2$. Since $R$ has 10 vertices, there exist $r_8, r_9 \in R - \{r, r_i | 1 \leq i \leq 7\}$. By Lemma 3.2, the vertex $r_4$ is not adjacent to $r_i$ for $1 \leq i \leq 7$. By Lemma 5.8, the vertex $r_4$ is adjacent to $r_8$ and $r_9$. By Lemma 3.2, the vertex $r_3$ is not adjacent to $r_i$ for $1 \leq i \leq 9$. Thus $r_3$ has degree 1 in $R$, a contradiction to Lemma 5.8. Hence Case

2 cannot occur.

Case 3 : $s_1$ is adjacent to exactly two vertices in $\{r_1, r_2, r_3\}$.

Without loss of generality, we may assume that $s_1$ is adjacent to $r_2$ and $r_3$. Then $s_1$ is not adjacent to $r_1$. By Lemma 3.2, $r_2$ is adjacent to $r_3$, and $s_3$ is not adjacent to $r_2$ and $r_3$. By Case 2 applied to $r$ and $s_3$, the vertex $s_3$ is not adjacent to $r_1$. By Lemma 3.2,the vertex $r_1$ is not adjacent to $s_2$ and $s_4$. So $r_1$ has at most two neighbors in $S$ by Lemma 5.5 that is $r_1$ has degree at least 3 in $R$. By Lemma 3.2, the vertex $r_1$ is not adjacent to $r_2$ and $r_3$. Then there exist $r_4, r_5 \in R - \{r, r_1, r_2, r_3\}$ such that $r_4, r_5$ are adjacent to $r_1$. Since each vertex in $R$ is adjacent to at least one vertex in $S$, we may assume that $r_1$ is adjacent to $s_5$. By Lemma 3.2, the vertex $s_3$ is not adjacent to $r_4$ and $r_5$. Since $s_3$ is adjacent to 3 vertices in $R$, there exist $r_6, r_7 \in R - \{r, r_1, r_2, r_3, r_4, r_5\}$ such that $r_6, r_7$ is adjacent to $s_3$. By Lemma 5.2 applied to $s_3$, the vertex $r_6$ is adjacent to $r_7$. By Lemma 3.2, $s_4$ is not adjacent to $r, r_1, r_2, r_3, r_6, r_7$, and $s_4$ is adjacent to at most one vertex in $\{r_4, r_5\}$. Since $s_4$ is adjacent to 3 vertices in $R$ and $|R| = 10$, we may assume that $s_4$ is adjacent to $r_4, r_8$ and $r_9$ where $\{r_8, r_9\} = R - \{r, r_1, r_2, \ldots, r_7\}$. Then $r_1$ and $r_8$ are not adjacent; otherwise $r_1, r_8, s_4$ and $r_4$ form a cycle in $\Delta$ and hence $r_1$ is adjacent to $s_4$, a contradiction. Similarly, the vertices $r_1$ and $r_9$ are not adjacent. By Lemma 3.2, the vertex $r_1$ is not adjacent to $r_6$ and $r_7$. Thus $r_1$ has degree 3 in $R$. By Lemma 3.2, we may assume that $r_1$ is adjacent to $s_7$. By Case 1 and Case 2 appiled to $r_1$ and $s_5$, we may assume that $s_5$ is adjacent to $r_4$ and $r_5$. Then $r_4$ and $r_5$ are adjacent by Lemma 3.2. Since $s_2$ is adjacent to 3 vertices in $R$ and by Lemma 3.2, the vertex $s_2$ is adjacent to one vertex in $\{r_4, r_5\}$, one vertex in $\{r_6, r_7\}$ and one vertex in $\{r_8, r_9\}$. Without loss of generality, we may assume that $s_2$ is adjacent to $r_6$ and $r_8$. Then $s_2$ and $r_4$ are not adjacent; otherwise $s_2, r_4, s_4$ and $r_8$ form a cycle in $\Delta$ and hence $s_2$ is adjacent to $s_4$, a contradiction. Thus $s_2$ is adjacent to $r_5$. The vertices $s_7$ and $r_4$ are not adjacent; otherwise the vertices $s_7, r_4, s_5$ and $r_1$ form a cycle in $\Delta$ and hence $s_5$ is adjacent to $s_7$, a contradiction. By Lemma 3.2, the vertex $r_4$ is not adjacent to $s_6$ and $s_8$. Thus $r_4$ has degree 3 in $R$. The vertex $r_4$ is not adjacent to $r_2$ and $r_3$; otherwise the vertices $r_4, r_i, r$ and

$r_1$ form a cycle in $\Delta$ where $i \in \{2,3\}$ and hence $r_4$ is adjcent to $r$, a contradiction. The vertices $r_4$ and $r_6$ are not adjcent; otherwise the vertices $r_4, r_6, s_3$ and $s_4$ form a cycle in $\Delta$ and hence $r_4$ is adjcent to $s_3$, a contradiction. Similarly, the vertex $r_4$ is not adjacent to $r_7$. Hence $r_4$ is adjacent to either $r_8$ or $r_9$. The vertices $r_4$ and $r_8$ are not adjacent; otherwise $r_4, r_8, s_2$ and $r_5$ form a cycle in $\Delta$ and hence $r_4$ is adjacent to $s_2$, a contradiction. It follows that $r_4$ is adjacent to $r_9$. By Case 2 appiled to $r_4$ and $s_4$, the vertex $s_4$ is adjacent to $r_5$. Hence $s_4$ has degree more than 5 in $\Delta$, a contradiction. Therefore Case 3 cannot occur.

By Case 1, Case 2 and Case 3, $|E(R)| \neq 13$. □

**Lemma 5.10.** $|E(R)| \neq 12$.

*Proof.* Suppose that $|E(R)| = 12$. Then the subgraph induced by $S$ contains 3 edges. Without loss of generality, we may assume that $s_{2i-1}$ and $s_{2i}$ are adjacent for $i \in \{1,2,3\}$ but $s_7$ and $s_8$ are not adjacent. By Lemma 5.7, the number of edges between $S$ and $R$ is 26. By Lemma 3.2 and Lemma 5.2, there are 10 distinct edges $e_1, e_2, \ldots, e_{10}$ in $R$ such that $s_i$ is adjacent to both ends of $e_i$ for $1 \leq i \leq 6$, $s_7$ is adjacent to both ends of $e_7$ and $e_8$ and $s_8$ is adjacent to both ends of $e_9$ and $e_{10}$. Let $T = \{e_1, e_2, \ldots, e_{10}\}$. By similar arguments as in Lemma 5.9, each vertex in $R$ has degree at most 4 in $R$.

Suppose that there exists a vertex $r$ in $R$ which has degree 4 in $R$. Let $r_1, r_2, r_3$ and $r_4$ denote distinct neighbors of $r$ in $R$. Since $|E(R) - T| = 2$, we may assume that $rr_1, rr_2 \in T$ and $r$ is adjacent to $s_7$. By Lemma 3.2, the vertex $r_1$ is adjacent to $r_2$. By construction, $r_1r_2 \notin T$. Since $rr_1$ and $rr_2$ are two edges with both ends adjacent to $s_7$, it follows that $rr_3, rr_4 \notin T$. Hence $13 = |T \cup \{r_1r_2, rr_3, rr_4\}| \leq |E(R)| = 12$, a contradiction. Thus there are no vertices in $R$ which has degree 4 in $R$.

By Table 5.3, there exist 6 vertices in $R$ with degree 2 in $R$, and 4 vertices in $R$ with degree 3 in $R$. By Lemma 5.6, each line in $G$ is incident with at least 3 points. Since $s_7$ and $s_8$ are not adjacent, $s_7$ and $s_8$ are lines of size 4 in $G$. By Lemma 5.5, the lines $s_7$ and $s_8$ have no common points. Let the point set of $G$

be $\{r_i | 1 \leq i \leq 10\}$ such that $r_3, r_4, r_5, r_6$ lie on $s_7$ and $r_7, r_8, r_9, r_{10}$ lie on $s_8$. Note that any line other than $s_7$ and $s_8$ must be incidence with either $r_1$ or $r_2$. If $r_1$ lies on exactly 2 lines, then $G$ has at most 7 lines, a contradiction. Since every vertex in $R$ is adjacent to 2 or 3 vertices in $S$, $r_1$ lies on 3 lines in $G$. Similarly, $r_2$ lies on 3 lines in $G$. The points $r_1$ and $r_2$ are not on the same line; otherwise $G$ has at most 7 lines, a contradiction. If there exist at least 3 points in $s_7$ each of which lies on exactly two lines, then $G$ has at most 7 lines, a contradiction. So there are 2 points on the line $s_7$ which lie on exactly two lines. Similarly, there are 2 points on the line $s_8$ which lie on exactly two lines. Without loss of generality, we may assume that each of $r_5, r_6, r_9$ and $r_{10}$ lies on exactly 2 lines and each of $r_3, r_4, r_7$ and $r_8$ lies on exactly 3 lines. Then there are 3 possibilities for the incidence geometry $G$ on 10 points and 8 lines satisfying these properties as shown in Figure 5.4.

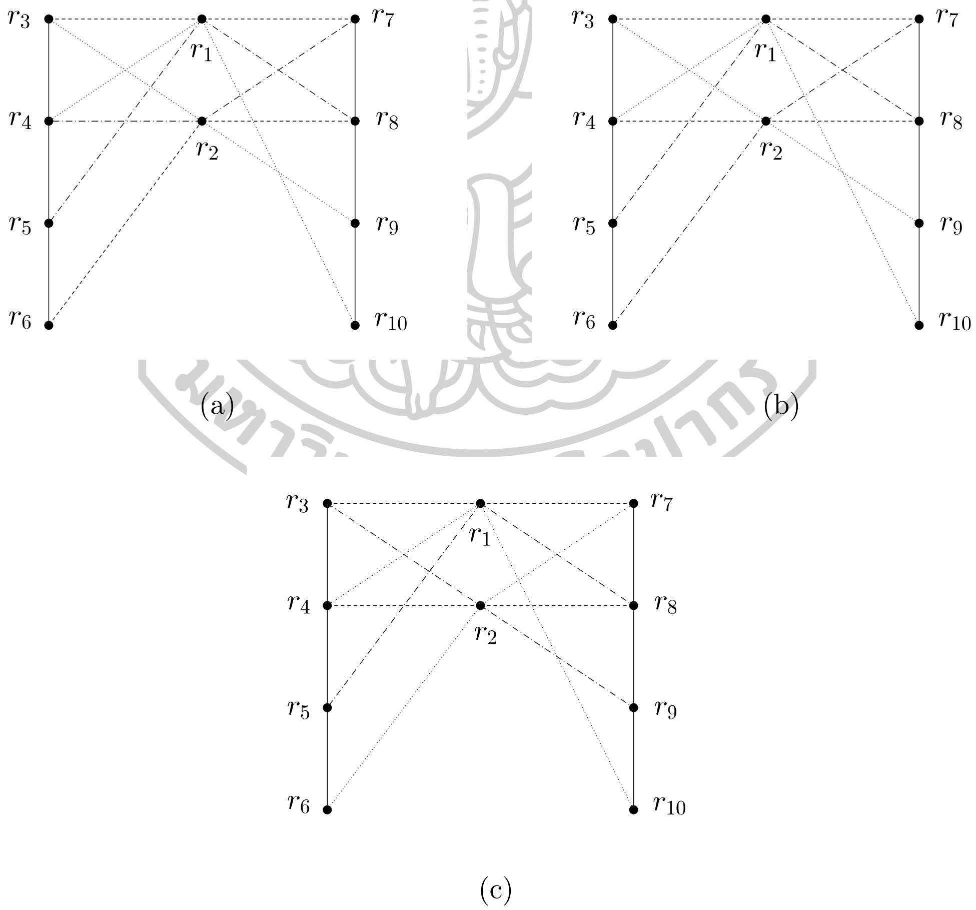


Figure 5.4: The 3 possibilities for the incidence geometry $G$.

In each figure a pair of solid lines represents $s_7$ and $s_8$, and each pair of nonsolid lines of same style represents $s_{2i-1}$ and $s_{2i}$ for $1 \leq i \leq 3$. If a point $r$ is on a line $s_{2i-1}$ and a point $r'$ is on a line $s_{2i}$, then the vertex $r$ is not adjacent to $r'$; otherwise $r, r', s_{2i}$ and $s_{2i-1}$ form a cycle in $\Delta$, and by Lemma 3.2, the point $r$ is on both $s_{2i-1}$ and $s_{2i}$, a contradiction. For convenience we call this the parallelity of lines.

In Figure 5.4a, by the parallelity of lines, the vertex $r_3$ is not adjacent to $r_4, r_6$, and the vertex $r_5$ is not adjacent to $r_4$. Suppose that the vertices $r_5$ and $r_6$ are adjacent. The vertices $r_3$ and $r_5$ are not adjacent; otherwise the vertices $r_3, r_5, r_6$ and $s_7$ form a cycle in $\Delta$, and by Lemma 3.2, the vertices $r_3$ and $r_6$ are adjacent, a contradiction. The vertices $r_4$ and $r_6$ are not adjacent; otherwise the vertices $r_4, r_6, r_5$ and $s_7$ form a cycle in $\Delta$, and by Lemma 3.2, the vertices $r_4$ and $r_5$ are adjacent, a contradiction. Thus the vertex $s_7$ is on exactly one subgraph $K_3$ of $\Delta$ which contradicts Lemma 5.2. Hence the vertices $r_5$ and $r_6$ are not adjacent. The vertex $r_6$ is not adjacent to $r_i$ for $i \in \{1, 2\}$; otherwise the vetices $r_6, r_i, s_j$ and $r_4$ form a cycle in $\Delta$ where $s_j$ is the line containing both $r_i$ and $r_4$, and by Lemma 3.2, the point $r_6$ is on $s_j$, a contradiction. Since $r_6$ has degree 3 in $R$, the vertex $r_6$ is adjacent to 2 vertices $u, v$ in $\{r_7, r_8, r_9, r_{10}\}$. Thus the vertices $r_6, u, s_8$ and $v$ form a cycle in $\Delta$, and by Lemma 3.2, the point $r_6$ is on $s_8$, a contradiction.

In Figure 5.4b, by the parallelity of lines, the vertex $r_3$ is not adjacent to $r_4$, and the vertex $r_5$ is not adjacent to $r_6$. Since $r_2$ has degree 2 in $R$, the vertex $r_2$ is adjacent to $r_6$ and $r_9$ by the parallelity of lines. The vertices $r_4$ and $r_6$ are not adjacent; otherwise the vertices $r_4, r_6, r_2$ and $s_j$ forms a cycle in $\Delta$ where $s_j$ is the line containing both $r_2$ and $r_4$, and by Lemma 3.2, the point $r_6$ is on $s_j$, a contradiction. Suppose that the vertices $r_3$ and $r_5$ are adjacent. The vertices $r_3$ and $r_6$ are not adjacent; otherwise the vertices $r_3, r_6, s_7$ and $r_5$ form a cycle in $\Delta$, and by Lemma 3.2, the vertices $r_5$ and $r_6$ are adjacent, a contradiction. The vertices $r_4$ and $r_5$ are not adjacent; otherwise the vertices $r_4, r_5, r_3$ and $s_7$ form a cycle in $\Delta$, and by Lemma 3.2, the vertices $r_3$ and $r_4$ are adjacent, a contradiction. Hence the vertex $s_7$ is on exactly one subgraph $K_3$ of $\Delta$ which contradicts Lemma

5.2. Thus the vertices $r_3$ and $r_5$ are not adjacent. The vertex $r_5$ is not adjacent to $r_i$ for $i \in \{1,2\}$; otherwise the vertices $r_5, r_i, s_j$ and $r_4$ form a cycle in $\Delta$ where $s_j$ is the line containing both $r_i$ and $r_4$, and by Lemma 3.2, the point $r_5$ is on $s_j$, a contradiction. Since $r_5$ has degree 3 in $R$, the vertex $r_5$ is adjacent to 2 vertices $u, v$ in $\{r_7, r_8, r_9, r_{10}\}$. Thus the vertices $r_5, u, s_8$ and $v$ form a cycle in $\Delta$, and by Lemma 3.2, the point $r_5$ is on $s_8$, a contradiction.

In Figure 5.4c, by the parallelity of lines, the vertex $r_7$ is not adjacent to $r_8, r_{10}$, and the vertex $r_9$ is not adjacent to $r_8$. Suppose that the vertices $r_9$ and $r_{10}$ are adjacent. The vertices $r_7$ and $r_9$ are not adjacent; otherwise the vertices $r_7, r_9, r_{10}$ and $s_8$ form a cycle in $\Delta$, and by Lemma 3.2, the vertices $r_7$ and $r_{10}$ are adjacent, a contradiction. The vertices $r_8$ and $r_{10}$ are not adjacent; otherwise the vertices $r_8, r_{10}, r_9$ and $s_8$ form a cycle in $\Delta$, and by Lemma 3.2, the vertices $r_8$ and $r_9$ are adjacent, a contradiction. Thus the vertex $s_8$ is on exactly one subgraph $K_3$ of $\Delta$ which contradicts Lemma 5.2. Hence the vertices $r_9$ and $r_{10}$ are not adjacent. The vertex $r_{10}$ is not adjacent to $r_i$ for $i \in \{1,2\}$; otherwise the vertices $r_{10}, r_i, s_j$ and $r_8$ form a cycle in $\Delta$ where $s_j$ is the line containing both $r_i$ and $r_8$, and by Lemma 3.2, the point $r_{10}$ is on $s_j$, a contradiction. Since $r_{10}$ has degree 3 in $R$, the vertex $r_6$ is adjacent to 2 vertices $u, v$ in $\{r_3, r_4, r_5, r_6\}$. Thus the vertices $r_{10}, u, s_7$ and $v$ form a cycle in $\Delta$, and by Lemma 3.2, the point $r_{10}$ is on $s_7$, a contradiction. Hence $|E(R)| \neq 12$. □

By Lemma 5.7, Lemma 5.9 and Lemma 5.10, we have our main result.

**Theorem 5.11.** *A distance-regular graph with intersection array* $\{22, 16, 5; 1, 2, 20\}$ *does not exist.*

# Chapter 6

# Conclusions

In this thesis, we study three intersection arrays from the list, $\{22, 16, 5; 1, 2, 20\}$, $\{27, 20, 10; 1, 2, 18\}$, and $\{36, 28, 4; 1, 2, 24\}$. These intersection arrays have $c_2 = 2$, which means that every two nonadjacent vertices have either 0 or 2 common neighbors. We give some combinatorial properties of the local graphs of distance-regular graphs. For a fixed vertex $x$ in a distance-regular graph, we give an upper bound of the number of triangles corresponding to $x$ in term of the intersection numbers $a_1$ and $b_0 = k$. We show that any two nonadjacent vertices in a local graph have at most one common neighbors. We prove that distance-regular graphs with given intersection arrays from the list do not exist by assuming such graphs exist. For the intersection array $\{27, 20, 10; 1, 2, 18\}$ we derive some combinatorial properties of its local graph to display a contradiction. For the intersection array $\{36, 28, 4; 1, 2, 24\}$ we show that its local graph is a disjoint union of completes $K_8$'s to get a contradiction. For the intersection array $\{22, 16, 5; 1, 2, 20\}$ we construct a partial linear space from its local graph to display the contradiction.

Potentially it might be possible to adapt our results to check feasibility of some other intersection arrays with $c_2 = 2$. However, more combinatorial properties of individual array need to be investigated.

# References


[1] Berlekamp, E. R., van Lint, J. H., and Seidel J. J. (1973). "A Strongly Regular Graph Derived from the Perfect Ternary Golay Code." In **A Survey of Combinatorial Theory, Symp. Colorado State Univ., 1971**, 25–30. Edited by Srivastava, Jagdish N., and others. Amsterdam: North Holland.

[2] Biggs, Norman L. (1974). **Algebraic Graph Theory**. Cambridge: Cambridge University Press.

[3] Bondy, John A., and Murty, Uppaluri S. R. (1982). **Graph Theory with Applications**. 5th ed. New York: Elsevire Science Publishing.

[4] Brouwer, Andries E. (2016). **Parameters of strongly regular graphs**. Accessed February 5. Available from https://www.win.tue.nl/ aeb/graphs/srg/srgtab.html.

[5] Brouwer, Andries E., Cohen, Arjeh M., and Neumaier Arnold. (1989). **Distance-regular graphs**, Berlin Heidelberg: Springer-Verlag.

[6] Brouwer, Andries E., and Pasechnik, Dmitrii V. (2012). "Two distance-regular graphs." **Journal of Algebraic Combinatorics** 36, 3 (November): 403–407.

[7] Coolsaet, Kris. (2005). "A distance-regular graph with intersection array (21,16,8;1,4,14) does not exist." **European Journal of Combinatorics** 26, 5 (July): 709–716.

[8] Coolsaet, Kris. (1995). "Local structure of graphs with $\lambda = \mu = 2, a_2 = 4$." **Combinatorica** 15, 4 (December): 481–487.

[9] Coolsaet, Kris, and Degraer, Jan. (2005). "A computer-assisted proof of the uniqueness of the Perkel graph." **Designs, Codes and Cryptography** 34, 2–3 (February): 155–171.

[10] Coolsaet, Kris, and Jurišić, Aleksandar. (2008). "Using equality in the Krein conditions to prove the nonexistence of certain distance-regular graphs." **Journal of Combinatorial Theory, Series A** 115, 6 (August): 1086–1095.

[11] van Dam, Edwin R., Koolen, Jack H., and Tanaka Hajime. (2016). "Distance-regular graphs." **The Electronic Journal of Combinatorics**, dynamic surveys 22 (April): 1–156.

[12] Gavrilyuk, Aleksandr L'vovich. (2011). "Distance-regular graphs with intersection arrays $\{55, 36, 11; 1, , 4, 45\}$ and $\{56, 36, 9; 1, , 3, 48\}$ do not exist." **Doklady Mathematics** 84, 18 (March): 444–446.

[13] Jurišić, Aleksandar, and Koolen, Jack H. (2000). "Nonexistence of some antipodal distance-regular graphs of diameter four." **European Journal of Combinatorics** 21, 1039–1046.

[14] Terwilliger, Paul. (1992). "The subconstituent algebra of an association scheme I." **Journal of Algebraic Combinatorics** 1, 20 (June): 363–388.

# Publications

[1] Brouwer, Andries E., Sumalroj, Supalak, and Worawannotai, Chalermpong. (2016). "The nonexistence of distance-regular graphs with intersection arrays $\{27, 20, 10; 1, 2, 18\}$ and $\{36, 28, 4; 1, 2, 24\}$." **Australasian Journal of Combinatorics** 66(2) (July): 330–332.

[2] Sumalroj, Supalak, and Worawannotai, Chalermpong. (2016). "The nonexistence of a distance-regular graph with intersection array $\{22, 16, 5; 1, 2, 20\}$." **The Electronic Journal of Combinatorics**, 23 (1) (February): #P1.32.

# Biography

| | |
|---|---|
| **Name** | Miss Supalak Sumalroj |
| **Date of Birth** | May 6, 1988 |
| **Address** | 26/3 Moo 5 Yaicha, Samphran, Nakhon Pathom, Thailand |
| **Institution Attended** | |
| 2009 | Bachelor of Science in Mathematics, Silpakorn University |
| 2012 | Master of Science in Mathematics, Silpakorn University |
| 2017 | Doctor of Philosophy in Mathematics, Silpakorn University |