EFFECT OF AN EDGE SUBDIVISION ON GAME DOMINATION NUMBERS

By

MISS Pakanun DOKYEESUN

A Thesis Submitted in Partial Fulfillment of the Requirements

for Master of Science (MATHEMATICS)

Department of MATHEMATICS

Graduate School, Silpakorn University

Academic Year 2018

ผลกระทบของการแบ่งเส้นย่อยต่อค่าโดมิเนชันเกม

โดย
นางสาวภคนันท์ ดอกยี่สุ่น

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรวิทยาศาสตรมหาบัณฑิต
สาขาวิชาคณิตศาสตร์ แผน ก แบบ ก 2 ระดับปริญญามหาบัณฑิต
ภาควิชาคณิตศาสตร์
บัณฑิตวิทยาลัย มหาวิทยาลัยศิลปากร
ปีการศึกษา 2561

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Title	Effect of an edge subdivision on game domination numbers
By	Pakanun DOKYEESUN
Field of Study	(MATHEMATICS)
Advisor	Chalermpong Worawannotai

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59305204 : Major (MATHEMATICS)

Keywords : Domination game, Game domination numbers, Edge subdivision

MISS PAKANUN DOKYEESUN : EFFECT OF AN EDGE SUBDIVISION ON GAME DOMINATION NUMBER. THESIS ADVISOR : ASSISTANT PROFESSOR CHALERMPONG WORAWANNOTAI, Ph.D.

Domination game is a game on a graph G played by two players called Dominator and Staller. They alternately choose a vertex of G such that each move dominates at least one new undominated vertex. The game ends when all vertices are dominated. Dominator's goal is to finish the game as soon as possible while Staller's goal is to prolong it as much as she can. When Dominator moves first, the game is called a Dominator-start game; when Staller moves first, it is called a Staller-start game. The game domination numbers $\gamma_g(G)$ and $\gamma_g'(G)$ are the sizes of the final dominating sets when both players play optimally for the Dominator-start game and for the Staller-start game, respectively. For a graph G and an edge e, we show that $0 \leq \gamma_g(G) - \gamma_g(G_e) \leq 2$ and $0 \leq \gamma_g'(G) - \gamma_g'(G_e) \leq 2$ where G_e is the graph resulting from subdividing the edge e in G. We also demonstrate that each difference satisfying the above bounds are realizable by infinitely many connected graphs.

ACKNOWLEDGEMENTS

First of all, I would like to thank you to my thesis advisor, Assistant Professor Dr.Chalermpong Worawannotai, for all his assistance. He introduced me to know domination game and provided suggestions throughout my time as his advisee. Without his advice, encouragement and supporting this thesis would hardly have been completed. I am really appreciated.

Second, I also deeply thank you to Assistant Professor Dr.Ratana Srithus and Dr.Nanthapath Trakultraipruk for agreeing to be valued members of my examination committee and providing interesting and valuable comments on this thesis.

Next, I am very grateful to the Development and Promotion of Science and Technology Talents Project (DPST) for the financial support throughout my education.

Moreover, I would like to thank my colleagues and all my friends for their friendship, emotional support and patience throughout these years.

Finally, I wish to express my love and my gratitude to my family for their unconditional support, encouragement and love.

Pakanun DOKYEESUN

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Chapter 1

Introduction

A *graph* G is a pair $(V(G), E(G))$ of sets, where each element of $E(G)$ is associated with an unordered pair of (not necessarily distinct) elements of $V(G)$. Each element of $V(G)$ is called a *vertex* and each element of $E(G)$ is called an *edge*. Two vertices are *adjacent* if there is an edge associated with the two vertices; in this case, we say that the edge *joins* the two vertices and the vertices are its *end vertices*. An edge is called a *loop* if it join a vertex to itself. *Multiple edges* are edges that join the same pair of vertices. In this thesis, all graphs we consider are *simple graphs*, graphs with no loops or multiple edges. In simple graphs, we may represent an edge with the set of its end vertices.

Domination on graphs is a widely studied topic. Many researchers are interested in this field because there are many useful applications. For example, in wifi router installation, we would like to install wifi routers in a building in such a way that we use as few routers as possible while all the area has wifi access. Here we can divide the building into smaller areas (e.g. rooms). Each area is represented by a vertex. Two vertices are joined by an edge if a wifi router in one area can cover the other. Then domination can be applied to solve this kind of resource allocation problem. See [5] and [6] for more information about domination.

We study a variation of domination called a domination game. It was introduced by Brešar, Klavžar, and Rall [3] in 2010, where the original idea was attributed to Henning in 2003. The domination game played on a graph G consists of two players, *Dominator* and *Staller*, who alternately choose a vertex from G. Unlike a typical game where players want to win the game, the goal of the domination game is not about winning or losing, but it is about doing the best toward each player's individual goal. We will discuss the formal definition of domination game in the next chapter.

Brešar, Klavžar and Rall [3] gave the bounds of the game domination number in terms of the domination number. They also proposed a useful theorem for comparing moves known as the Continuation Principle which was later proved by Kinnersley, West, and Zamani in [8]. Brešar, Klavžar and Rall [3] also gave a bound of the game domination numbers of a product of graphs. Afterwards, the effect of some graph operations on game domination numbers has been studied. In 2014, Brešar, Dorbec, Klavžar, and Košmrlj [1] showed the effect of edge-removal and a vertex-removal on domination games. In 2015, Dorbec, Košmrlj, and Renault [4] studied the effect of graph union on domination games. Recently, Onphaeng, Ruksasakcha and Worawannotai [11] studied the game domination numbers of a disjoint union of paths and cycles.

In this thesis, we study the effect of an edge subdivision on game domination numbers. In Chapter 2, we recall necessary definitions, well-known bounds and results of the game domination numbers that we will use. In Chapter 3, we show that subdividing an edge could increase the game domination numbers of a graph by at most 2. Finally, in Chapter 4 we show that each outcome in Chapter 3 is realizable by an infinite family of connected graphs.

Chapter 2
Preliminaries

In this chapter, we mention basic definitions, some useful lemmas and bounds of game domination numbers.

Definition 2.1. The *open neighborhood* $N_G(v)$ of vertex v in graph G is the set of vertices adjacent to v, and the *closed neighborhood* of v is $N_G[v] := N_G(v) \cup \{v\}$. The order of G is the number of vertices of G, denoted by $|G|$. The *degree* of the vertex v, written as $deg(v)$, is the number of edges which connect to v. A vertex v of a graph is said to be *isolated vertex* if $deg(v) = 0$. A vertex v of a graph is said to be a *pendant* or *leaf* if $deg(v) = 1$.

Definition 2.2. Let $S \subset V(G)$ be any subset of vertices of G. Then graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The subgraph H with vertex set $V(H) = S$ is called an *induced subgraph* of G, denoted by $G[S]$, if its edge set consists of all of the edges in $E(G)$ that have both end vertices in S.

Definition 2.3. Let G be a graph and let $e = \{u, v\}$ be an edge. To *subdivide* an edge $e = \{u, v\}$ of graph G is to add one new vertex w, and to replace e by two new edges $\{u, w\}$ and $\{w, v\}$. A *subdivision* G_e of G is the graph resulting from subdividing the edge e in G.

Example 2.4. In Figure 2.1, we show an example of a subdivision of a graph.

A *path* P_n is a graph of order n whose vertices can be listed in the order $v_1, v_2, \ldots, v_n$ such that the vertices v_i and v_{i+1} are adjacent for $i = 1, 2, \ldots, n-1$ and no other pairs of vertices are adjacent. The number of edges in a path is called the *path length*. A *connected graph* is a graph such that there is a path between every pair of vertices. A *cycle* C_n is a graph of order n whose vertices can

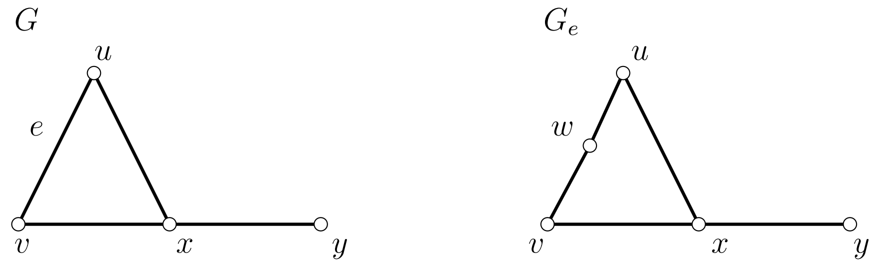

Figure 2.1: Graphs G and G_e

be listed in the order $v_1, v_2, \ldots, v_n$ such that the vertices v_i and v_{i+1} are adjacent for $i = 1, 2, \ldots, n-1$ and v_1 is adjacent to v_n and no other pairs of vertices are adjacent. A *complete graph* K_n is a graph on n vertices in which every pair of distinct vertices are adjacent. A *bipartite graph* is a graph such that its vertex set can be partitioned into two disjoint subsets X and Y such that every edge has the form $e = \{u, v\}$ where one vertex is in set X and the other one is in set Y. Then (X, Y) is called the *bipartition* of the graph. A *complete bipartite graph* $K_{m,n}$ is a bipartite graph with bipartition (X, Y) such that every vertex in X is adjacent to every vertex in Y and $|X| = m$, $|Y| = n$. A complete bipartite graph $K_{1,n}$ is called a *star*. Next, we describe some constructions of graphs from two given graphs G and H. The *union* $K = G \cup H$ has $V(K) = V(G) \cup V(H)$ and $E(K) = E(G) \cup E(H)$. In the case of $V(G) \cap V(H) = \varnothing$, we called K a *disjoint union* of G and H. The *(cartesian) product* $K = G \times H$ has $V(K) = V(G) \times V(H)$ and vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $\{v_1, v_2\} \in E(H)$ or $v_1 = v_2$ and $\{u_1, u_2\} \in E(G)$. In the case of $G = P_m$ and $H = P_n$, we call K a *grid graph* $P_m \times P_n$.

Example 2.5. In Figure 2.2, we show some examples of a path, a cycle, a complete graph, a complete bipartite graph, a star, and a grid graph, respectively.

Definition 2.6. A set S of vertices of a graph G is a *dominating set* if every vertex not in S is adjacent to some vertex in S. The *domination number* $\gamma(G)$ of G is the number of vertices in a minimum dominating set for G.

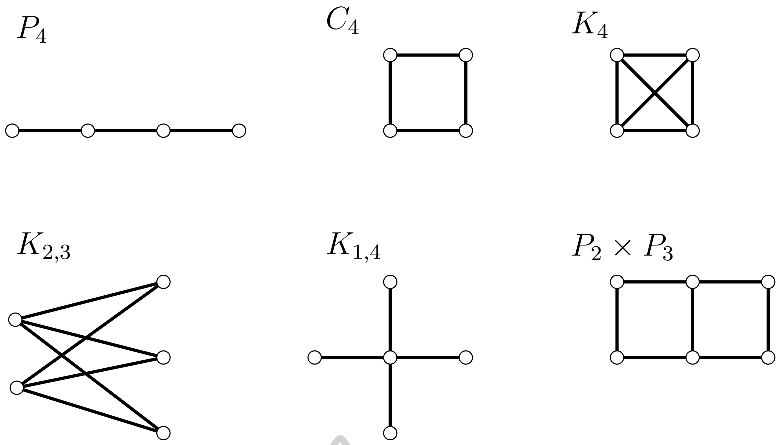

Figure 2.2: Graph P_4, C_4, K_4, $K_{2,3}$, $K_{1,4}$, and $P_2 \times P_3$ respectively

In others word, a set S of vertices of a graph G is a dominating set if every vertex in G is either an element in S or is adjacent to an element in S. If S is a dominating set of G, then we say S *dominates* G.

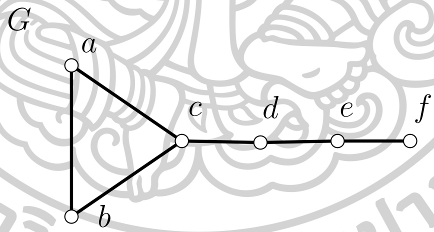

Figure 2.3: Graph G

Example 2.7. For the graph G shown in Figure 2.3, the set $\{c, e\}$ is a dominating set of G. Since any single vertex cannot dominate all vertices in this graph, we have that $\gamma(G) = 2$.

The domination game played on a graph G consists of two players, *Dominator* and *Staller*, who alternate taking turns choosing a vertex from G. Playing a vertex makes all adjacent vertices and itself dominated. A vertex is *valid* to choose if there are at least one undominated vertex in its closed neighborhood. The game ends when all vertices are dominated, i.e., the chosen vertices form a dominating set. Dominator's goal is to finish the game as soon as possible, and Staller's goal is to prolong it as much as possible. For this thesis, A *turn* is the phase of the game consisting of one player's action to select a vertex. When a player completes that action, we say that he *moves*; when that action is selecting a vertex v, we refer to v as the *move* and say that he *plays* v. The move v is *legal* if v is a valid vertex. When Dominator moves first, the game is a *Dominator-start* game; when Staller moves first, it is a *Staller-start* game.

Definition 2.8. The *game domination number* is the size of the final dominating set when both players play optimally, denoted by $\gamma_g(G)$ for the Dominator-start game and by $\gamma_g'(G)$ for the Staller-start game.

Example 2.9. For the graph G shown in Figure 2.3, we can calculate the game domination numbers as follows. In the Dominator-start game, notice that no matter how Dominator starts, he cannot force Staller to end the game in the next move. Therefore $\gamma_g(G) \geq 3$. If Dominator starts with vertex c, then the number of turns in the game is 3. Thus $\gamma_g(G) = 3$. Now consider the Staller-start game. No matter how Staller starts the game, Dominator can always force the game to end within 3 turns. Then $\gamma_g'(G) \leq 3$. If Staller starts at vertex d, then the number of turns in the game is 3. Thus $\gamma_g'(G) = 3$.

Suppose Dominator has a strategy that can end the game within k moves no matter how Staller plays. This Dominator's strategy might not be optimal so the game domination number is at most k. Similarly, suppose Staller has a strategy that can end the game using at least k moves no matter how Dominator plays. This Staller's strategy might not be optimal so the game domination number is at least k. Therefore, we have the following lemma.

Lemma 2.10. *Let G be a graph. Then the following statements hold.*

(i) *For Dominator-start game, if Dominator has a strategy that can end the game within k moves, then $\gamma_g(G) \leq k$.*

(ii) *For Staller-start game, if Dominator has a strategy that can end the game within k moves, then $\gamma'_g(G) \leq k$.*

(iii) *For Dominator-start game, if Staller has a strategy that can end the game with at least k moves, then $\gamma_g(G) \geq k$.*

(iv) *For Staller-start game, if Staller has a strategy that can end the game with at least k moves, then $\gamma'_g(G) \geq k$.*

Lemma 2.10 can help us to determine game domination numbers. We can prove an upper bound by providing a Dominator's strategy, and we can prove a lower bound by providing a Staller's strategy.

The relationship between the game domination number and the domination number of a graph is given below.

Theorem 2.11. ([3, Theorem 1]). For any graph G, we have $\gamma(G) \leq \gamma_g(G) \leq 2\gamma(G) - 1$.

The next theorem shows that the game domination numbers for Dominator-start game and Staller-start game of a graph differ by at most 1.

Theorem 2.12. ([3, Theorem 6], [8, Corollary 4.2]). For any graph G, we have $|\gamma_g(G) - \gamma'_g(G)| \leq 1$.

Definition 2.13. For a graph G and a subset A of V, a *partially-dominated graph* $G|A$ is the graph G whose vertices in A are declared dominated from the beginning. In particular, if $A = \{x\}$, we write $G|x$. The notion of game domination number extends naturally to partially-dominated graphs by considering the number of moves to dominate the remaining undominated vertices.

Example 2.14. In Figure 2.4, we show an example of the partially-dominated graph $G|\{e, f\}$ of graph G shown in Figure 2.3 where the vertex f and e are already dominated before the game starts. Then the game domination numbers $\gamma_g(G|\{e, f\}) = 1$ and $\gamma'_g(G|\{e, f\}) = 2$.

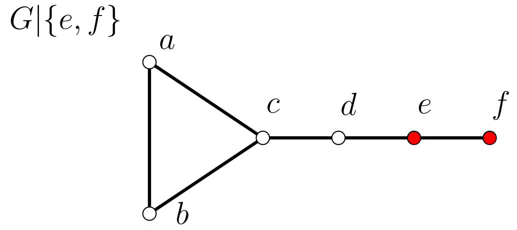

Figure 2.4: Graph $G|\{e, f\}$

Definition 2.15. The *residual graph* H of a partially-dominated graph G is obtained from G by removing all vertices which are invalid moves and removing all edges such that both end vertices are dominated.

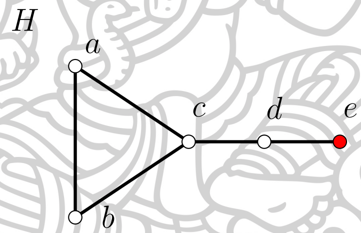

Figure 2.5: Graph H

Example 2.16. In Figure 2.5, we show an example of the residual graph H of graph $G|\{e, f\}$ as shown in Figure 2.4. Then the game domination numbers $\gamma_g(H) = 1$ and $\gamma'_g(H) = 2$.

Let H be the residual graph of a partially-dominated graph G. Since removing invalid vertices and edges joining dominated vertices does not affect the game, we have that $\gamma_g(G) = \gamma_g(H)$ and $\gamma'_g(G) = \gamma'_g(H)$.

Lemma 2.17. ([8, Lemma 2.1 - (Continuation Principle)]). Let G be a graph, and let $A, B \subseteq V(G)$. If $B \subseteq A$, then $\gamma_g(G|A) \leq \gamma_g(G|B)$ and $\gamma'_g(G|A) \leq \gamma'_g(G|B)$.

According to the Continuation Principle whenever u and v are valid moves and $N[u] \subseteq N[v]$, then Dominator prefers v to u. On the other hand, Staller prefers u to v.

Although paths and cycles are so simple, their game domination numbers are not easy to compute. The authors in [7, 10] determined the game domination numbers of paths and cycles as follows.

Theorem 2.18. ([7], [10, Theorem 2.2, 2.4]). For $n \geq 3$, we have

$$\gamma_g(C_n) = \gamma_g(P_n) = \begin{cases} \lceil \frac{n}{2} \rceil - 1 & \text{if} \quad n \equiv 3 \pmod 4, \\ \lceil \frac{n}{2} \rceil & \textit{otherwise}, \end{cases}$$

$$\gamma_g'(P_n) = \left\lceil \frac{n}{2} \right\rceil,$$

$$\gamma_g'(C_n) = \begin{cases} \lceil \frac{n-1}{2} \rceil - 1 & \text{if} \quad n \equiv 2 \pmod 4, \\ \lceil \frac{n-1}{2} \rceil & \textit{otherwise}. \end{cases}$$

We can add a new vertex to a graph without changing its domination number and game domination numbers as follows.

Lemma 2.19. ([2, Proposition 1.4]). Let G be a graph and $u \in V(G)$. Let G' be the graph obtained from G by adding a new vertex u' and joining it to every vertex in the closed neighborhood of u. Then $\gamma(G) = \gamma(G')$, $\gamma_g(G) = \gamma_g(G')$ and $\gamma_g'(G) = \gamma_g'(G')$.

Proof. First, we show that $\gamma(G) = \gamma(G')$. To prove $\gamma(G) \geq \gamma(G')$, it suffices to show that a dominating set of G is a dominating set of G'. Let S be a dominating set of G. Then there exists a vertex $w \in S \cap N_G[u]$. In G', vertex u' is adjacent to w. Thus S is a dominating set of G'.

To prove $\gamma(G) \leq \gamma(G')$, we show that there exists a dominating set of G of size $\gamma(G')$. Let S' be a dominating set of G' of size $\gamma(G')$. If u' is not in S', then S' is a dominating set of G. Suppose that $u' \in S'$. Since $N_{G'}[u'] = N_G[u] \cup \{u'\}$, we have $(S' \setminus \{u'\}) \cup \{u\}$ is a dominating set of G.

Next, consider the game domination numbers of G. Note that in G' the closed neighborhood of u and the closed neighborhood of u' are the same so u and u' are dominated simultaneously when a vertex in the common neighborhood is played. Thus $\gamma_g(G) = \gamma_g(G')$ and $\gamma'_g(G) = \gamma'_g(G')$. □

A general technique for proving results on the domination game is *imagination strategy* [3]. The main idea is that one of the players imagines another appropriate game being played at the same time; this game is called the *imagined game*, and it is usually played on a copy of the same graph. The basic procedure of his strategy in the real game is to copy each move of the opponent to the imagined game, respond optimally there, and finally copy back this move to the real game. If there is a move in the imagined game that cannot be copied to the real game, then he plays an arbitrary valid vertex in the real game. If there is a move in the real game that cannot be copied to the imagined game, then he plays an arbitrary valid vertex in the imagined game for his opponent. The overall aim is to ensure that the number of moves in the real game is bounded by the number of moves in the imagined game with some offset, which can be lead the bound on the corresponding game domination number of the graph.

Chapter 3

Effect of an edge subdivision on game domination numbers

In this chapter, we show that subdividing an edge of a graph could increase the game domination numbers by at most 2. First, we show that subdividing an edge of a graph will not decrease the game domination numbers.

Lemma 3.1. *For a graph G and an edge e, we have $\gamma_g(G) \leq \gamma_g(G_e)$ and $\gamma_g'(G) \leq \gamma_g'(G_e)$.*

Proof. Let G be a graph and let $e = \{u, v\}$ be an edge. Let w represent the new vertex in G_e that is adjacent to u and v.

First, consider the Dominator-start game. We will show that $\gamma_g(G) \leq \gamma_g(G_e)$. It suffices to show that Dominator has a strategy on G such that at most $\gamma_g(G_e)$ moves will be played. His strategy is to play on graph G as follows. Dominator imagines another game being played on G_e. In parallel to the real game, he copies every move of Staller into this game. Dominator plays optimally on G_e and copies the moves back to G if possible; otherwise Dominator plays an arbitrary valid vertex in G. He can always copy each move of Staller into G_e because at any moment if a vertex in G is undominated, then the corresponding vertex in G_e is undominated.

If all first $\gamma_g(G)$ moves are legal in the both games, then after $\gamma_g(G)$ moves all vertices of G are dominated and all vertices of G_e except maybe w are dominated. Thus Dominator ensures that G is dominated no later than G_e. Since the number of turns for G is at most $\gamma_g(G_e)$, we have $\gamma_g(G) \leq \gamma_g(G_e)$ by Lemma 2.10.

Now suppose that at some point during the first $\gamma_g(G)$ moves, a move of Dominator on G_e is not legal on G. Then Dominator can choose a new legal vertex on G. Thus the set of dominated vertices on G is a superset of the set of dominated vertices on G_e excluding w. Hence Dominator ensures that G is dominated no later than G_e. Since the number of turns for G_e is at most $\gamma_g(G_e)$, we have $\gamma_g(G) \leq \gamma_g(G_e)$ by Lemma 2.10.

Next, we show that $\gamma'_g(G) \leq \gamma'_g(G_e)$. For Staller-start game, Dominator uses the same strategy on G as above. Thus Dominator ensures that G is dominated no later than G_e. Since the number of turns for G_e is at most $\gamma'_g(G_e)$, we have $\gamma'_g(G) \leq \gamma'_g(G_e)$ by Lemma 2.10. $\square$

Theorem 3.2. *For a graph G and an edge e, we have $0 \leq \gamma_g(G_e) - \gamma_g(G) \leq 2$.*

Proof. Let G be a graph and let $e = \{u, v\}$ be an edge. Let w represent the new vertex in G_e that is adjacent to u and v. By Lemma 3.1, we have that $\gamma_g(G) \leq \gamma_g(G_e)$.

To prove the bound $\gamma_g(G_e) \leq \gamma_g(G)+2$, it suffices to show that Dominator has a strategy on G_e such that at most $\gamma_g(G) + 2$ moves will be played. His strategy is to play on graph G_e as follows. Dominator imagines another game being played on G. In parallel to the real game, he copies every move of Staller into this game which are legal; otherwise Dominator imagines that Staller plays an arbitrary valid vertex in G. Dominator plays optimally on G and copies the moves back to G_e. Dominator can always copy his move into G_e because at any moment if a vertex in G is undominated, then the corresponding vertex in G_e is undominated. Consider the following possibilities during the first $\gamma_g(G)$ moves.

Case 1 No one plays on u, v or w in the both games. Then after $\gamma_g(G)$ moves, all vertices of G are dominated and all of vertices of G_e except w are dominated. Thus the number of turns for G_e is $\gamma_g(G) + 1$.

Case 2 One of the players plays vertex u or v. Without loss of generality, assume vertex u is played.

Case 2.1 This move is legal in the other game and all the remaining moves are legal in both games. Then after $\gamma_g(G)$ moves all vertices of G_e except maybe v are dominated. Thus the number of turns for G_e is at most $\gamma_g(G) + 1$.

Case 2.2 This move is legal in the other game and some moves are not legal in G. That is Staller plays the k^{th} move which only v is dominated in G_e and this is not legal in G. Then the set of dominated vertices after $k - 1$ moves are played in G is equal to the set of dominated vertices after k moves are played in G_e without w. Let D be the set of dominated vertices after $k-1$ moves are played in G. Then the residual graphs of $G|D$ and of $G_e|(D \cup \{w\})$ are the same. Since Dominator plays optimally on the imagination game on G, we have

$$(k-1) + \gamma_g'(G|D) \leq \gamma_g(G). \tag{3.1}$$

Since Staller plays optimally on the real game on G_e, we have

$$\begin{aligned}
\gamma_g(G_e) &\leq k + \gamma_g(G_e|(D \cup \{w\})) \\
&= k + \gamma_g(G|D) && \text{(since the residual graphs of } G|D \\
& && \text{and of } G_e|(D \cup \{w\}) \text{ are the same.)} \\
&\leq k + (1 + \gamma_g'(G|D)) && \text{(by Theorem 2.12)} \\
&= (k-1) + \gamma_g'(G|D) + 2 \\
&\leq \gamma_g(G) + 2. && \text{(by (3.1))}
\end{aligned}$$

Case 2.3 This move is not legal in the other game. That is Staller plays the k^{th} move on vertex u in G_e but this move is not legal in G. Then vertex w is the only new vertex dominated by this move. Then the set of dominated vertices after $k-1$ moves are played in G is equal to the set of dominated vertices after k moves are played in G_e without w. By the same argument as Case 2.2, we have $\gamma_g(G_e) \leq \gamma_g(G) + 2$.

Case 3 One of the players plays vertex w.

That is Staller plays the k^{th} move on vertex w in G_e. This move is not legal in G. Then the set of dominated vertices after $k-1$ moves are played in G is a subset of the set of dominated vertices after k moves are played in G_e. Let D be the set of dominated vertices after $k-1$ moves are played in G. Thus the set of dominated vertices after k moves are played in G_e is $D \cup \{u, v, w\}$. Then the residual graph of $G_e|(D \cup \{u, v, w\})$ and the residual graph of $G|(D \cup \{u, v\})$ are the same. Since Dominator plays optimally on the imagination game on G, we have

$$(k-1) + \gamma_g'(G|D) \leq \gamma_g(G). \tag{3.2}$$

Since Staller plays optimally on the real game on G_e, we have

$$
\begin{aligned}
\gamma_g(G_e) &\le k + \gamma_g(G_e|(D \cup \{u,v,w\})) \\
&= k + \gamma_g(G|(D \cup \{u,v\})) && \text{(since the residual graphs of } G|(D \cup \{u,v,w\}) \\
& && \text{and of } G_e|(D \cup \{u,v\}) \text{ are the same.)} \\
&\le k + (1 + \gamma_g'(G|(D \cup \{u,v\}))) && \text{(by Theorem 2.12)} \\
&\le k + (1 + \gamma_g'(G|D)) && \text{(by the Continuation Principle)} \\
&= (k-1) + \gamma_g'(G|D) + 2 \\
&\le \gamma_g(G) + 2. && \text{(by (3.2))}
\end{aligned}
$$

From the above cases, Dominator has a strategy on G_e such that at most $\gamma_g(G) + 2$ moves will be played. Therefore, we have $\gamma_g(G_e) \le \gamma_g(G) + 2$.

□

Theorem 3.3. *For a graph G and an edge e, we have $0 \le \gamma_g'(G_e) - \gamma_g'(G) \le 2$.*

Proof. Let G be a graph and let $e = \{u, v\}$ be an edge. Let w represent the new vertex in G_e that is adjacent to u and v. By Lemma 3.1, we have that $\gamma_g'(G) \le \gamma_g'(G_e)$.

To prove the bound $\gamma_g'(G_e) \le \gamma_g'(G) + 2$, we consider an optimal first move of Staller on G_e in Staller-start game.

Case 1 w is an optimal first move of Staller on G_e. Since $G \cap N_{G_e}[w] = \{u, v\}$, the residual graph of $G_e|N_{G_e}[w]$ and the residual graph of $G|\{u,v\}$ are the same. Therefore,

$$
\begin{aligned}
\gamma_g'(G_e) &= 1 + \gamma_g(G_e|N_{G_e}[w]) && (w \text{ is an optimal first move of Staller}) \\
&= 1 + \gamma_g(G|\{u,v\}) \\
&\le 1 + \gamma_g(G) && \text{(by the Continuation Principle)} \\
&= 1 + (1 + \gamma_g'(G)) && \text{(by Theorem 2.12)} \\
&= \gamma_g'(G) + 2.
\end{aligned}
$$

Case 2 $x \in V(G_e) \setminus \{w\}$ is an optimal first move of Staller on G_e. Since $G \cap N_{G_e}[w] = \{u, v\}$, the residual graph of $G_e|(N_{G_e}[x] \cup N_{G_e}[w])$ and the residual graph of $G|(N_G[x] \cup \{u, v\})$ are the same. Therefore,

$$\begin{aligned}
\gamma_g'(G_e) &= 1 + \gamma_g(G_e|N_{G_e}[x]) && (x \text{ is an optimal first move of Staller}) \\
&\leq 2 + \gamma_g'(G_e|(N_{G_e}[x] \cup N_{G_e}[w])) && (w \text{ may not be an optimal} \\
& && \text{first move of Dominator}) \\
&= 2 + \gamma_g'(G|(N_G[x] \cup \{u, v\})) \\
&\leq 2 + \gamma_g'(G). && (\text{by the Continuation Principle})
\end{aligned}$$

From above cases, we have $\gamma_g'(G_e) \leq \gamma_g'(G) + 2$. $\square$

Chapter 4

Realization

We begin this chapter by showing the effect of an edge subdivision on game domination numbers of paths and cycles resulting from Theorem 2.18.

Corollary 4.1. *For $n \geq 2$, the graph obtained from a path P_n by subdividing one of its edges is a path P_{n+1}. Moreover,*

$$\gamma_g(P_{n+1}) - \gamma_g(P_n) = \begin{cases} 1 & \text{if} \quad n \equiv 0 \quad \text{or} \quad n \equiv 3 \pmod 4; \\ 0 & \text{otherwise.} \end{cases}$$

$$\gamma'_g(P_{n+1}) - \gamma'_g(P_n) = \begin{cases} 1 & \text{if} \quad n \text{ is even}; \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 4.2. *For $n \geq 3$, the graph obtained from a cycle C_n by subdividing one of its edges is a cycle C_{n+1}. Moreover,*

$$\gamma_g(C_{n+1}) - \gamma_g(C_n) = \begin{cases} 1 & \text{if} \quad n \equiv 3 \pmod 4; \\ 0 & \text{otherwise.} \end{cases}$$

$$\gamma'_g(C_{n+1}) - \gamma'_g(C_n) = \begin{cases} 1 & \text{if} \quad n \equiv 0 \quad \text{or} \quad n \equiv 3 \pmod 4; \\ 0 & \text{otherwise.} \end{cases}$$

In this chapter, we demonstrate that all differences satisfying the bounds in Theorem 3.2 and Theorem 3.3 are realizable by some infinite families of connected graphs. In particular, we show that for any positive integers x and y such that $0 \leq y - x \leq 2$ there exists a connected graph G and an edge e with $(\gamma_g(G), \gamma_g(G_e)) = (x, y)$ except for $(1, 3)$, $(2, 2)$ and $(2, 4)$. We will also show the analogous result for Staller-start game.

4.1 $\gamma_g(G_e) - \gamma_g(G) = 0$

Proposition 4.3. *For any positive integer $n \neq 2$, there exists a connected graph G with an edge e such that $\gamma_g(G) = n = \gamma_g(G_e)$.*

Proof. Let n be a positive integer such that $n \neq 2$. For $n \in \{1, 3\}$, let $G = P_{2n}$. Then for any edge e of G, we have $G_e = P_{2n+1}$. Note that $2n \equiv 2 \pmod 4$ and $2n + 1 \equiv 3 \pmod 4$. By Theorem 2.18, we have $\gamma_g(P_{2n}) = n = \gamma_g(P_{2n+1})$.

For $n \geq 4$, let $k = n - 4 \geq 0$. We construct the family of graphs $G_0, G_1, G_2, \ldots$ as follows. Let G_0 be the graph obtained by joining a vertex u of a cycle C_4 to an end vertex v of a path P_4 (see Figure 4.1).

For $k \geq 1$, let G_k be the graph obtained from G_0 by identifying an end vertex of each of k copies of P_3 with u. Let $e = \{u, v\}$ (see Figure 4.2).

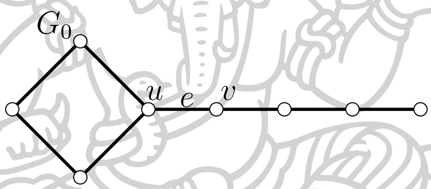

Figure 4.1: Graph G_0

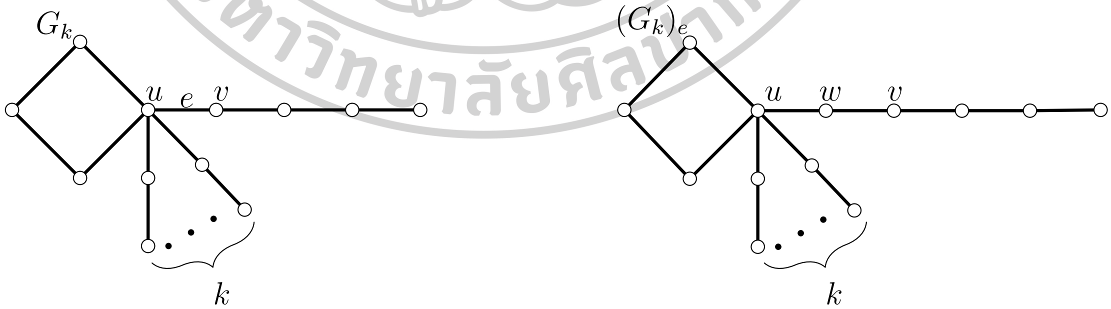

Figure 4.2: Graph G_k and Subdivision $(G_k)_e$

Let $G = G_k$. We claim that $\gamma_g(G) = \gamma_g(G_e) = k + 4 = n$. First, we prove that $\gamma_g(G) = k+4$. Note that if Dominator plays his first move on u, then there remain

3 undominated vertices in the path P_4 and $k+1$ isolated undominated vertices. Aside from u, at most 2 moves will be played to dominate the path P_4 and $k+1$ moves will be played to dominate the $k+1$ isolated vertices. Hence, at most $k+4$ moves are required to dominate G. By Lemma 2.10, we have that $\gamma_g(G) \leq k+4$. Next, we present a strategy for Staller which ensures that at least $k+4$ moves are needed to end the game in G. Note that $\gamma_g(G_0) = 4$ and $\gamma_g'(G_0|u) = 4$. Then no matter how Dominator starts the game in G, at least 4 moves will be played to dominate the subgraph G_0, and at least k moves will be played to dominate the k attached paths. By Lemma 2.10, we conclude that $\gamma_g(G) \geq k+4$. The above strategies can be used to prove that $\gamma_g(G_e) = k+4 = n$. In particular, an optimal first move of Dominator is to play on u. □

4.2 $\gamma_g(G_e) - \gamma_g(G) = 1$

Proposition 4.4. *For any positive integer n, there exists a connected graph G with an edge e such that $\gamma_g(G) = n$ and $\gamma_g(G_e) = n+1$.*

Proof. Let n be a positive integer. We divide n into two cases.

Case 1 n is odd. Let $G = P_{2n+1}$. Then for any edge e of G, we have $G_e = P_{2n+2}$. Note that $2n+1 \equiv 3 \pmod 4$ and $2n+2 \equiv 0 \pmod 4$.
By Theorem 2.18,

$$\gamma_g(P_{2n+1}) = \left\lceil \frac{2n+1}{2} \right\rceil - 1 = n,$$
$$\gamma_g(P_{2n+2}) = \left\lceil \frac{2n+2}{2} \right\rceil = n+1.$$

Case 2 n is even. Let $G = P_{2n}$. Then for any edge e of G, we have $G_e = P_{2n+1}$. Note that $2n \equiv 0 \pmod 4$ and $2n+1 \equiv 1 \pmod 4$.
By Theorem 2.18,

$$\gamma_g(P_{2n}) = \left\lceil \frac{2n}{2} \right\rceil = n,$$
$$\gamma_g(P_{2n+1}) = \left\lceil \frac{2n+1}{2} \right\rceil = n+1.$$

□

4.3 $\gamma_g(G_e) - \gamma_g(G) = 2$

We will consider the families of graphs from [1, p.5]. Two graphs that frequently appear in the constructions are a cycle C_6 and the graph Z from Figure 4.3. Recall that $\gamma_g(C_6) = 3 = \gamma_g(C_6|x)$ and $\gamma_g'(C_6) = 2 = \gamma_g'(C_6|x)$, where x is an arbitrary vertex of C_6. In addition, $\gamma_g(Z) = 4 = \gamma_g(Z|z)$ and $\gamma_g'(Z) = 3 = \gamma_g'(Z|z)$, where z is the vertex of Z as shown in Figure 4.3.

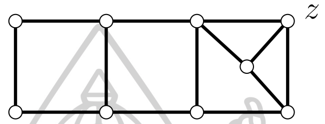

Figure 4.3: Graph Z

Proposition 4.5. *For any positive integer $n \geq 3$, there exists a connected graph G with an edge e such that $\gamma_g(G) = n$ and $\gamma_g(G_e) = n + 2$.*

Proof. We present two infinite families of connected graphs U_k and V_k which realize odd and even n, respectively. These families of graphs were introduced in [1, p.5]. Let B be the graph obtained by adding an edge to a star $K_{1,4}$ and denote its central vertex by u. Let v be an end vertex of the new edge (see Figure 4.4).

Case 1 n is odd. We construct the family of graphs $U_0, U_1, U_2, \ldots$ as follows. Let U_0 be the graph obtained from the disjoint union of cycle C_6 and the graph B by connecting an arbitrary vertex x of C_6 to vertex u of B. For $k \geq 1$, let U_k be the graph obtained from U_0 by identifying the left end vertices of k copies of P_6 with u. Let $e = \{u, v\}$ (see Figure 4.5).

Let $k = \frac{n-3}{2} \geq 0$ and $G = U_k$. We claim that $\gamma_g(G) = 2k + 3 = n$ and $\gamma_g(G_e) = 2k + 5 = n + 2$. By Theorem 3.2, it suffices to show that $\gamma_g(G) \leq 2k + 3$ and $\gamma_g(G_e) \geq 2k + 5$.

For the first inequality we present a strategy for Dominator that guarantees at most $2k + 3$ moves are played on G. Dominator starts by playing u. Then the undominated vertices form a graph with $k + 1$ components. Dominator responds

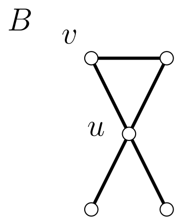

Figure 4.4: Graph B

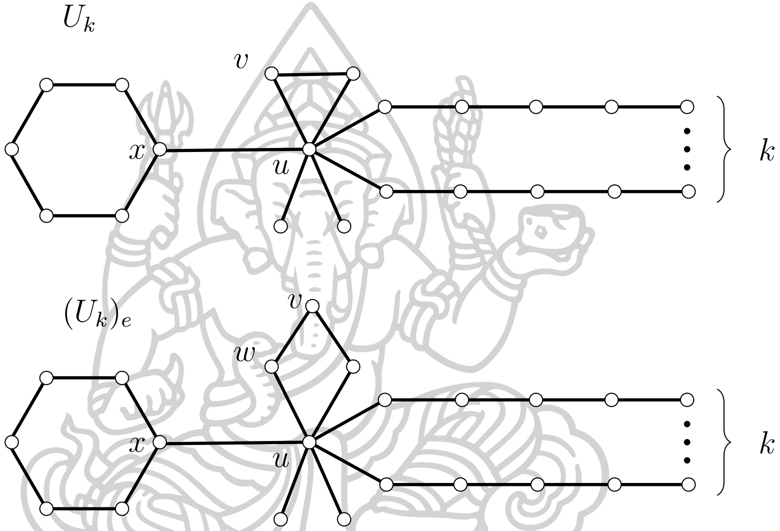

Figure 4.5: Graph U_k and Subdivision $(U_k)_e$

to a Staller's move by playing in the same component and ensuring two moves are played in each of the components. Thus at most $2k+3$ moves are required to dominate G. By Lemma 2.10, we have that $\gamma_g(G) \leq 2k+3$.

For the second inequality we present a strategy for Staller that ensures at least $2k+5$ moves are played on G_e. Staller's strategy is, if possible, not to be the first to play in the 6-cycle. Note that at least 2 moves will be played in each of the k attached paths, and at least 2 moves will be played in B_e. If exactly $2k+2$ moves are played before a move in the 6-cycle is played, then Dominator is the first to play in the 6-cycle, yielding 3 additional moves. Otherwise, at least $2k+3$ moves are played elsewhere, and 2 additional moves are played in the 6-cycle. Thus at least $2k+5$ moves are required to dominate G_e. By Lemma 2.10, $\gamma_g(G_e) \geq 2k+5$. This concludes the proof for the case when n is odd.

Case 2 n is even. We construct an infinite family of graphs $V_0, V_1, V_2, \ldots$ as follows. Let V_0 be the graph obtained from the disjoint union of the graph Z in Figure 4.3 and the graph B by connecting vertex z of the graph Z to vertex u of the graph B. For $k \geq 1$, let V_k be the graph obtained from V_0 by identifying the left end vertices of k copies of P_6 with u. Let $e = \{u, v\}$ (see Figure 4.6). In other words, V_k is obtained from U_k by replacing the 6-cycle with the graph Z.

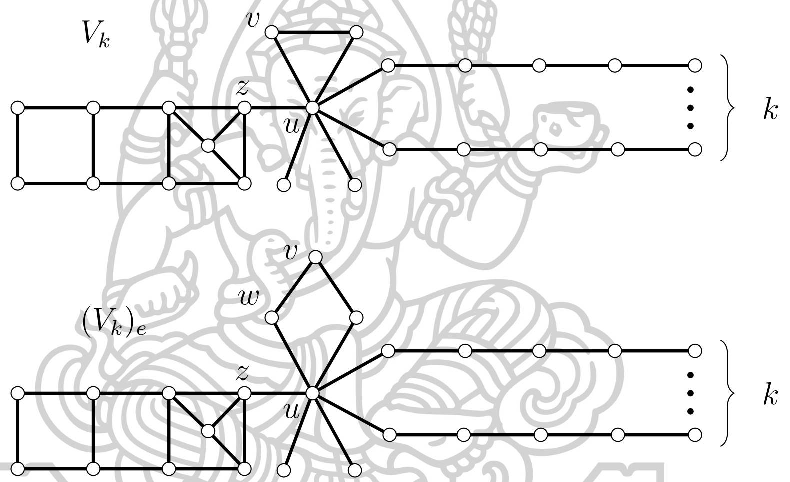

Figure 4.6: Graph V_k and subdivision $(V_k)_e$

Let $k = \frac{n-4}{2} \geq 0$ and $G = V_k$. We claim that $\gamma_g(G) = 2k+4 = n$ and $\gamma_g(G_e) = 2k+6 = n+2$. By Theorem 3.2, it suffices to show that $\gamma_g(G) \leq 2k+4$ and $\gamma_g(G_e) \geq 2k+6$.

For the first inequality we present a strategy for Dominator that guarantees at most $2k+4$ moves are played on G. Dominator starts by playing u. Then the undominated vertices form a graph with $k+1$ components. Dominator responds to a Staller's move by playing in the same component and ensures that each component is dominated using 2 moves except for the component corresponding to Z which he ensures that at most 3 moves are played. Thus at most $2k+4$ moves will are required to dominate G. By Lemma 2.10, we have that $\gamma_g(G) \leq 2k+4$.

For the second inequality we present a strategy for Staller that ensures at least $2k+6$ moves are played on G_e. Staller's strategy is, if possible, not to be the first to play in the subgraph Z. Note that at least 2 moves will be played in each of the k attached paths, and 2 additional moves will be played in B_e. If exactly $2k+2$ moves are played before a move in the subgraph Z is played, then Dominator is the first to play in the subgraph Z, yielding 4 additional moves. Otherwise, at least $2k+3$ moves are played elsewhere, and 3 additional moves will be played in the subgraph Z. Thus at least $2k+6$ moves will be played in G_e. By Lemma 2.10, we have that $\gamma_g(G_e) \geq 2k+6$. This concludes the proof for the case when n is even.

$\square$

Now, we show that when $n = 1$ or $n = 2$, there exists no graph G and an edge e such that $\gamma_g(G) = n$ and $\gamma_g(G_e) = n+2$.

Proposition 4.6. *Let G be a graph. If $\gamma_g(G) = 1$, then $\gamma_g(G_e) \leq 2$ for every edge e of G.*

Proof. Let $e = \{u, v\}$ be an edge of G. Let w represent the new vertex in G_e that is adjacent to u and v. Assume that $\gamma_g(G) = 1$. Then G has a universal vertex (a vertex that is adjacent to all other vertices). We divide our argument into cases according to the universal vertex.

Case 1 u or v is a universal vertex. Without loss of generality, assume that u is a universal vertex. In G_e, u is adjacent to all vertices except v. Dominator can play u and dominate all but vertex v. Hence, Staller has to dominate v in any legal move.

Case 2 u and v are not the universal vertices. Then there exists a universal vertex $x \in V(G) \setminus \{u, v\}$. In G_e, x is adjacent to all vertices except w. Dominator can play x and in this way dominate all but vertex w. Hence, Staller has to dominate w in any legal move.

In both cases, Dominator ensures that at most 2 moves are required to dominate G_e. Thus, $\gamma_g(G_e) \leq 2$.

□

Proposition 4.7. *Let G be a graph. If $\gamma_g(G) = 2$, then $\gamma_g(G_e) = 3$ for every edge e of G.*

Proof. Let $e = \{u, v\}$ be an edge of G. Let w represent the new vertex in G_e that is adjacent to u and v. Assume that $\gamma_g(G) = 2$. Let x be an optimal start vertex for Dominator in G. Since $\gamma_g(G) = 2$, the residual graph of $G|N_G[x]$ is a complete graph (here we join all pairs of dominated vertices). To prove $\gamma_g(G_e) \leq 3$, it suffices to show that Dominator has a strategy on G_e such that at most 3 moves will be played. Dominator starts by playing x.

Case 1 $x \in \{u, v\}$. Without loss of generality, assume that $x = u$. Then w are dominated by x but v is not dominated. Since $G|N_G[x]$ is complete, the residual graph of $G_e|N_{G_e}[x]$ is the union of a complete graph and a vertex. Thus at most 2 moves are played in $G_e|N_{G_e}[x]$.

Case 2 $x \notin \{u, v\}$. Then w is an undominated vertex. Thus the residual graph of $G_e|N_{G_e}[x]$ is the graph obtained from the residual graph of $G|N_G[x]$ by subdividing edge e. Since $G|N_G[x]$ is complete, at most 2 moves are played in $G_e|N_{G_e}[x]$.

By above cases, we conclude that $\gamma_g(G_e) \leq 3$.

To prove $\gamma_g(G_e) \geq 3$, it suffices to present a strategy for Staller that ensures at least 3 moves will be played on G_e. Suppose that Dominator plays first at a vertex y.

Case 1 $y = w$. Without loss of generality, assume that u has undominated neighbors. Then Staller plays u, and the residual graph of $G_e|(N_{G_e}[u] \cup N_{G_e}[w])$ is the residual graph of $G|N_G[u]$. Since $\gamma_g(G) = 2$, at least an additional move is required to end the game. Thus at least 3 moves are required to end the game.

Case 2 $y \in \{u, v\}$. Without loss of generality, assume that $y = u$. Then Staller plays w and we are in the same situation as Case 1.

Case 3 $y \notin \{u, v, w\}$. We divide our argument into 2 subcases.

Case 3.1 $\{u, v\} \subset N_G(y)$. Then Staller plays w and the residual graph of $G_e|(N_{G_e}[y] \cup N_{G_e}[w])$ is the residual graph of $G|N_G[y]$. Thus at least 3 moves are required to end the game.

Case 3.2 $\{u, v\} \not\subset N_G(y)$. Without loss of generality, assume that $v \notin N_G(y)$. Then v and w are not dominated. Then Staller responds on u, so v is undominated. Thus at least 3 moves are required to end the game.

We conclude that $\gamma_g(G_e) = 3$.

□

Our realization results for Domination-start game can be summarized in the following theorem.

Theorem 4.8. *For any positive integers x and y such that $0 \leq y - x \leq 2$ there exists a connected graph G with an edge e such that $(\gamma_g(G), \gamma_g(G_e)) = (x, y)$ except for $(1, 3)$, $(2, 2)$ and $(2, 4)$.*

Proof. The proof is the direct result of Propositions 4.3 – 4.7. □

Next, we show that for any positive integers x and y such that $0 \leq y - x \leq 2$ there exists a connected graph G with an edge e such that $(\gamma_g'(G), \gamma_g'(G_e)) = (x, y)$ except for $(1, 1)$ and $(1, 3)$.

4.4 $\gamma_g'(G_e) - \gamma_g'(G) = 0$

Proposition 4.9. *For any positive integer $n \geq 2$, there exists a graph G with an edge e such that $\gamma_g'(G) = n = \gamma_g'(G_e)$.*

Proof. Let n be a positive integer such that $n \geq 2$. Let $G = P_{2n-1}$. Then for any edge e of G, we have $G_e = P_{2n}$. By Theorem 2.18,

$$\gamma_g'(P_{2n-1}) = \left\lceil \frac{2n-1}{2} \right\rceil = n,$$
$$\gamma_g'(P_{2n}) = \left\lceil \frac{2n}{2} \right\rceil = n.$$

□

4.5 $\gamma'_g(G_e) - \gamma'_g(G) = 1$

Proposition 4.10. *For any positive integer n, there exists a connected graph G with an edge e such that $\gamma'_g(G) = n$ and $\gamma'_g(G_e) = n + 1$.*

Proof. Let n be a positive integer. We divide n into two cases.

Case 1 n is odd. Let $G = C_{2n+1}$. Then for any edge e of G, we have $G_e = C_{2n+2}$. Note that $2n + 1 \equiv 3 \pmod 4$ and $2n + 2 \equiv 0 \pmod 4$.
By Theorem 2.18,

$$\gamma'_g(C_{2n+1}) = \left\lceil \frac{(2n+1)-1}{2} \right\rceil = n,$$
$$\gamma'_g(C_{2n+2}) = \left\lceil \frac{(2n+2)-1}{2} \right\rceil = n+1.$$

Case 2 n is even. Let $G = C_{2n+2}$. Then for any edge e of G, we have $G_e = C_{2n+3}$. Note that $2n + 2 \equiv 2 \pmod 4$ and $2n + 3 \equiv 3 \pmod 4$.
By Theorem 2.18,

$$\gamma'_g(C_{2n+2}) = \left\lceil \frac{(2n+2)-1}{2} \right\rceil - 1 = n,$$
$$\gamma'_g(C_{2n+3}) = \left\lceil \frac{(2n+3)-1}{2} \right\rceil = n+1.$$

□

4.6 $\gamma'_g(G_e) - \gamma'_g(G) = 2$

Proposition 4.11. *For any positive integer $n \geq 2$, there exists a connected graph G with an edge e such that $\gamma'_g(G) = n$ and $\gamma'_g(G_e) = n + 2$.*

Proof. For the case $n = 2$, consider a cycle C on 5 vertices and denote a pair of adjacent vertices by u and v. Let G be the graph obtained from C by adding two new adjacent vertices u' and v' and joining u' and v' to all vertices in the closed neighborhoods of u and v, respectively. Let $e = \{u', v'\}$ and let w represent the new vertex in G_e that is adjacent to u' and v' (see Figure 4.7).

By Lemma 2.19, we have that $\gamma'_g(G) = \gamma'_g(C_5) = 2$. If Staller starts with w in G_e, then all vertices in C are undominated. Thus Staller ensures that at least

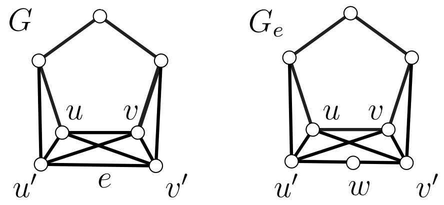

Figure 4.7: Graph G and Subdivision G_e

3 additional moves are required to end the game. By Lemma 2.10, we have that $\gamma_g'(G_e) \geq 4$. By Theorem 3.3, we have that $\gamma_g'(G_e) = 4$.

For the case $n = 3$, consider the grid graph $P_2 \times P_4$ [9, Therem 3.3] and denote its vertices by x_1, x_2, y_1, y_2, z_1, z_2, u, v (see Figure 4.8). We claim that $\gamma_g'(P_2 \times P_4) \leq 3$. By symmetry, there are two ways for Staller to make his first move. If Staller plays first at x_1, then Dominator responds on z_2. If Staller plays first at y_1, then Dominator responds on v. Then Dominator ensures at most 3 moves are played on $P_2 \times P_4$ and the claim is proved.

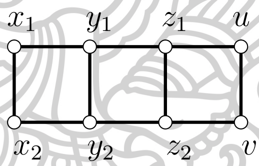

Figure 4.8: Graph $P_2 \times P_4$

Let G be the graph obtained from grid graph $P_2 \times P_4$ by adding two new adjacent vertices u' and v' and joining u' and v' to all vertices in the closed neighborhoods of u and v, respectively. Let $e = \{u', v'\}$ and let w represent the new vertex in G_e that is adjacent to u' and v' (see Figure 4.9).

By Lemma 2.19, we have that $\gamma_g'(G) = \gamma_g'(P_2 \times P_4) \leq 3$. By Theorem 3.3, it suffices to show that $\gamma_g'(G_e) \geq 5$. Staller starts by playing on w. Then all of vertices in $P_2 \times P_4$ are undominated. By symmetry, we may assume that Dominator responds with x_1, y_1, z_1, u or u'. Then Staller can respond by playing x_2, z_1, y_1, v or v', respectively. In each case, at least two additional moves are required to end the game. Thus Staller ensures that at least 5 moves are required to end the

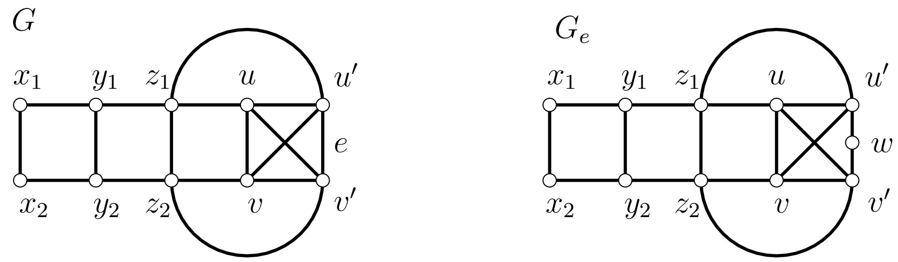

Figure 4.9: Graph G and Subdivision G_e

game. By Lemma 2.10, we have that $\gamma_g'(G_e) \geq 5$.

For the case $n \geq 4$, we consider the graphs U_k and V_k from Proposition 4.5.

Case 1 n is even. Let $k = \frac{n-4}{2} \geq 0$ and $G = U_k$. We claim that $\gamma_g'(G) = 2k+4 = n$ and $\gamma_g'(G_e) = 2k+6 = n+2$. By Theorem 3.2, it suffices to show that $\gamma_g'(G) \leq 2k+4$ and $\gamma_g'(G_e) \geq 2k+6$.

For the first inequality we present a strategy for Dominator that guarantees at most $2k+4$ moves are played on G. Dominator responds to a Staller's move by following Staller in the 6-cycle, in B, and in each of the k attached paths. Thus ensuring at most two moves in each part (Staller might be able to force 3 moves on a path once but this allows Dominator to dominate B with one move). Thus at most $2k+4$ moves are required to dominate G. By Lemma 2.10, we have that $\gamma_g(G) \leq 2k+4$.

For the second inequality we present a strategy for Staller that ensures at least $2k+6$ moves are played on G_e. Staller starts by playing on a pendant adjacent to u and if possible she tries not to be the first to play in the 6-cycle. Note that at least two moves will be played in each of the k attached paths and 3 additional moves will be played in B_e. If exactly $2k+3$ moves are played before a move in the 6-cycle is played, then Dominator plays first in the 6-cycle, yielding 3 additional moves. Otherwise, at least $2k+4$ moves are played elsewhere, and 2 additional moves will be played in the 6-cycle. Thus at least $2k+6$ moves are required to dominate G_e. By Lemma 2.10, we have that $\gamma_g(G_e) \geq 2k+5$. This concludes the proof for the case when n is even.

Case 2 n is odd. Let $k = \frac{n-5}{2} \geq 0$ and $G = V_k$. One can verify that $\gamma'_g(G) = 2k+5$ and $\gamma'_g(V_{k_e}) = 2k+7$ for any $k \geq 0$ by using similar arguments to the above case. □

The following result shows that for $n = 1$, there are no graph G and an edge e such that $\gamma'_g(G) = n$ and $\gamma'_g(G_e) = n+2$.

Proposition 4.12. *Let G be a graph. If $\gamma'_g(G) = 1$, then $\gamma'_g(G_e) = 2$ for every edge $e \in E(G)$.*

Proof. Suppose that $\gamma'_g(G) = 1$. Then G must be a complete graph. Clearly, $\gamma'_g(G_e) = 2$ for every edge e of G.

□

Our realization results for Staller-start game can be summarized in the following theorem.

Theorem 4.13. *For any positive integers x and y such that $0 \leq y - x \leq 2$ there exists a connected graph G with and an edge e such that $(\gamma'_g(G), \gamma'_g(G_e)) = (x, y)$ except for $(1,1)$ and $(1,3)$.*

Proof. The proof is the direct result of Propositions 4.9 – 4.12. □

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DISSEMINATIONS

Publications

1. Pakanun Dokyeesusun, and Chalermpong Worawannotai. “Effect of and edge subdivision on game domination numbers.” **Asian-European Journal of Mathematics** (accepted).

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