MERGING IN BIPARTITE DISTANCE-REGULAR GRAPHS

By
MISS Siwaporn MAMART

A Thesis Submitted in Partial Fulfillment of the Requirements
for Doctor of Philosophy (MATHEMATICS)
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Graduate School, Silpakorn University
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Merging the first and third classes in a connected graph is the operation of adding edges between all vertices at distance 3 in the original graph while keeping the original edges. Merging the first m classes in a connected graph is joining all the pairs of vertices with distance at most m in the graph with edges. In this thesis, we determine when merging the first and third classes in a bipartite distance-regular graph produces a distance-regular graph. We also determine when merging the first m classes in a bipartite distance-regular graph produces a distance-regular graph.

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Siwaporn MAMART

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Chapter 1
Introduction

In graph theory, there are many ways to construct some new graphs from given graphs. For example, deleting vertices or edges from a graph produces a smaller graph. There are also methods for constructing a new larger graph from given graphs such as the union of graphs and the Cartesian product of graphs. The union of graphs is the graph obtained by taking the union of the vertex sets and the union of the edge sets of those graphs. Sometimes we call the union of graphs "the merging of graphs". If those graphs have a common vertex set, the merging of graphs is joining the pair of vertices with edges from those given graphs. In this thesis, we focus on merging some classes in distance-regular graphs.

Sometimes, merging some classes in a distance-regular graph yields again a distance-regular graph. Merging can be used to construct some new distance-regular graphs [1, Section 11.4 F]. For example, Clebsch graph is defined as the graph obtained from merging the first and fourth classes in the Hamming graph $H(4, 2)$. More generally, merging classes of association schemes were studied by Kageyama [2], [3], [4], [5] and Kageyama et al. [6]. In [1, Proposition 4.2.18], Brouwer et al. characterized when merging the first and second classes in a distance-regular graph produces a distance-regular graph. Merging can be used to characterize certain families of distance-regular graphs. Brouwer [7] used merging to characterize certain antipodal distance-regular graphs of diameter 3 with generalized quadrangles containing a spread. Jurišić [8] determined when merging the first and last classes in an antipodal distance-regular graph produces a distance-regular graph. He gave a characterization of a class of antipodal distance-regular graphs with a class of regular near polygons containing a certain spread. This generalizes Brouwer's characterization of a class of distance-regular graphs of diameter 3 with generalized quadrangles containing a spread.

In this thesis, we determine when merging the first and third classes in a bipartite distance-regular graph produces a distance-regular graph. We also determine when merging the first m classes in a bipartite distance-regular graph produces a distance-regular graph.

Chapter 2

Preliminaries

This chapter contains basic definitions and notations used in this thesis. In general, we follow [1] and [9].

All graphs considered here are finite undirected simple graphs. For a graph Γ, we denote its vertex set by $V(\Gamma)$ and denote its edge set by $E(\Gamma)$. A graph is called *r-regular* if all its vertices have degree r. A *complete graph* is a graph in which every pair of vertices are adjacent. A complete graph on n vertices is denoted by K_n. A *cycle* C_n is a graph on n vertices containing a single cycle through all vertices. A graph Γ is *bipartite* if $V(\Gamma)$ can be partitioned into two non-empty parts V_1 and V_2 (i.e., $V_1 \cup V_2 = V(\Gamma)$ and $V_1 \cap V_2 = \varnothing$) in such a way that each edge of Γ has one end in V_1 and the other in V_2. The partition $V(\Gamma) = V_1 \cup V_2$ is called a *bipartition* of Γ. A *complete bipartite graph* is a bipartite graph with a bipartition $V_1 \cup V_2$ in which every vertex in V_1 is joined to every vertex in V_2; it is denoted by $K_{m,n}$ where $|V_1| = m$ and $|V_2| = n$. The *line graph* of graph Γ is the graph whose vertex set is the edge set of Γ and two of these vertices are adjacent if and only if the corresponding edges in Γ have a common vertex. A *matching* in Γ is a set of pairwise non-adjacent edges; that is, no two edges share a common vertex. A *perfect matching* of a graph is a matching in which every vertex of the graph is incident to exactly one edge of the matching. The *distance* $d_\Gamma(u, v)$ of two vertices u and v in a connected graph Γ is the length of a shortest path between u and v in Γ. The *diameter* of a connected graph Γ, denoted by diam(Γ), is the maximum distance of any two vertices of Γ.

For a vertex x in a graph Γ with diameter d and for i $(0 \leq i \leq d)$, define $\Gamma_i(x)$ to be the set of vertices at distance i from x. For convenience, set $\Gamma_{-1}(x) = \Gamma_{d+1}(x) = \varnothing$. For any connected graph Γ with diameter d, we denote by Γ_i $(0 \leq i \leq d)$ the graph whose vertices are those of Γ and two vertices are adjacent if they have distance i in Γ. We call Γ_i the *i*th class of Γ. In particular, $\Gamma_1 = \Gamma$. For any connected graph Γ with diameter $d \geq 3$, we let $\Gamma_{1,3} := \Gamma_1 \cup \Gamma_3$ denote the graph whose vertices are those of Γ and two vertices x and y are adjacent if $d_\Gamma(x, y) = 1$ or 3; in other words $\Gamma_{1,3}$ is the graph obtained by merging the first and third classes of Γ.

Example 2.1. Figure 2.1 shows the graphs Γ_1, Γ_3 and $\Gamma_{1,3}$ of the graph $\Gamma = C_6$.

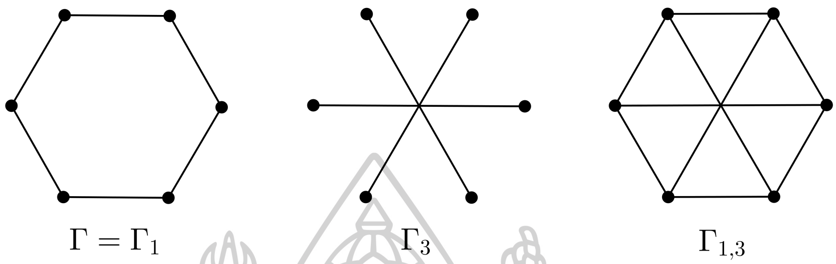


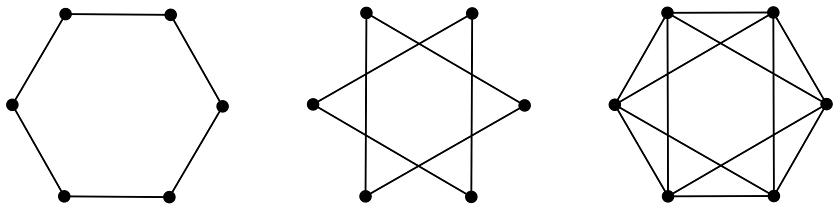
Figure 2.1: The graphs Γ_1, Γ_3 and $\Gamma_{1,3}$ where $\Gamma = C_6$

Example 2.2. Figure 2.2 shows the graphs Γ_1, Γ_3 and $\Gamma_{1,3}$ of the graph $\Gamma = C_{12}$.

Figure 2.2: The graphs Γ_1, Γ_3 and $\Gamma_{1,3}$ where $\Gamma = C_{12}$

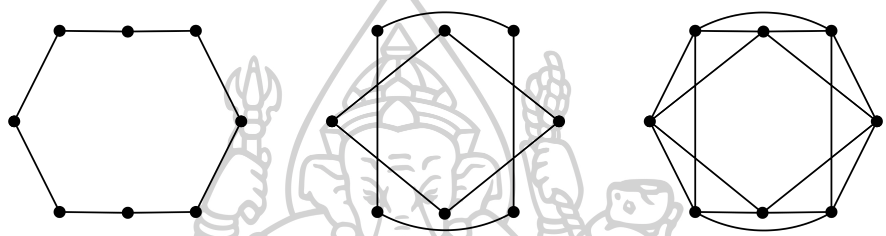
For any connected graph Γ of diameter $d \geq 2$ and for integer $m \leq d$, we let $\Gamma_{1,2,\ldots,m} := \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_m$ denote the graph whose vertices are those of Γ and two vertices x and y are adjacent if $d_\Gamma(x, y) \in \{1, 2, \ldots, m\}$; in other words $\Gamma_{1,2,\ldots,m}$ is the graph obtained by merging the first m classes of Γ.

Example 2.3. Figure 2.3 shows the graphs Γ_1, Γ_2 and $\Gamma_{1,2}$ of the graph $\Gamma = C_6$.



$\Gamma = \Gamma_1$ Γ_2 $\Gamma_{1,2}$

Figure 2.3: The graphs Γ_1, Γ_2 and $\Gamma_{1,2}$ where $\Gamma = C_6$

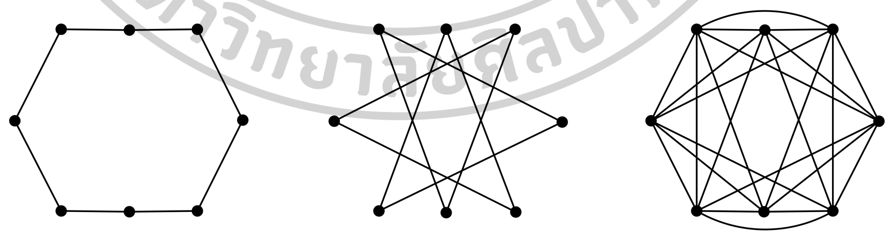


$\Gamma = \Gamma_1$ Γ_2 $\Gamma_{1,2}$

Figure 2.4: The graphs Γ_1, Γ_2 and $\Gamma_{1,2}$ where $\Gamma = C_8$

Example 2.4. Figure 2.4 shows the graphs Γ_1, Γ_2 and $\Gamma_{1,2}$ of the graph $\Gamma = C_8$.

Example 2.5. Figure 2.5 shows the graphs Γ_1, Γ_3 and $\Gamma_{1,2,3}$ of the graph $\Gamma = C_8$.



$\Gamma = \Gamma_1$ Γ_3 $\Gamma_{1,2,3}$

Figure 2.5: The graphs Γ_1, Γ_3 and $\Gamma_{1,2,3}$ where $\Gamma = C_8$

Example 2.6. Figure 2.6 shows the graphs Γ_1, Γ_2, Γ_3 and $\Gamma_{1,2,3}$ of the graph $\Gamma = C_{10}$.

For vertices x and y of Γ at distance i ($0 \leq i \leq d$), we define $c_i(x, y) =$

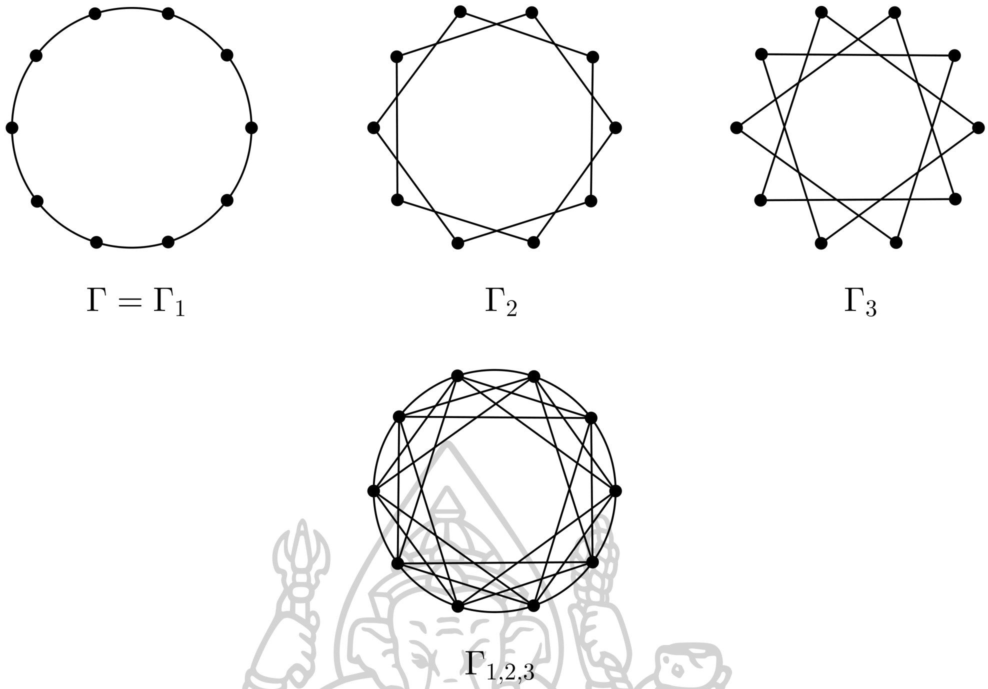


Figure 2.6: The graphs Γ_1, Γ_2, Γ_3 and $\Gamma_{1,2,3}$ where $\Gamma = C_{10}$

$|\Gamma_{i-1}(x) \cap \Gamma_1(y)|$, $a_i(x, y) = |\Gamma_i(x) \cap \Gamma_1(y)|$ and $b_i(x, y) = |\Gamma_{i+1}(x) \cap \Gamma_1(y)|$ (see Figure 2.7).

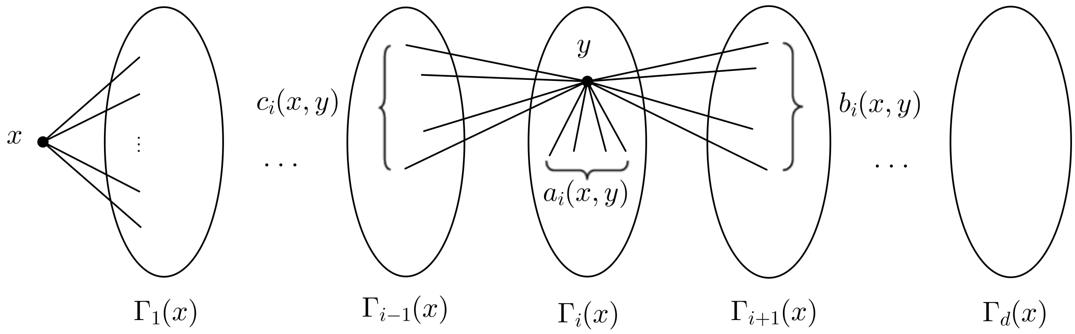


Figure 2.7: The illustrations for the $c_i(x, y)$, $a_i(x, y)$ and $b_i(x, y)$

Definition 2.7. A *multipartite graph* is a graph whose vertex set can be partitioned into at least two parts so that no edge has both ends in any one part. If the number of the parts is n, then the graph is called an *n-partite graph.* A *complete multipartite graph* is a multipartite graph in which each vertex is joined to every vertex that is not in the same part. If the partition contains n parts whose sizes are $k_1, k_2, \ldots, k_n$, then the complete multipartite graph is denoted by $K_{k_1,k_2,\ldots,k_n}$ where $k_i \geq 2$ for some $1 \leq i \leq n$. We denote the complete n-partite graph in which each part in the partition has size r by $K_{n\times r}$. In particular, the complete n-partite graph $K_{n\times 2}$ is also known as the *n-cocktail party graph*, the *hyperoctahedral graph* or *Roberts graph*.

Note that a complete n-partite graph $K_{n\times r}$ has nr vertices and diameter 2.

Example 2.8. Figure 2.8 shows the graph $K_{4\times 2}$ or the 4-cocktail party graph.

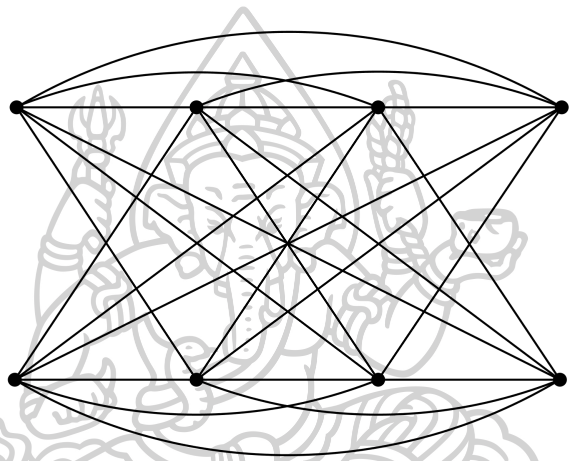


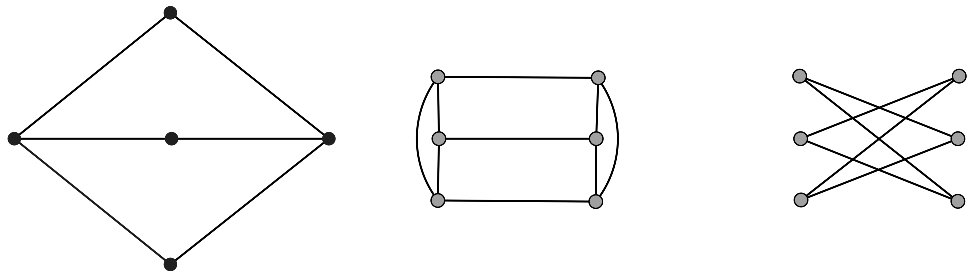
Figure 2.8: The complete multipartite graph $K_{4\times 2}$ or the 4-cocktail party graph

Definition 2.9. For positive integers m and n, the *$(m \times n)$-grid* is the line graph of the complete bipartite graph $K_{m,n}$ (sometimes known as a *rook graph* or a *lattice graph*).

Note that the complement of a $(2 \times m)$-grid where $m \geq 3$ (sometimes known as a *crown graph*) has $2m$ vertices and diameter 3 and it can be viewed as a complete bipartite graph $K_{m,m}$ from which the edges of a perfect matching have been removed.

Example 2.10. Figure 2.9 shows the graph $K_{2,3}$, the (2×3)-grid, and the complement of a (2×3)-grid.

Definition 2.11. For integers $d > 1$ and $q > 1$, the *Hamming graph* $H(d, q)$ is the graph whose vertex set consists of the words of length d from an alphabet of size q, where two vertices are adjacent if they differ in precisely one position.



$K_{2,3}$ $\qquad$ (2×3)-grid $\qquad$ the complement of a (2×3)-grid

Figure 2.9: The graph $K_{2,3}$, the (2×3)-grid, and the complement of a (2×3)-grid

Note that a Hamming graph $H(d, q)$ has q^d vertices and diameter d.

The Hamming graph $H(d, 2)$ is also called a *(hyper)cube* or d-*cube* and it is a bipartite graph.

Example 2.12. Figure 2.10 shows the Hamming graph $H(3, 2)$.

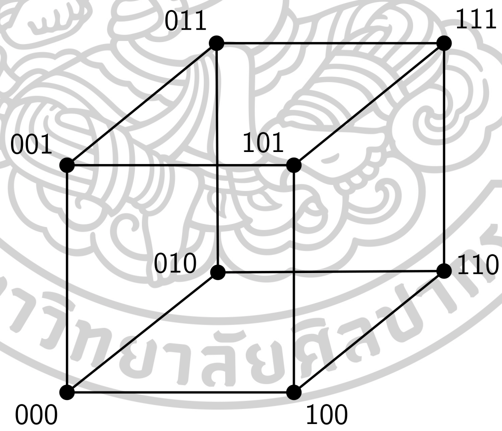


Figure 2.10: The Hamming graph $H(3, 2)$

Definition 2.13. For an integer $d > 1$, let Y_d denote the graph with vertex set $V(H(d, 2))$, and two vertices are adjacent in Y_d if they are at distance 2 in $H(d, 2)$.

Note that Y_d is not connected, but it contains two isomorphic components on 2^{d-1} vertices, each of which is called a *halved* d-*cube*.

Note that a halved d-cube has 2^{d-1} vertices and diameter $\left\lfloor \frac{d}{2} \right\rfloor$.

Example 2.14. Figure 2.11 shows Y_3 and a halved 3-cube.

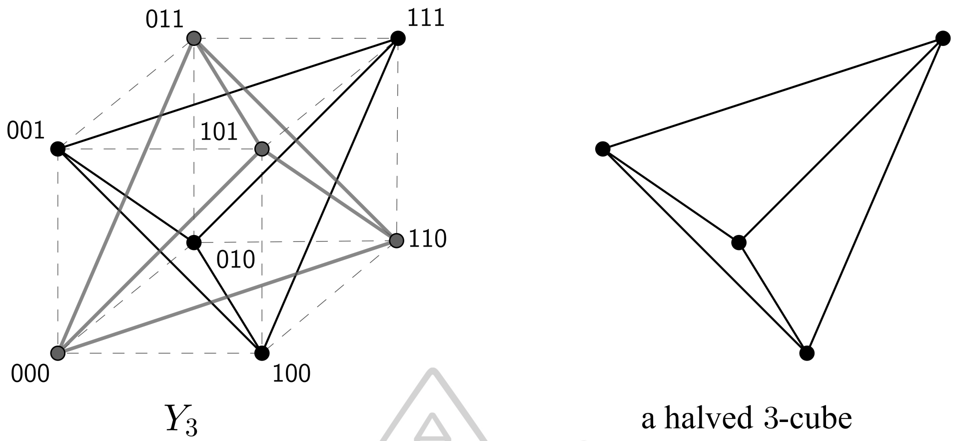


Figure 2.11: The graph Y_3 and a halved 3-cube

Definition 2.15. For integers $n > 1$ and $k > 1$, the *Johnson graph* $J(n, k)$ is the graph whose vertex set consists of the k-subsets of $\{1, ..., n\}$, where two vertices are adjacent if their intersection has cardinality $k - 1$.

Note that a Johnson graph $J(n, k)$ has $\binom{n}{k}$ vertices and diameter $\min(k, n - k)$.

Example 2.16. Figure 2.12 shows the Johnson graph $J(4, 2)$.

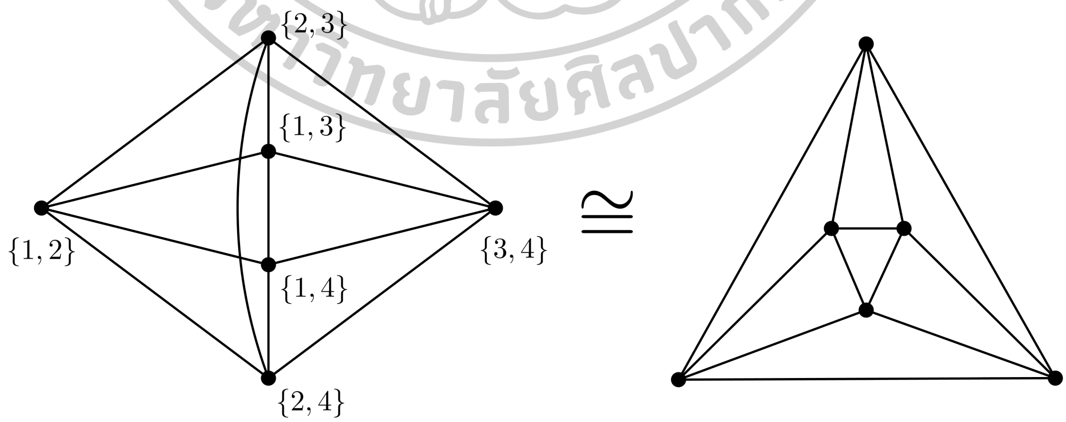


Figure 2.12: The Johnson graph $J(4, 2)$

Definition 2.17. Let X be the set of cardinality $2m+1$ where $m \geq 3$.

The *doubled Odd graph* on X is the graph Γ whose vertices are the m-subsets and $(m+1)$-subsets of X, and distinct vertices $A, B \in \Gamma$ are adjacent if $A \subset B$ or $B \subset A$.

Note that a doubled Odd graph on $2m+1$ points is a bipartite graph with diameter $2m+1$ and has $\binom{2m+1}{m} + \binom{2m+1}{m+1}$ vertices.

Example 2.18. Figure 2.13 shows a doubled Odd graph with $X = \{1, 2, 3\}$.

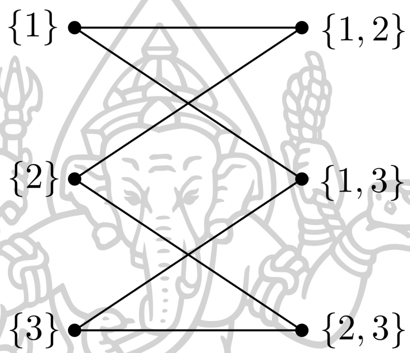


Figure 2.13: The doubled Odd graph on $X = \{1, 2, 3\}$

Chapter 3
Distance-regular graphs

In this chapter, we review some basic concepts and relevant results about distance-regular graphs. In general, we follow [1], [10] and [11].

A connected graph Γ of diameter d is said to be *distance-regular* whenever for any two vertices x and y in Γ at distance i, the numbers

$$c_i := |\Gamma_{i-1}(x) \cap \Gamma_1(y)|,\ a_i := |\Gamma_i(x) \cap \Gamma_1(y)| \text{ and } b_i := |\Gamma_{i+1}(x) \cap \Gamma_1(y)|$$

depend only on the distance $d_\Gamma(x, y) = i$ rather than on individual vertices. When this is the case we call numbers c_i, a_i and b_i the *intersection numbers* of Γ. Observe that $a_0 = 0$, $c_1 = 1$ and $c_0 = b_d = 0$. We denote $k_i = |\Gamma_i(x)|$ $(0 \leq i \leq d)$. In particular, $k_0 = 1$ and $k_{i+1} = b_i k_i / c_{i+1}$ $(0 \leq i \leq d-1)$. By [1, p.127],

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \qquad (1 \leq i \leq d). \tag{3.1}$$

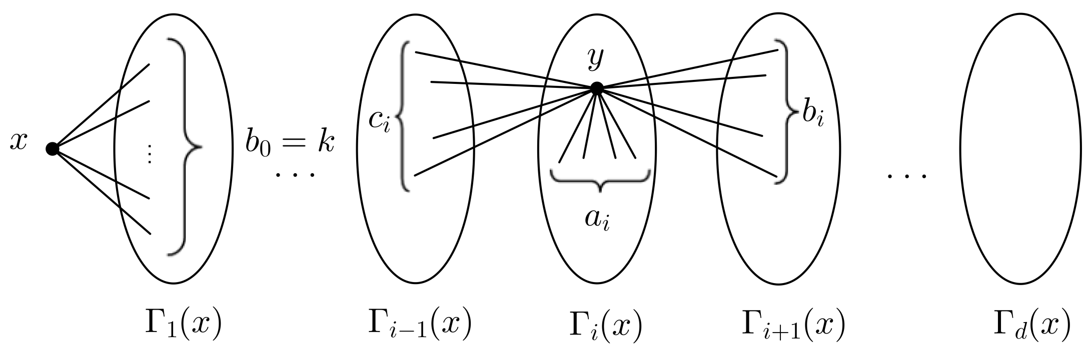


Figure 3.1: The illustrations for the c_i, a_i and b_i

Observe that Γ is regular of degree $k = k_1 = b_0$ and has $k_0 + k_1 + k_2 + \ldots + k_d$ vertices. Moreover, $c_i + a_i + b_i = k$ $(0 \leq i \leq d)$. We call $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$ the *intersection array* of Γ.

More generally, let x and y be two vertices at distance h in a distance-regular graph Γ with diameter d. Then the numbers

$$p^h_{ij} := |\Gamma_i(x) \cap \Gamma_j(y)| = |\{z \in V(\Gamma) \mid d_\Gamma(x,z) = i \text{ and } d_\Gamma(y,z) = j\}|$$

exist (i.e., they only depend on h, i and j) for integers $0 \le h, i, j \le d$. Note that $p^h_{ij} = p^h_{ji}$ $(0 \le h, i, j \le d)$. Observe that $c_i = p^i_{1,i-1}$ $(1 \le i \le d)$, $a_i = p^i_{1,i}$ $(0 \le i \le d)$, $b_i = p^i_{1,i+1}$ $(0 \le i \le d-1)$, and $k_i = p^0_{ii}$ $(0 \le i \le d)$.

Lemma 3.1. [1, p.127, Lemma 4.1.7] *Let Γ be a distance-regular graph with diameter d. Then for all integers $0 \le h, i, j \le d$ the followings hold.*

(i) $p^h_{ij} = 0$ *if one of h, i, j is greater than the sum of the other two.*

(ii) $p^h_{ij} \neq 0$ *if one of h, i, j is equal to the sum of the other two.*

Assume that Γ is a bipartite distance-regular graph with diameter d and has bipartition $V_1 \cup V_2$. Let x be a vertex in V_1. Since any path in Γ has its vertices alternating between V_1 and V_2, we have $V_1 = \bigcup_{i \text{ is even}} \Gamma_i(x)$ and $V_2 = \bigcup_{i \text{ is odd}} \Gamma_i(x)$. Thus for any i $(1 \le i \le d)$, the set $\Gamma_i(x)$ is contained in a bipartition V_1 or V_2. So no two vertices in $\Gamma_i(x)$ are adjacent, which means $a_i = 0$ for all i $(0 \le i \le d)$. Hence $b_i + c_i = k$ for all i $(0 \le i \le d)$. In particular $k = b_0 = c_d$. Since bipartite graphs contain no odd cycles, we have the following lemma.

Lemma 3.2. *A distance-regular graph is bipartite if and only if $p^h_{ij} = 0$ for all $h + i + j$ is odd.*

Example 3.3. The following graphs are distance-regular.

(i) For $n \ge 2$, a complete graph K_n is a distance-regular graph with diameter 1 and has intersection array $\{n-1; 1\}$.

(ii) For $n > 1$ and $r > 1$, a complete n-partite graph $K_{n \times r}$ is a distance-regular graph with diameter 2 and has intersection array $\{r(n-1), r-1; 1, r(n-1)\}$.

(iii) For $n \ge 3$, a cycle C_n is a distance-regular graph with diameter $d = \left\lfloor \frac{n}{2} \right\rfloor$ and has intersection array $\{2, 1, 1, \ldots, 1; 1, 1, 1, \ldots, c_d\}$ where $c_d = 1$ if n is odd and $c_d = 2$ if n is even. Moreover, if n is even, C_n is bipartite.

(iv) A complement of a $(2 \times m)$-grid where $m \geq 3$ is a distance-regular graph with diameter 3 and has intersection array $\{m-1, m-2, 1; 1, m-2, m-1\}$ (See [1, p.432]).

(v) A Hamming graph $H(d, q)$ is a distance-regular graph with diameter d and has intersection array $\{d(q-1), (d-1)(q-1), (d-2)(q-1), \ldots, q-1; 1, 2, 3, \ldots, d\}$ (See [1, Theorem 9.2.1]).

(vi) For $d \geq 3$, a halved d-cube is a distance-regular graph with diameter $\left\lfloor \frac{d}{2} \right\rfloor$ and its intersection array is given by

$$b_i = \frac{1}{2}(d-2i)(d-2i-1), \quad c_i = i(2i-1) \quad \text{for } 0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor$$

(See [1, p.264]).

(vii) A Johnson graph $J(n, k)$ is a distance-regular graph with diameter $d = \min(k, n-k)$ and its intersection array is given by

$$b_i = (k-i)(n-k-i), \quad c_i = i^2 \quad \text{for } 0 \leq i \leq d$$

(See [1, Theorem 9.1.2]).

(viii) A doubled Odd graph on $2m+1$ points is a distance-regular graph with diameter $2m+1$ and has intersection array $\{m+1, m, m, \ldots, 1, 1; 1, 1, \ldots, m, m, m+1\}$ (See [1, Theorem 9.1.8]).

Example 3.4. Referring to the Example 2.1, the graph $\Gamma_{1,3}$ of the graph $\Gamma = C_6$ is isomorphic to the graph in Figure 3.2 and it is a distance-regular graph with intersection array $\{3, 2; 1, 3\}$.

Example 3.5. Referring to the Example 2.4, the graph $\Gamma_{1,3}$ of the graph $\Gamma = C_{12}$ is isomorphic to the graph in Figure 3.3 and it is not a distance-regular graph because $b_2(u_1, u_3) = 1 \neq 2 = b_2(u_1, u_5)$.

Example 3.6. Referring to the Example 2.3, the graph $\Gamma_{1,2}$ of the graph $\Gamma = C_6$ is isomorphic to the graph in Figure 3.4 and it is a distance-regular graph with intersection array $\{4, 1; 1, 4\}$.

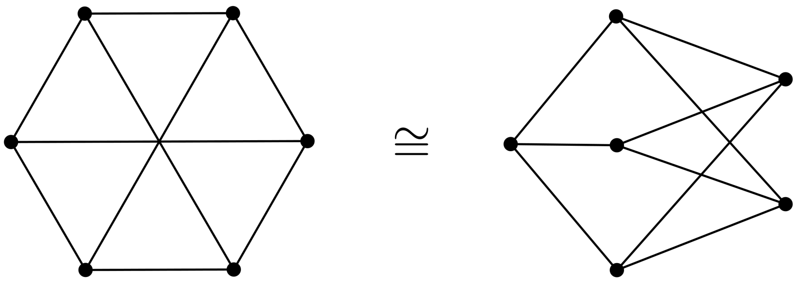


Figure 3.2: The graph $\Gamma_{1,3}$ of the graph $\Gamma = C_6$

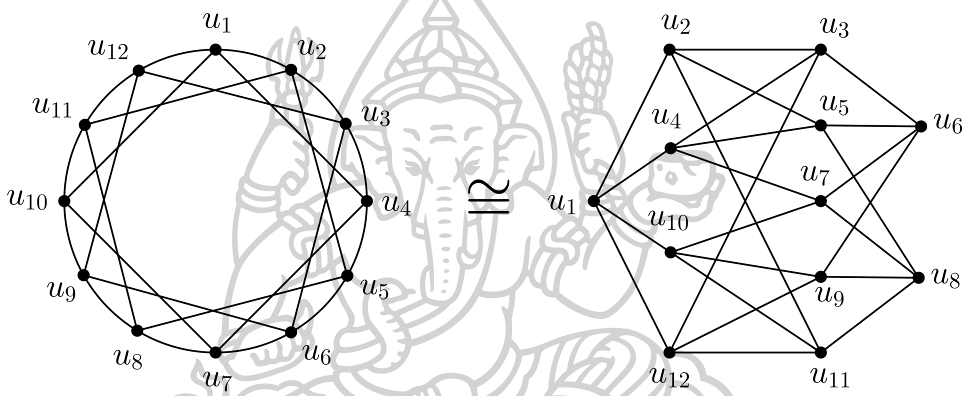


Figure 3.3: The graph $\Gamma_{1,3}$ of the graph $\Gamma = C_{12}$

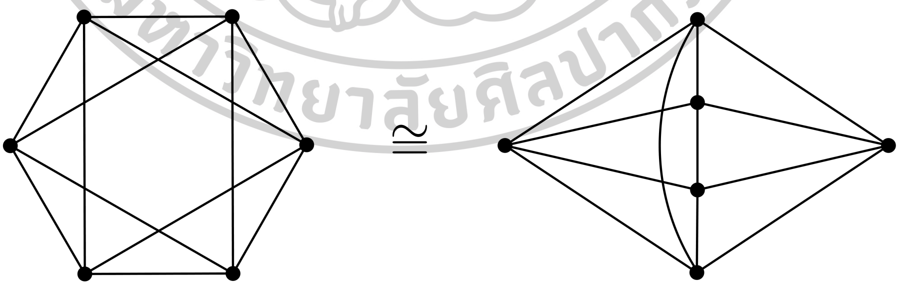


Figure 3.4: The graph $\Gamma_{1,2}$ of the graph $\Gamma = C_6$

Example 3.7. Referring to the Example 2.4, the graph $\Gamma_{1,2}$ of the graph $\Gamma = C_8$ is isomorphic to the graph in Figure 3.5 and it is not a distance-regular graph because $b_1(u_1, u_2) = 1 \neq 2 = b_1(u_1, u_3)$.

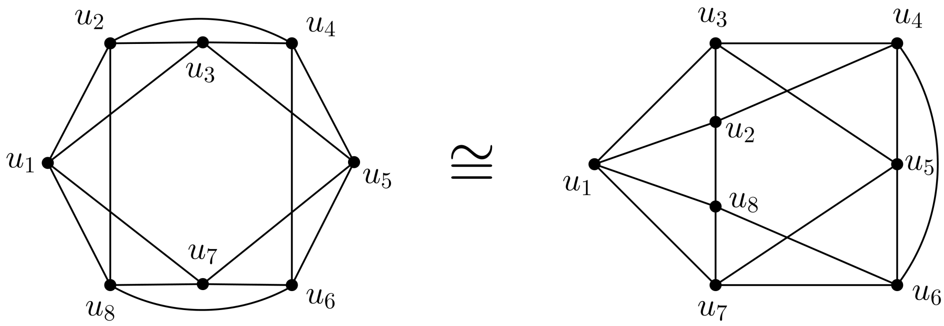


Figure 3.5: The graph $\Gamma_{1,2}$ of the graph $\Gamma = C_8$

Example 3.8. Referring to the Example 2.5, the graph $\Gamma_{1,2,3}$ of the graph $\Gamma = C_8$ is isomorphic to the graph in Figure 3.6 and it is a distance-regular graph with intersection array $\{6, 1; 1, 6\}$.

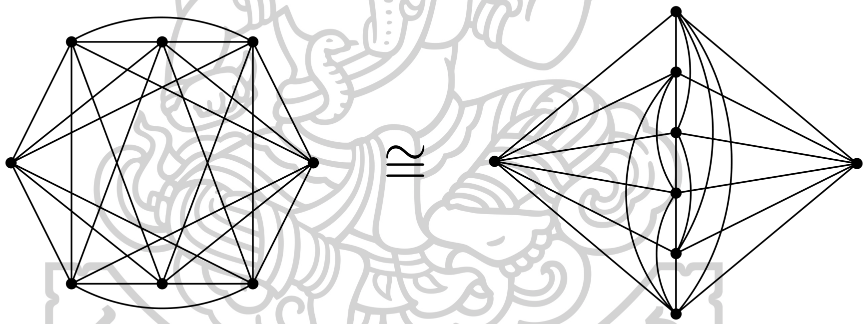


Figure 3.6: The graph $\Gamma_{1,2,3}$ of the graph $\Gamma = C_8$

Example 3.9. Referring to the Example 2.6, the graph $\Gamma_{1,2,3}$ of the graph $\Gamma = C_{10}$ is isomorphic to the graph in Figure 3.7 and it is not a distance-regular graph because $b_1(u_1, u_2) = 1 \neq 3 = b_1(u_1, u_4)$.

A k-regular graph with v vertices is *strongly regular* if there exist positive integers λ and μ such that every two adjacent vertices have λ common neighbors, and every two non-adjacent vertices have μ common neighbors. A graph of this kind is sometimes said to be a strongly regular graph with parameters (v, k, λ, μ). Connected strongly regular graphs are precisely distance-regular graphs with diameter 2. In terms

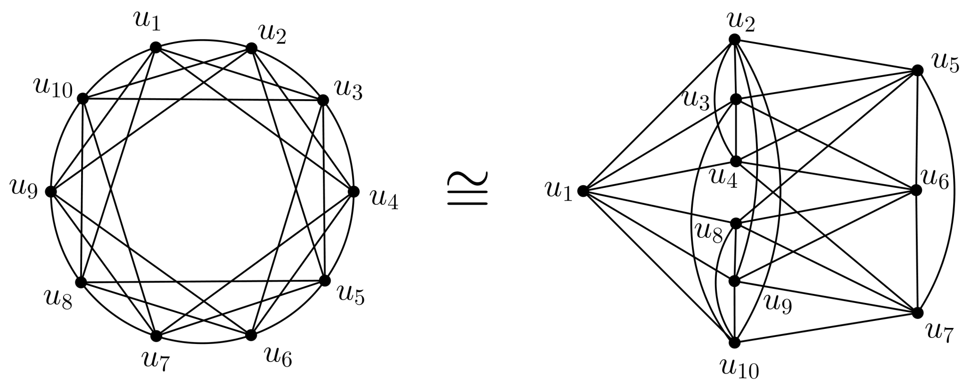


Figure 3.7: The graph $\Gamma_{1,2,3}$ of the graph $\Gamma = C_{10}$

of the parameters (v, k, λ, μ), the intersection array is given by $\{k, k-1-\lambda; 1, \mu\}$.

Observe that the complete multipartite graph $K_{n\times r}$ is a strongly regular graph with parameters $(v, k, \lambda, \mu) = \big(nr, (n-1)r, (n-2)r, (n-1)r\big)$. Moreover, it is the only strongly regular graph with $\mu = k$ (see [1, Theorem 1.3.1]). In particular, the n-cocktail party graph is a strongly regular graph with parameters $(2n, 2n-2, 2n-4, 2n-2)$.

Example 3.10. Referring to the Example 3.4, Example 3.6, and Example 3.8, these graphs are distance-regular graphs with diameter 2, so they are strongly regular graphs.

(i) Referring to the Example 3.4, the graph $\Gamma_{1,3}$ of the graph $\Gamma = C_6$ is a strongly regular graph with parameters $(6, 3, 0, 3)$ which is a complete bipartite graph $K_{3,3}$.

(ii) Referring to the Example 3.6, the graph $\Gamma_{1,2}$ of the graph $\Gamma = C_6$ is a strongly regular graph with parameters $(6, 4, 2, 4)$ which is a 3-cocktail party graph.

(iii) Referring to the Example 3.8, the graph $\Gamma_{1,2,3}$ of the graph $\Gamma = C_8$ is a strongly regular graph with parameters $(8, 6, 4, 6)$ which is a 4-cocktail party graph.

Let Γ be a distance-regular graph with diameter d. For each integer i $(0 \le i \le d)$, let A_i denote the $|V(\Gamma)| \times |V(\Gamma)|$ symmetric matrix with (x, y)-entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } d(x,y) = i \\ 0 & \text{if } d(x,y) \neq i \end{cases} \qquad (x, y \in V(\Gamma)).$$

We call A_i the ith *distance matrix* of Γ. We abbreviate $A = A_1$, and call this the *adjacency matrix* of Γ. Observe

$$\text{(i) } A_0 = I\,, \quad \text{(ii) } \sum_{i=0}^{d} A_i = J\,, \quad \text{(iii) } A_i^t = A_i\,, \quad \text{(iv) } A_iA_j = \sum_{h=0}^{d} p_{ij}^h A_h$$

where J is the all 1's matrix. We recall some results that will be used later. Brouwer et al. provided short proofs for the following results ([1, Lemma 2.1.1], [1, Lemma 4.1.7]) for which we extend and give more detailed proofs.

Lemma 3.11. [1, Lemma 2.1.1] *For a distance-regular graph with diameter d and for $0 \le h, i, j, r \le d$,*

$$\sum_{s=0}^{d} p_{ri}^s p_{sj}^h = \sum_{t=0}^{d} p_{it}^h p_{rj}^t$$

.

Proof. Expanding each side of the equation $A_j(A_rA_i) = (A_jA_r)A_i$, we have

$$A_j(A_rA_i) = A_j\Big(\sum_{s=0}^{d} p_{ri}^s A_s\Big) = \sum_{s=0}^{d} p_{ri}^s (A_jA_s) = \sum_{s=0}^{d} p_{ri}^s\Big(\sum_{h=0}^{d} p_{js}^h A_h\Big)$$

$$(A_jA_r)A_i = \Big(\sum_{t=0}^{d} p_{jr}^t A_t\Big)A_i = \sum_{t=0}^{d} p_{jr}^t (A_tA_i) = \sum_{t=0}^{d} p_{jr}^t\Big(\sum_{h=0}^{d} p_{ti}^h A_h\Big)$$

Since $A_0, A_1, A_2, \ldots, A_d$ are linearly independent, we obtain $\sum_{s=0}^{d} p_{ri}^s p_{sj}^h = \sum_{t=0}^{d} p_{it}^h p_{rj}^t$ as desired. $\square$

Lemma 3.12. [1, Lemma 4.1.7] *For a distance-regular graph with diameter d and for $0 \le h, i, j \le d$,*

$$p_{0j}^h = \delta_{hj}, \quad p_{i0}^h = \delta_{hi}, \quad p_{i,d+1}^h = 0,$$

$$p_{i+1,j}^h = \frac{1}{c_{i+1}}\Big(p_{i,j-1}^h b_{j-1} + p_{ij}^h(a_j - a_i) + p_{i,j+1}^h c_{j+1} - p_{i-1,j}^h b_{i-1}\Big). \tag{3.2}$$

In particular,

$$p_{ij}^{i+j} = \frac{c_{i+1}c_{i+2}\,\cdots\,c_{i+j}}{c_1c_2\,\cdots\,c_j}, \qquad p_{ij}^{i-j} = \frac{b_{i-1}b_{i-2}\,\cdots\,b_{i-j}}{c_1c_2\,\cdots\,c_j}.$$

Proof. Clearly, $p^h_{0j} = \delta_{hj}$, $p^h_{i0} = \delta_{hi}$ and $p^h_{i,d+1} = 0$.

Taking $r = 1$ in Lemma 3.11, we have $\sum_{s=0}^{d} p^s_{1i} p^h_{sj} = \sum_{t=0}^{d} p^h_{it} p^t_{1j}$. By Lemma 3.1, we have
$\sum_{s=0}^{d} p^s_{1i} p^h_{sj} \;=\; p^{i-1}_{1\,i} p^h_{i-1,j} + p^i_{1i} p^h_{ij} + p^{i+1}_{1\,i} p^h_{i+1,j} \;=\; b_{i-1} p^h_{i-1,j} + a_i p^h_{ij} + c_{i+1} p^h_{i+1,j}$ and $\sum_{t=0}^{d} p^h_{it} p^t_{1j} = p^h_{i,j-1} p^{j-1}_{1\,j} + p^h_{i\,j} p^j_{1\,j} + p^h_{i,j+1} p^{j+1}_{1\,j} = p^h_{i,j-1} b_{j-1} + p^h_{ij} a_j + p^h_{i,j+1} c_{j+1}$. Then the result follows.

In particular, taking $h = i+j$ and $j = j-1$ in (3.2). By Lemma 3.1(i), we have

$$c_{i+1} p^{i+j}_{i+1,j-1} = c_j p^{i+j}_{i\,j}.$$

In other words,

$$p^{i+j}_{i\,j} \;=\; \frac{c_{i+1}}{c_j} \cdot p^{i+j}_{i+1,j-1}.$$

Thus

$$p^{i+j}_{i\,j} \;=\; \frac{c_{i+1} c_{i+2} \;\cdots\; c_{i+j}}{c_j c_{j-1} \;\cdots\; c_1} \cdot p^{i+j}_{i+j,0} \;=\; \frac{c_{i+1} c_{i+2} \;\cdots\; c_{i+j}}{c_1 c_2 \;\cdots\; c_j}.$$

Next, taking $h = i-j$ and $j = j-1$ in (3.2). By Lemma 3.1(i) we have

$$b_{i-1} p^{i-j}_{i-1,j-1} = c_j p^{i-j}_{i\,j}.$$

In other words,

$$p^{i-j}_{i\,j} \;=\; \frac{b_{i-1}}{c_j} \cdot p^{i-j}_{i-1,j-1}.$$

Thus

$$p^{i-j}_{i\,j} \;=\; \frac{b_{i-1} b_{i-2} \;\cdots\; b_{i-j}}{c_j c_{j-1} \;\cdots\; c_1} \cdot p^{i-j}_{i-j,0} \;=\; \frac{b_{i-1} b_{i-2} \;\cdots\; b_{i-j}}{c_1 c_2 \;\cdots\; c_j}.$$

□

For convenience of use we state some special cases of the above formula.

Corollary 3.13. *For a distance-regular graph with diameter d and for $0 \le i \le d$,*

$$p^i_{i-2,2} = \frac{c_{i-1} c_i}{c_2}, \qquad p^{i+1}_{i\,2} = \frac{c_{i+1}(a_i + a_{i+1} - a_1)}{c_2},$$

$$p^{i-1}_{i+1,2} = \frac{b_{i-1} b_i}{c_2}, \qquad p^{i-1}_{i\,2} = \frac{b_{i-1}(a_i + a_{i-1} - a_1)}{c_2}.$$

Proposition 3.14. [10, 12, 13] *For a distance-regular graph with diameter d, the following restrictions on the intersection array hold.*

(i) *$k = b_0 > b_1 \geq b_2 \geq \ldots \geq b_{d-1} > b_d = 0$ and $1 = c_1 \leq c_2 \leq \ldots \leq c_d \leq k$.*

(ii) *If $i + j \leq d$, then $c_i \leq b_j$.*

(iii) *All parameters p_{ij}^h are nonnegative integers.*

(iv) *There is an i such that $k_0 \leq k_1 \leq \ldots \leq k_i$ and $k_{i+1} \geq k_{i+2} \geq \ldots \geq k_d$.*

An *antipodal* graph is a connected graph Γ with diameter $d > 1$ for which Γ_d is a disjoint union of complete graphs.

Example 3.15. Referring to Definition 2.7, the complete n-partite graph $K_{n\times r}$ is an antipodal graph with diameter 2. For instance, the complete 3-partite graph $K_{3\times 3}$ in Figure 3.8 is an antipodal distance-regular graph with intersection array $\{6, 2; 1, 6\}$. Observe that Γ_2 of $K_{3\times 3}$ is a disjoint union of K_3's.

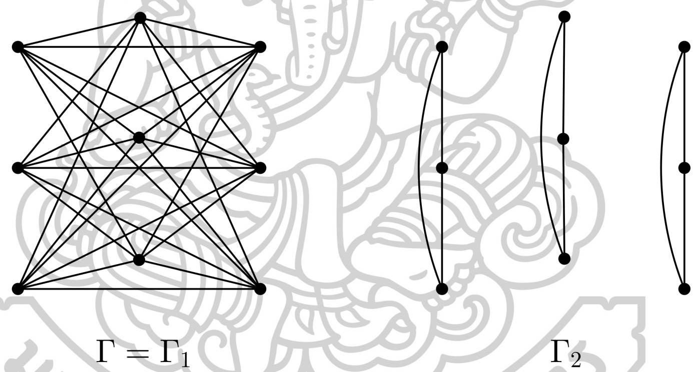


Figure 3.8: The graphs Γ_1 and Γ_2 where Γ is the complete 3-partite graph $K_{3\times 3}$

Example 3.16. Referring to Definition 2.9, the complement of a $(2 \times m)$-grid is an antipodal graph with diameter 3. For instance, the complement of a (2×5)-grid in Figure 3.9 is an antipodal distance-regular graph with intersection array $\{4, 3, 1; 1, 3, 4\}$. Observe that Γ_3 of the complement of a (2×5)-grid is a disjoint union of K_2's.

Example 3.17. Referring to Definition 2.11, the Hamming graph $H(d, 2)$ is an antipodal graph with diameter d. For instance, the Hamming graph $H(3, 2)$ in Figure 3.10 is an antipodal distance-regular graph with intersection array $\{3, 2, 1; 1, 2, 3\}$. Observe that Γ_3 of $H(3, 2)$ is a disjoint union of K_2's.

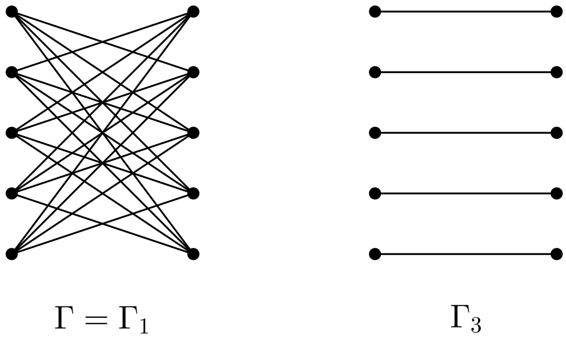


Figure 3.9: The graphs Γ_1 and Γ_3 where Γ is the complement of a (2×5)-grid

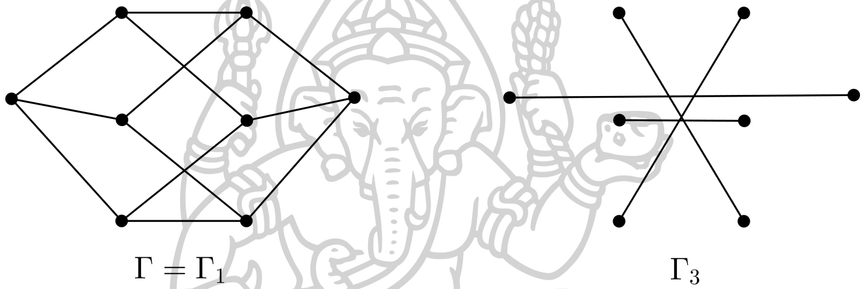


Figure 3.10: The graphs Γ_1 and Γ_3 where Γ is the Hamming graph $H(3, 2)$

Proposition 3.18. *A distance-regular graph with diameter d is antipodal if and only if $p^i_{dd} = 0$ unless $i = 0$ or d.*

Proof. Let Γ be a distance-regular graph with diameter d. By the definition, Γ is antipodal if and only if Γ_d is a disjoint union of complete graphs. Equivalently, the vertices at distance d from a given vertex are all at distance d from each other. In other words, for vertices x, y in Γ with $d_\Gamma(x, y) = i$ $(1 \leq i \leq d-1)$, $p^i_{dd} = |\{z \in V(\Gamma) \mid d_\Gamma(x, z) = d \text{ and } d_\Gamma(y, z) = d\}| = 0$. $\square$

Proposition 3.19. [14] *Let Γ be a distance-regular graph with diameter $d \in \{2m, 2m+1\}$. Then Γ is antipodal if and only if $b_i = c_{d-i}$ for all i $(0 \leq i \leq d-1, i \neq m)$.*

Corollary 3.20. *Let Γ be a bipartite distance-regular graph with odd diameter d. Then Γ is antipodal if and only if $b_i = c_{d-i}$ for all i $(0 \leq i \leq d-1)$.*

Proof. For a positive integer m, let Γ be a bipartite distance-regular graph with diameter $d = 2m+1$. Assume that Γ is antipodal, by Proposition 3.19, we have $b_i = c_{d-i}$ for all i $(0 \leq i \leq d-1, i \neq m)$. In particular, $b_{m+1} = c_{d-(m+1)} = c_{2m+1-(m+1)} = c_m$. Since Γ is bipartite, $b_j + c_j = k$ for $0 \leq j \leq d$. Hence $b_m = k - c_m = k - b_{m+1} = c_{m+1}$. The other direction follows by Proposition 3.19. $\square$

Corollary 3.21. *Let Γ be an antipodal bipartite distance-regular graph with odd diameter d. Then $k_i = k_{d-i}$ for all i $(0 \leq i \leq d)$. In particular, $k_d = 1$.*

Proof. For $0 \leq i \leq d$, we have

$$k_{d-i} = \frac{b_0 b_1 \cdots b_{i-1} b_i b_{i+1} \cdots b_{d-i-1}}{c_1 c_2 \cdots c_i c_{i+1} c_{i+2} \cdots c_{d-i}} = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} = k_i$$

where the second equality follows from Corollary 3.20. In particular, $k_d = k_0 = 1$. $\square$

Corollary 3.22. *Let Γ be an antipodal distance-regular graph with even diameter d. Then $k_i = k_{d-i} b_{\frac{d}{2}} / c_{\frac{d}{2}}$ for all i $(\frac{d}{2} < i \leq d)$. In particular, $k_d = b_{\frac{d}{2}} / c_{\frac{d}{2}}$.*

Proof. For $\frac{d}{2} < i \leq d$, we have

$$k_i = \frac{b_0 b_1 \cdots b_{\frac{d}{2}-1} b_{\frac{d}{2}} b_{\frac{d}{2}+1} \cdots b_{i-1}}{c_1 c_2 \cdots c_{\frac{d}{2}} c_{\frac{d}{2}+1} c_{\frac{d}{2}+2} \cdots c_i} = \frac{b_0 b_1 \cdots b_{d-i-1}}{c_1 c_2 \cdots c_{d-i}} \cdot \frac{b_{\frac{d}{2}}}{c_{\frac{d}{2}}} = k_{d-i} \cdot \frac{b_{\frac{d}{2}}}{c_{\frac{d}{2}}}$$

where the second equality follows from Proposition 3.19. In particular, $k_d = k_0 \cdot \frac{b_{\frac{d}{2}}}{c_{\frac{d}{2}}} = \frac{b_{\frac{d}{2}}}{c_{\frac{d}{2}}}$. $\square$

Chapter 4

Merging the first and third classes in bipartite distance-regular graphs

In this chapter, we start by investigating some properties of $\Gamma_{1,3}$, where Γ is bipartite or distance-regular. Then, we determine when merging the first and third classes in a bipartite distance-regular graph produces a distance-regular graph.

For a connected bipartite graph, the distance between two vertices in the same part is even and the distance between two vertices in different parts is odd. In particular, let x be a vertex of a connected bipartite graph Γ. Then for vertices y and z of Γ, $d_\Gamma(y,z)$ is even if and only if $d_\Gamma(x,y)$ and $d_\Gamma(x,z)$ have the same parity, and $d_\Gamma(y,z)$ is odd otherwise.

Throughout this chapter, we denote $\Gamma' := \Gamma_{1,3}$, $c'_i(x,y) = |\Gamma'_{i-1}(x) \cap \Gamma'_1(y)|$, $a'_i(x,y) = |\Gamma'_i(x) \cap \Gamma'_1(y)|$ and $b'_i(x,y) = |\Gamma'_{i+1}(x) \cap \Gamma'_1(y)|$ for $i \in \{0, 1, 2, \ldots, \mathrm{diam}(\Gamma')\}$.

Proposition 4.1. *If Γ is a bipartite graph, then $\Gamma_{1,3}$ is also bipartite with the same bipartition.*

Proof. Let Γ be a bipartite graph. Let $V_1 \cup V_2$ be a bipartition of Γ. Let x and y be two adjacent vertices in $\Gamma_{1,3}$. Thus $d_\Gamma(x,y) = 1$ or 3, so $x \in V_i$ and $y \in V_j$ where $i \neq j$. Consequently, $\Gamma_{1,3}$ is a bipartite graph with bipartition $V(\Gamma_{1,3}) = V_1 \cup V_2$. $\square$

Proposition 4.2. *If Γ is a distance-regular graph, then $\Gamma' := \Gamma_{1,3}$ is $(k_1 + k_3)$-regular.*

Proof. Let Γ be a distance-regular graph. Let $x \in V(\Gamma)$. Then $\Gamma'_1(x) = \Gamma_1(x) \cup \Gamma_3(x)$ so the degree of x in Γ' is $|\Gamma_1(x) \cup \Gamma_3(x)| = k_1 + k_3$. $\square$

A cycle of even length is a bipartite distance-regular graph. We first characterize when merging the first and third classes in a cycle produces a distance-regular graph.

Proposition 4.3. *Let* Γ *be a cycle* C_{2d} *where* $d \geq 3$*. Then* $\Gamma' := \Gamma_{1,3}$ *is distance-regular (with diameter* $\lceil \frac{d+2}{3} \rceil$*) if and only if* $d \leq 5$*.*

Proof. For $\Gamma \in \{C_6, C_8, C_{10}\}$, it is easy to see that $\Gamma_{1,3}$ is distance-regular. Now let $\Gamma = C_{2d}$ where $d \geq 6$. Then $|V(\Gamma)| \geq 12$. Let $V(\Gamma) = \{v_1, v_2, v_3, ..., v_{2d}\}$ where v_i, v_j are adjacent if $|i - j| \equiv 1 (\text{mod } 2d)$. Note that $d_{\Gamma'}(v_1, v_3) = d_{\Gamma'}(v_1, v_5) = 2$. We have $c_2'(v_1, v_3) = |\{v_2, v_4, v_{2d}\}| = 3$ and $c_2'(v_1, v_5) = |\{v_2, v_4\}| = 2$. Hence c_2' does not exist, so $\Gamma_{1,3}$ is not a distance-regular graph. $\square$

The graphs obtained by merging the first and third classes in the cycles C_6 and C_8 are the complete bipartite graphs $K_{3,3}$ and $K_{4,4}$, respectively. For C_{10}, the resulting graph is the complement of a (2×5)-grid.

We now characterize when merging the first and third classes of a bipartite distance-regular graph produces a distance-regular graph. We divide our results according to the diameter of the original graph.

For diameter 3, Brouwer showed the following characterization. We provide the proof of this lemma that is omitted in [7].

Lemma 4.4. [7] *Let* Γ *be a distance-regular graph with diameter* 3*. Then* $\Gamma' := \Gamma_{1,3}$ *is a distance-regular graph with diameter* 2 *if and only if* $c_3(a_3 + a_2 - a_1) = b_1 a_2$.

Proof. We first prove the necessity. Assume that Γ' is a distance-regular graph. Then b_1' exists. Let $x \in V(\Gamma)$. Then $\Gamma_1'(x) = \Gamma_1(x) \cup \Gamma_3(x)$ and $\Gamma_2'(x) = \Gamma_2(x)$. We calculate b_1' from vertices in $\Gamma_1(x)$ and $\Gamma_3(x)$. Let $y \in \Gamma_1(x)$ and $z \in \Gamma_3(x)$. We have $b_1'(x, y) = p^1_{21} + p^1_{23}$ and $b_1'(x, z) = p^3_{21} + p^3_{23}$. Thus $b_1' = p^1_{21} + p^1_{23} = p^3_{21} + p^3_{23}$. By Lemma 3.12, we have $p^1_{21} = b_1$, $p^3_{21} = c_3$, $p^1_{23} = \frac{b_2 b_1}{c_1 c_2}$ and $p^3_{23} = \frac{1}{c_2}(c_3 b_2 + a_3^2 - a_1 a_3 - b_0)$. Hence

$$b_1 + \frac{b_2 b_1}{c_1 c_2} = c_3 + \frac{1}{c_2}(c_3 b_2 + a_3^2 - a_1 a_3 - b_0)$$

Using the fact that $a_i + b_i + c_i = k$ $(1 \leq i \leq 3)$, we have $a_1 + b_1 = k - 1$, $a_2 + b_2 + c_2 = k$ and $a_3 + c_3 = k$. Simplifying the above equation, we obtain $c_3(a_3 + a_2 - a_1) = b_1 a_2$ as desired.

For the sufficiency, we assume that $c_3(a_3 + a_2 - a_1) = b_1 a_2$. From the above proof, we know that $c_3(a_3 + a_2 - a_1) = b_1 a_2$ implies b_1' exists. Observe that Γ' has diameter 2. We see that $b_0' = k_1 + k_3$, $c_1' = 1$ and $c_2' = c_2 + 2b_2 + p_{33}^2$. Hence Γ' is a distance-regular graph. $\square$

For a bipartite distance-regular graph of diameter 3, the condition above always holds, so we obtain the following result.

Theorem 4.5. *Let* Γ *be a bipartite distance-regular graph with diameter* 3. *Then* $\Gamma' := \Gamma_{1,3}$ *is a distance-regular graph with diameter* 2. *Moreover,* $\Gamma_{1,3}$ *is a complete bipartite graph* $K_{m,m}$*, where* $m = 1 + k_2 = k_1 + k_3$.

Proof. Let $x \in V(\Gamma)$. Then $\Gamma_1'(x) = \Gamma_1(x) \cup \Gamma_3(x)$ and $\Gamma_2'(x) = \Gamma_2(x)$. By Lemma 4.4 and since $a_1 = a_2 = a_3 = 0$, the graph Γ' is a distance-regular graph with diameter 2. Moreover, Γ' is bipartite with bipartition $\big(\{x\} \cup \Gamma_2(x)\big) \cup \big(\Gamma_1(x) \cup \Gamma_3(x)\big)$. Since $b_0' = c_2' = k_1 + k_3$, each vertex in $\{x\} \cup \Gamma_2(x)$ is adjacent to every vertex in $\Gamma_1(x) \cup \Gamma_3(x)$. Consequently, Γ' is a complete bipartite graph K_{1+k_2, k_1+k_3}. Since Γ' is regular, we have $1 + k_2 = k_1 + k_3$. $\square$

Theorem 4.6. *Let* Γ *be a bipartite distance-regular graph with diameter* 4. *Then* $\Gamma' := \Gamma_{1,3}$ *is a distance-regular graph with diameter* 2. *Moreover,* $\Gamma_{1,3}$ *is a complete bipartite graph* $K_{m,m}$*, where* $m = 1 + k_2 + k_4 = k_1 + k_3$.

Proof. Let $x \in V(\Gamma)$. Then $\Gamma_1'(x) = \Gamma_1(x) \cup \Gamma_3(x)$ and $\Gamma_2'(x) = \Gamma_2(x) \cup \Gamma_4(x)$. It is easy to see that $b_0' = k_1 + k_3$ and $c_1' = 1$. It remains to show that b_1' and c_2' exist.

Let $y \in \Gamma_1(x)$. Since $d_\Gamma(y, u) = 1$ or 3 for all $u \in \Gamma_2(x)$ and $d_\Gamma(y, v) = 3$ for all $v \in \Gamma_4(x)$, we have $b_1'(x, y) = k_2 + k_4$. Let $z \in \Gamma_3(x)$. Since Γ is bipartite with diameter 4, we have $d_\Gamma(z, v) = 1$ or 3 for all $v \in \Gamma_2(x) \cup \Gamma_4(x) = \Gamma_2'(x)$. Thus $b_1'(x, z) = k_2 + k_4$. It follows that $b_1' = k_2 + k_4$.

Let $u \in \Gamma_2(x)$. Since Γ is bipartite with diameter 4, we have $d_\Gamma(u, y) = 1$ or 3 for all $y \in \Gamma_1(x) \cup \Gamma_3(x) = \Gamma_1'(x)$. Thus $c_2'(x, u) = k_1 + k_3$. Let $v \in \Gamma_4(x)$. Since $d_\Gamma(v, y) = 3$ for all $y \in \Gamma_1(x)$ and $d_\Gamma(v, z) = 1$ or 3 for all $z \in \Gamma_3(x)$, we have $c_2'(x, v) = k_1 + k_3$. Thus $c_2' = k_1 + k_3$.

Thus Γ' is a distance-regular graph with diameter 2. Moreover, the graph Γ' is bipartite with bipartition $\big(\{x\} \cup \Gamma_2(x) \cup \Gamma_4(x)\big) \cup \big(\Gamma_1(x) \cup \Gamma_3(x)\big)$. Since $b_0' = c_2' = k_1 + k_3$, each vertex in $\{x\} \cup \Gamma_2(x) \cup \Gamma_4(x)$ is adjacent to every vertex in $\Gamma_1(x) \cup \Gamma_3(x)$.

Consequently, Γ' is a complete bipartite graph $K_{1+k_2+k_4,k_1+k_3}$. Since Γ' is regular, we have $1 + k_2 + k_4 = k_1 + k_3$. □

Theorem 4.7. *Let* Γ *be a bipartite distance-regular graph with diameter* 5*. Then* $\Gamma' := \Gamma_{1,3}$ *is a distance-regular graph (with diameter* 3*) if and only if* $b_2 = b_4c_3$*. In this case,* $\Gamma_{1,3}$ *has the intersection array* $\{k_1 + k_3, k_1 + k_3 - 1, p^2_{53}; 1, k_1 + k_3 - p^2_{53}, k_1 + k_3\}$.

Proof. We first prove the necessity. Assume that Γ' is a distance-regular graph. Then b'_2 exists. Let $x \in V(\Gamma)$. Then $\Gamma'_1(x) = \Gamma_1(x) \cup \Gamma_3(x)$, $\Gamma'_2(x) = \Gamma_2(x) \cup \Gamma_4(x)$ and $\Gamma'_3(x) = \Gamma_5(x)$. We calculate b'_2 from vertices in $\Gamma_2(x)$ and $\Gamma_4(x)$. Let $y \in \Gamma_2(x)$ and $z \in \Gamma_4(x)$. We have $b'_2(x,y) = p^2_{53}$ and $b'_2(x,z) = b_4 + p^4_{53}$. Thus $b'_2 = p^2_{53} = b_4 + p^4_{53}$. By Lemma 3.12, we have $p^2_{53} = \frac{b_2b_3b_4}{c_1c_2c_3}$ and $p^4_{53} = \frac{1}{c_3}\left[\frac{1}{c_2}(c_4b_3 + b_4c_5 - b_0)b_4 - b_1b_4\right]$. Simplifying the above equation, we have $b_2b_3 = c_2c_3 + c_4b_3 + b_4c_5 - b_0 - b_1c_2$. Since Γ is bipartite, $c_i = k - b_i$ $(1 \le i \le 5)$. Substituting $c_i = k - b_i$ and $b_1 = k - 1$ in the above equation, we have $b_2 = b_4(k - b_3) = b_4c_3$ as desired.

For the sufficiency, we assume that $b_2 = b_4c_3$. From the above proof, we know that $b_2 = b_4c_3$ implies b'_2 exists and $b'_2 = p^2_{53}$. Observe that Γ' has diameter 3. We see that $b'_0 = k_1 + k_3$ and $c'_1 = 1$. By Proposition 4.1 and Proposition 4.2, Γ' is bipartite and $(k_1 + k_3)$-regular. Therefore $k_1 + k_3 = b'_i(x,y) + c'_i(x,y)$ for $0 \le i \le 3$ and $x \in V(\Gamma)$, $y \in \Gamma'_i(x)$. So $c'_3 = k_1 + k_3$. It remains to show that b'_1 and c'_2 exist. Let $x \in V(\Gamma)$. We compare $b'_1(x,y)$ and $b'_1(x,z)$, where $y \in \Gamma_1(x)$ and $z \in \Gamma_3(x)$. Since $c'_1 = 1$, we have $b'_1(x,y) = k_1 + k_3 - c'_1(x,y) = k_1 + k_3 - 1 = k_1 + k_3 - c'_1(x,z) = b'_1(x,z)$. It follows that $b'_1 = k_1 + k_3 - 1$. Also, we compare $c'_2(x,u)$ and $c'_2(x,v)$, where $u \in \Gamma_2(x)$ and $v \in \Gamma_4(x)$. Since $b'_2 = p^2_{53}$, we have $c'_2(x,u) = k_1 + k_3 - b'_2(x,u) = k_1 + k_3 - p^2_{53} = k_1 + k_3 - b'_2(x,v) = c'_2(x,v)$. It follows that $c'_2 = k_1 + k_3 - p^2_{53}$. Hence Γ' is a distance-regular graph. □

Lemma 4.8. *If* Γ *is an antipodal bipartite distance-regular graph with diameter* 5*, then* $b_2 = c_3$ *and* $b_4 = 1$*. In particular,* $\Gamma_{1,3}$ *is distance-regular.*

Proof. By Corollary 3.20 and Theorem 4.7. □

Lemma 4.9. *Let* Γ *be a bipartite distance-regular graph with diameter* 5 *such that* $b_2 = b_4c_3$*. Then* $b_4 = 1$ *if and only if* Γ *is antipodal.*

Proof. The sufficiency result follows from Lemma 4.8. To prove the necessity, let Γ be a bipartite distance-regular graph with diameter 5 such that $b_2 = b_4c_3$ and $b_4 = 1$. Then $b_2 = c_3$ and $b_3 = c_2$. Since $b_4 = 1 = c_1$, we have $b_1 = k - 1 = c_4$. Since Γ is a bipartite distance-regular graph, $b_0 = c_5$. Thus Γ is antipodal. $\square$

Corollary 4.10. *Let Γ be a bipartite distance-regular graph with diameter 5 such that $b_4 = 1$. Then $\Gamma_{1,3}$ is a distance-regular graph (with diameter 3) if and only if Γ is antipodal. In this case, $\Gamma_{1,3}$ is the complement of a $2 \times (k_1 + k_3 + 1)$-grid.*

Proof. Let Γ be a bipartite distance-regular graph with diameter 5 such that $b_4 = 1$. By Theorem 4.7, Lemma 4.8 and Lemma 4.9, the graph $\Gamma_{1,3}$ is distance-regular if and only if Γ is antipodal. In this case, by Theorem 4.7, the graph $\Gamma_{1,3}$ is a bipartite antipodal graph with diameter 3 having intersection array $\{k_1 + k_3, k_1 + k_3 - 1, 1; 1, k_1+k_3-1, k_1+k_3\}$. By [1, p.432], we have $\Gamma_{1,3}$ is the complement of a $2\times(k_1+k_3+1)$-grid. $\square$

Theorem 4.11. *Let Γ be a bipartite distance-regular graph with diameter $d \geq 6$. Then $\Gamma' := \Gamma_{1,3}$ is not a distance-regular graph.*

Proof. Let $x \in V(\Gamma)$. Since $d \geq 6$, then $\Gamma'_1(x) = \Gamma_1(x) \cup \Gamma_3(x)$, $\Gamma'_2(x) = \Gamma_2(x) \cup \Gamma_4(x) \cup \Gamma_6(x)$ and $\Gamma_5(x) \subseteq \Gamma'_3(x)$. Suppose that Γ' is a distance-regular graph. Then b'_2 exists. We calculate b'_2 from vertices in $\Gamma_2(x)$ and $\Gamma_4(x)$. Let $y \in \Gamma_2(x)$ and $z \in \Gamma_4(x)$. By Lemma 3.12, we have $p^2_{53} = \frac{b_2b_3b_4}{c_1c_2c_3}$ and $p^4_{53} = \frac{1}{c_3}\left[\frac{1}{c_2}(c_4b_3 + b_4c_5 - b_0)b_4 + \frac{b_4b_5c_6}{c_1c_2} - b_1b_4\right]$ for $d \geq 6$ and $p^4_{73} = \frac{b_4b_5b_6}{c_1c_2c_3}$ for $d \geq 7$. We distinguish two cases.

Case 1 : $d = 6$.

We have $b'_2(x, y) = p^2_{53}$ and $b'_2(x, z) = b_4 + p^4_{53}$. Thus $b'_2 = p^2_{53} = b_4 + p^4_{53}$. Simplifying the above equation, we have

$$b_2b_3 = c_2c_3 + c_4b_3 + b_4c_5 - b_0 + b_5c_6 - b_1c_2. \tag{4.1}$$

Case 2 : $d \geq 7$.

We have $b'_2(x, y) = p^2_{53}$ and $b'_2(x, z) = p^4_{73} + b_4 + p^4_{53}$. Thus $b'_2 = p^2_{53} = p^4_{73} + b_4 + p^4_{53}$. Simplifying the above equation, we have

$$b_2b_3 = b_5b_6 + c_2c_3 + c_4b_3 + b_4c_5 - b_0 + b_5c_6 - b_1c_2. \tag{4.2}$$

Since Γ is bipartite, $b_i = k - c_i$ $(0 \leq i \leq d)$. Substituting b_i by $k - c_i$ in equations (4.1) and (4.2), we have $c_4(c_3 + c_5) = (c_4 + c_3 - 1)k + c_2$ for $d \geq 6$. Observe that

$$\begin{aligned} c_4(c_3+c_5) &\leq c_4(c_3+k-1) && = c_4k + c_4(c_3-1) \\ &\leq c_4k + k(c_3-1) && < c_4k + k(c_3-1) + c_2 \\ &= (c_4+c_3-1)k + c_2. \end{aligned}$$

It follows that $c_4(c_3 + c_5) < (c_4 + c_3 - 1)k + c_2$, a contradiction. Hence, $\Gamma_{1,3}$ is not a distance-regular graph. □

We summarize our results in the last theorem.

Theorem 4.12. *Let Γ be a bipartite distance-regular graph with diameter $d \geq 3$. Then $\Gamma_{1,3}$ is a distance-regular graph if and only if one of the following conditions holds:*

(i) $d \leq 4$,

(ii) $d = 5$ *and* $b_2 = b_4c_3$.

Chapter 5

Merging the first m classes in bipartite distance-regular graphs

In this chapter, we start by investigating some properties of $\Gamma_{1,2}$, where Γ is a bipartite distance-regular. Then, we characterize when merging the first and second classes in a bipartite distance-regular graph produces a distance-regular graph.

Next, we describe some properties of $\Gamma_{1,2,\ldots,m}$, where Γ is bipartite distance-regular and $m \geq 3$. Then we determine when merging the first m classes in a bipartite distance-regular graph produces a distance-regular graph.

Throughout this chapter, we denote $\Gamma' := \Gamma_{1,2,\ldots,m}$, $c'_i(x,y) = |\Gamma'_{i-1}(x) \cap \Gamma'_1(y)|$, $a'_i(x,y) = |\Gamma'_i(x) \cap \Gamma'_1(y)|$ and $b'_i(x,y) = |\Gamma'_{i+1}(x) \cap \Gamma'_1(y)|$ for $i \in \{0, 1, 2, \ldots, \mathrm{diam}(\Gamma')\}$.

5.1 Merging the first and second classes in bipartite distance-regular graphs

In this section, we determine when merging the first and second classes in a bipartite distance-regular graph produces a distance-regular graph.

Observe that for a bipartite distance-regular graph Γ with diameter d, the graph $\Gamma' = \Gamma_{1,2}$ is $(k_1 + k_2)$-regular with diameter $\lfloor \frac{d+1}{2} \rfloor$ and $\Gamma'_i(x) = \Gamma_{2i-1}(x) \cup \Gamma_{2i}(x)$ for $1 \leq i \leq \lfloor \frac{d+1}{2} \rfloor$.

For the case $d = 2$, the graph $\Gamma_{1,2}$ is a complete graph of order $k_1 + k_2 + 1$ which is a distance-regular graph. Now we consider the case $d \geq 3$.

Throughout this section, we denote $\mu = c_2$ and $\lambda = a_1$.

First we discuss the characterization result from [1]. We provide the proof of this theorem that is omitted in [1].

Proposition 5.1. [1, Proposition 4.2.18] *Let Γ be a distance-regular graph with diam-*

eter d. Then $\Gamma' := \Gamma_{1,2}$ is distance-regular if and only if we have

$$b_{j-1} + c_{j+1} - a_j = k + \mu - \lambda \qquad \text{for } 2 \le j \le d-1. \tag{5.1}$$

If this is the case, then $\Gamma_{1,2}$ has diameter $\lfloor \frac{d+1}{2} \rfloor$ and parameters

$$b'_j = \begin{cases} b_{2j-1}b_{2j}/\mu & \text{if } 1 \le j \le \lfloor \frac{d+1}{2} \rfloor \\ k_1 + k_2 & \text{if } j = 0, \end{cases},$$

$$c'_j = \begin{cases} c_d(k+\mu-\lambda+a_d-b_{d-1})/\mu & \text{if } j = \frac{d+1}{2} \text{ and } d \text{ is odd} \\ c_{2j-1}c_{2j}/\mu & \text{otherwise} \end{cases}.$$

Proof. We first prove the necessity. Assume that Γ' is a distance-regular graph with diameter $d' = \lfloor \frac{d+1}{2} \rfloor$. Then b'_i and c'_i exist for $1 \le i \le d'$. Let $x \in V(\Gamma)$. Then $\Gamma'_i(x) = \Gamma_{2i-1}(x) \cup \Gamma_{2i}(x)$ and $\Gamma_d(x) \subseteq \Gamma'_{d'}(x)$ for $1 \le i \le d'$. We calculate b'_i $(1 \le i \le d'-2)$ from vertices in $\Gamma_{2i-1}(x)$ and $\Gamma_{2i}(x)$. Let $y \in \Gamma_{2i-1}(x)$ and $z \in \Gamma_{2i}(x)$. We have $b'_i(x,y) = p^{2i-1}_{2i+1,2}$ and $b'_i(x,z) = p^{2i}_{2i+1,1} + p^{2i}_{2i+1,2} + p^{2i}_{2i+2,2}$. Thus

$$b'_i = p^{2i-1}_{2i+1,2} = p^{2i}_{2i+1,1} + p^{2i}_{2i+1,2} + p^{2i}_{2i+2,2}. \tag{5.2}$$

By Lemma 3.12, we have $p^{2i}_{2i+1,1} = b_{2i}$. By Corollary 3.13, we have $p^{2i-1}_{2i+1,2} = \frac{b_{2i-1}b_{2i}}{c_2}$, $p^{2i}_{2i+1,2} = \frac{b_{2i}(a_{2i+1}+a_{2i}-a_1)}{c_2}$ and $p^{2i}_{2i+2,2} = \frac{b_{2i}b_{2i+1}}{c_2}$. Simplifying equation (5.2) by using the fact that $a_i + b_i + c_i = k$ $(1 \le i \le d)$, we obtain $b_{2i-1} + c_{2i+1} - a_{2i} = k + c_2 - a_1$ $(1 \le i \le d'-2)$. That is $b_{j-1} + c_{j+1} - a_j = k + \mu - \lambda$ for $2 \le j \le d-1$. We also calculate c'_i $(2 \le i \le d'-1)$ from vertices in $\Gamma_{2i-1}(x)$ and $\Gamma_{2i}(x)$. Let $y \in \Gamma_{2i-1}(x)$ and $z \in \Gamma_{2i}(x)$. We have $c'_i(x,y) = p^{2i-1}_{2i-2,1} + p^{2i-1}_{2i-2,2} + p^{2i-1}_{2i-3,2}$ and $c'_i(x,z) = p^{2i}_{2i-2,2}$. Thus

$$c'_i = p^{2i-1}_{2i-2,1} + p^{2i-1}_{2i-2,2} + p^{2i-1}_{2i-3,2} = p^{2i}_{2i-2,2}. \tag{5.3}$$

By Lemma 3.12, we have $p^{2i-1}_{2i-2,1} = c_{2i-1}$. By Corollary 3.13, we have $p^{2i-1}_{2i-2,2} = \frac{c_{2i-1}(a_{2i-2}+a_{2i-1}-a_1)}{c_2}$, $p^{2i-1}_{2i-3,2} = \frac{c_{2i-2}c_{2i-1}}{c_2}$ and $p^{2i}_{2i-2,2} = \frac{c_{2i-1}c_{2i}}{c_2}$. Simplifying equation (5.3) by using the fact that $a_i + b_i + c_i = k$ $(1 \le i \le d)$, we obtain $b_{2i-2} + c_{2i} - a_{2i-1} = k + c_2 - a_1$ $(2 \le i \le d'-1)$. That is $b_{j-1} + c_{j+1} - a_j = k + \mu - \lambda$ for $2 \le j \le d-1$.

For the sufficiency, we assume that $b_{j-1} + c_{j+1} - a_j = k + \mu - \lambda$ for $2 \le j \le d-1$. Observe that Γ' has diameter $\lfloor \frac{d+1}{2} \rfloor$. From the above proof, we know that

$b_{j-1}+c_{j+1}-a_j = k+\mu-\lambda$ for $2 \le j \le d-1$ implies $b'_j = b_{2j-1}b_{2j}/\mu$ for $1 \le j \le \lfloor \frac{d+1}{2} \rfloor$ and $c'_j = c_{2j-1}c_{2j}/\mu$ for $2 \le j \le \frac{d}{2}$. We see that $b'_0 = k_1 + k_2$, $c'_1 = 1$ and if $j = \frac{d+1}{2}$ and d is odd, $c'_j = c_d(k+\mu-\lambda+a_d-b_{d-1})/\mu$. Hence, Γ' is distance-regular. $\square$

Note that (5.1) is equivalent to

$$b_j + c_{j+1} = k+1 \qquad \text{where } j \text{ is even} \quad (0 \le j \le d-1),$$

and

$$b_j + c_{j+1} = b_1 + \mu \qquad \text{where } j \text{ is odd} \quad (1 \le j \le d-1).$$

For bipartite graphs, we can rewrite Proposition 5.1 into the following simpler form.

Corollary 5.2. *Let Γ be a bipartite distance-regular graph with diameter $d \ge 3$. Then the following statements are equivalent.*

(i) $\Gamma_{1,2}$ *is a distance-regular graph.*

(ii) $b_j = \mu + b_{j+2}$ *for* $1 \le j \le d-2$.

(iii) $c_j = \mu + c_{j-2}$ *for* $3 \le j \le d$.

Proof. (i) $\Longleftrightarrow$ (ii) By Proposition 5.1 and the fact that $a_i = 0$ and $b_i = k - c_i$ $(1 \le i \le d)$.
(ii) $\Longleftrightarrow$ (iii) By the fact that $a_i = 0$ and $b_i = k - c_i$ $(1 \le i \le d)$. $\square$

Corollary 5.3. *Let Γ be a bipartite distance-regular graph with even diameter $d \ge 4$. Then $\Gamma_{1,2}$ is a distance-regular graph if and only if Γ has intersection array*

$$b_{2j} = \tfrac{d-2j}{2}\mu, \;\; b_{2j+1} = \tfrac{d-2j}{2}\mu - 1 \quad (0 \le j \le \tfrac{d}{2}-1)$$

$$c_{2j-1} = (j-1)\mu + 1, \;\; c_{2j} = j\mu \quad (1 \le j \le \tfrac{d}{2})$$

Proof. From Corollary 5.2, the graph $\Gamma_{1,2}$ is a distance-regular graph if and only if $c_j = \mu + c_{j-2}$ for $3 \le j \le d$. In this case, since $c_1 = 1$ and $c_2 = \mu$, we have $c_{2j-1} = (j-1)\mu + 1$ and $c_{2j} = j\mu$ for $1 \le j \le \frac{d}{2}$. In particular, $k = c_d = \frac{d}{2}\mu$. Since $b_i = k - c_i$ $(0 \le i \le d)$, we have $b_{2j} = k - c_{2j} = \frac{d}{2}\mu - j\mu = \frac{d-2j}{2}\mu$ and $b_{2j+1} = k - c_{2j+1} = \frac{d}{2}\mu - (j\mu+1) = \frac{d-2j}{2}\mu - 1$. Hence, Γ has the desired intersection array. $\square$

Corollary 5.4. *Let Γ be a bipartite distance-regular graph with odd diameter $d \geq 3$. Then $\Gamma_{1,2}$ is a distance-regular graph if and only if Γ has intersection array*

$$b_{2j} = \tfrac{d-2j-1}{2}\mu + 1, \;\; b_{2j+1} = \tfrac{d-2j-1}{2}\mu \quad (0 \leq j \leq \tfrac{d-1}{2})$$

$$c_{2j} = j\mu, \;\; c_{2j+1} = j\mu + 1 \quad (0 \leq j \leq \tfrac{d-1}{2})$$

Proof. From Corollary 5.2, the graph $\Gamma_{1,2}$ is a distance-regular graph if and only if $c_j = \mu + c_{j-2}$ for $3 \leq j \leq d$. In this case, since $c_1 = 1$ and $c_2 = \mu$, we have $c_{2j} = j\mu$ and $c_{2j+1} = j\mu + 1$ for $0 \leq j \leq \frac{d-1}{2}$. In particular, $k = c_d = \frac{d-1}{2}\mu + 1$. Since $b_i = k - c_i$ $(0 \leq i \leq d)$, we have $b_{2j} = k - c_{2j} = (\frac{d-1}{2}\mu + 1) - j\mu = \frac{d-2j-1}{2}\mu + 1$ and $b_{2j+1} = k - c_{2j+1} = (\frac{d-1}{2}\mu + 1) - (j\mu + 1) = \frac{d-2j-1}{2}\mu$. Hence, Γ has the desired intersection array. □

We recall one useful necessary condition of a distance-regular graph.

Theorem 5.5. [1, Theorem 5.4.1] *Let Γ be a distance-regular graph with diameter $d \geq 3$. If $\mu > 1$, then one of the following statements holds.*

(i) $c_3 \geq \frac{3}{2}\mu$.

(ii) $c_3 \geq \mu + b_2, d = 3$.

We now characterize when merging the first and second classes in a bipartite distance-regular graph produces a distance-regular graph.

Theorem 5.6. *Let Γ be a bipartite distance-regular graph with diameter d where $d \geq 3$. Then $\Gamma_{1,2}$ is distance-regular if and only if Γ is either the complement of a $2 \times (\mu + 2)$-grid, a doubled Odd graph of odd points or a Hamming d-cube. Moreover, the following statements hold.*

(i) *If Γ is the complement of a $2 \times (\mu + 2)$-grid, then $\Gamma_{1,2}$ is a strongly regular graph with parameters $(2\mu + 4, 2\mu + 2, 2\mu, 2\mu + 2)$ which is a $(\mu + 2)$-cocktail party graph.*

(ii) *If Γ is a doubled Odd graph of d points where d is odd, then $\Gamma_{1,2}$ is a Johnson graph $J(d + 1, \frac{d+1}{2})$.*

(iii) *If* Γ *is a Hamming* d*-cube, then* $\Gamma_{1,2}$ *is a halved* $(d+1)$*-cube.*

Proof. Let Γ be a bipartite distance-regular graph with diameter $d \geq 3$. Suppose that $\Gamma_{1,2}$ is distance-regular. We consider two cases.

Case 1: $d = 3$

By Corollary 5.4, the graph Γ has the intersection array $\{\mu+1, \mu, 1; 1, \mu, \mu+1\}$. From [1, p.432], the graph Γ is the complement of a $2 \times (\mu+2)$-grid.

Moreover, by Proposition 5.1, the graph $\Gamma_{1,2}$ has intersection array $\{2\mu+2, 1; 1, 2\mu+2\}$. That is $\Gamma_{1,2}$ is a strongly regular graph with parameters $(2\mu+4, 2\mu+2, 2\mu, 2\mu+2)$ which is a $(\mu+2)$-cocktail party graph.

Case 2: $d > 3$

Case 2.1: $\mu = 1$

If d is even, then by Corollary 5.3, the value $b_{d-1} = 0$, a contradiction. Thus d is odd. By Corollary 5.4, the graph Γ has the intersection array $\left\{\frac{d-1}{2}+1, \frac{d-1}{2}, \frac{d-3}{2}+1, \frac{d-3}{2}, ..., 2, 1, 1; 1, 1, 2, ..., \frac{d-3}{2}, \frac{d-3}{2}+1, \frac{d-1}{2}, \frac{d-1}{2}+1\right\}$ which is the same as the intersection array of a doubled Odd graph on d points [1, p.414]. Since doubled Odd graphs are characterized by their intersection array [1, Proposition 9.1.8], Γ must be a doubled Odd graph on d points.

Moreover, by Proposition 5.1, the graph $\Gamma_{1,2}$ has intersection array

$$b'_i = \left(\frac{d+1}{2} - i\right)^2 \quad \text{and} \quad c'_i = i^2 \quad \text{for} \quad 0 \leq i \leq \frac{d+1}{2}$$

which is the same as the intersection array of a Johnson graph $J(d+1, \frac{d+1}{2})$. Since Johnson graphs are characterized by their intersection array (see [15] and [16]), the graph $\Gamma_{1,2}$ must be a Johnson graph.

Case 2.2: $\mu > 1$

By Theorem 5.5, we have $c_3 \geq \frac{3}{2}\mu$. By Corollary 5.3 and Corollary 5.4, we have $c_3 = \mu + 1$. Thus $\mu \leq 2$. Since we are in the case $\mu > 1$, so we consider only the case $\mu = 2$. By Corollary 5.3 and Corollary 5.4, the graph Γ has the intersection array $\{d, d-1, d-2, d-3, ..., 3, 2, 1; 1, 2, 3, ..., d-3, d-2, d-1, d\}$ for all $d > 3$ which is the same as the intersection array of a Hamming graph d-cube [1, p.413]. Since the

Hamming graph d-cube is bipartite, by [17] it is characterized by its intersection array. Thus Γ must be the Hamming graph d-cube.

Moreover, by Proposition 5.1, the graph $\Gamma_{1,2}$ has intersection array

$$b'_i = \frac{(d-2i)(d-2i+1)}{2} \text{ and } c'_i = i(2i-1) \quad \text{for } 0 \leq i \leq \left\lfloor \frac{d+1}{2} \right\rfloor$$

which is the same as the intersection array of a halved $(d+1)$-cube. Since halved $(d+1)$-cubes are characterized by their intersection array (see [15] and [18]), the graph $\Gamma_{1,2}$ must be a halved $(d+1)$-cube.

The arguments above also show that the sufficiency holds. $\square$

5.2 Merging the first m classes in bipartite distance-regular graphs

In this section, we determine when merging the first m classes in a bipartite distance-regular graph produces a distance-regular graph where $m \geq 3$. Observe that for a bipartite distance-regular graph Γ with diameter $d = m$, the graph $\Gamma_{1,2,\ldots,m}$ is a complete graph of order $k_1 + k_2 + \ldots + k_m + 1$ which is a distance-regular graph. Now we consider the case $d \geq m+1$.

We first compute some p^h_{ij} in terms of the intersection array.

Lemma 5.7. *Let Γ be a bipartite distance-regular graph with diameter $d \geq 4$. For $3 \leq n \leq d-1$, we have*

$$p^3_{n+1,n} = \frac{b_n b_{n-1} \cdots b_3}{c_1 c_2 \cdots c_n}(b_{n+1}c_{n+2} + b_n c_{n+1} + b_{n-1}c_n - b_0 c_1 - b_1 c_2).$$

Proof. We will prove by induction on n. For $n = 3$, by Lemma 3.12 and since $a_i = 0$ for all i $(0 \leq i \leq d)$, we have

$$\begin{aligned} p^3_{43} &= \tfrac{1}{c_3}(p^3_{23}b_3 + p^3_{25}c_5 - p^3_{14}b_1) \\ &= \tfrac{1}{c_3}\left[\tfrac{b_3}{c_2}(c_3 b_2 + b_3 c_4 - b_0 c_1) + \tfrac{b_4 b_3}{c_1 c_2}c_5 - b_3 b_1\right] \\ &= \tfrac{b_3}{c_1 c_2 c_3}(b_2 c_3 + b_3 c_4 - b_0 c_1 + b_4 c_5 - b_1 c_2). \end{aligned}$$

For $n \geq 3$, assume that $p^3_{n+1,n} = \frac{b_n b_{n-1} \cdots b_3}{c_1 c_2 \ldots c_n}(b_{n+1}c_{n+2} + b_n c_{n+1} + b_{n-1}c_n - b_0 c_1 - b_1 c_2)$.

Then by Lemma 3.12, we have

$$\begin{aligned}
p^3_{n+2,n+1} &= \tfrac{1}{c_{n+1}}(p^3_{n,n+1}b_{n+1} + p^3_{n,n+3}c_{n+3} - p^3_{n-1,n+2}b_{n-1}) \\
&= \tfrac{1}{c_{n+1}}\Big[\tfrac{b_{n+1}b_n\cdots b_3}{c_1c_2\cdots c_n}(b_{n+1}c_{n+2} + b_nc_{n+1} + b_{n-1}c_n - b_0c_1 - b_1c_2) \\
&\quad + \tfrac{b_{n+2}b_{n+1}\cdots b_3}{c_1c_2\cdots c_n}c_{n+3} - \tfrac{b_{n+1}b_n\cdots b_3}{c_1c_2\cdots c_{n-1}}b_{n-1}\Big] \\
&= \tfrac{b_{n+1}b_n\cdots b_3}{c_1c_2\cdots c_{n+1}}[(b_{n+1}c_{n+2} + b_nc_{n+1} + b_{n-1}c_n - b_0c_1 - b_1c_2) \\
&\quad + b_{n+2}c_{n+3} - b_{n-1}c_n] \\
&= \tfrac{b_{n+1}b_n\cdots b_3}{c_1c_2\cdots c_{n+1}}(b_{n+2}c_{n+3} + b_{n+1}c_{n+2} + b_nc_{n+1} - b_0c_1 - b_1c_2).
\end{aligned}$$

This completes the proof of our lemma. □

Next we show that $\Gamma_{1,2,\ldots,m}$ is not distance-regular for a bipartite distance-regular graph with diameter $d \geq m+2$.

Theorem 5.8. *For $m \geq 3$, let Γ be a bipartite distance-regular graph with diameter $d \geq m+2$. Then $\Gamma' := \Gamma_{1,2,\ldots,m}$ is not distance-regular.*

Proof. Let $x \in V(\Gamma)$. We consider two cases.

Case 1: $d = m+2$

Then $\Gamma'_1(x) = \Gamma_1(x) \cup \Gamma_2(x) \cup \ldots \cup \Gamma_m(x)$ and $\Gamma'_2(x) = \Gamma_{m+1}(x) \cup \Gamma_{m+2}(x)$. Suppose that $\Gamma_{1,2,\ldots,m}$ is a distance-regular graph. Then b'_1 exists. We can calculate b'_1 from vertices in $\Gamma_1(x)$ and $\Gamma_3(x)$. Let $y \in \Gamma_1(x)$ and $z \in \Gamma_3(x)$. Since Γ is bipartite, $p^h_{ij} = 0$ whenever $i+j+h$ is odd. So we have $b'_1(x,y) = p^1_{m+1,m}$ and $b'_1(x,z) = p^3_{m+2,m-1} + p^3_{m+1,m-2} + p^3_{m+1,m}$. Thus

$$p^1_{m+1,m} = p^3_{m+2,m-1} + p^3_{m+1,m-2} + p^3_{m+1,m}. \tag{5.4}$$

By Lemma 3.12, we have $p^1_{m+1,m} = \frac{b_mb_{m-1}\cdots b_1}{c_1c_2\cdots c_m}$, $p^3_{m+2,m-1} = \frac{b_{m+1}b_m\cdots b_3}{c_1c_2\cdots c_{m-1}}$, and $p^3_{m+1,m-2} = \frac{b_mb_{m-1}\cdots b_3}{c_1c_2\cdots c_{m-2}}$. By Lemma 5.7, we have $p^3_{m+1,m} = \frac{b_mb_{m-1}\cdots b_3}{c_1c_2\cdots c_m}(b_{m+1}c_{m+2} + b_mc_{m+1} + b_{m-1}c_m - b_0c_1 - b_1c_2)$. Simplifying equation (5.4) by using $b_i = k - c_i$ $(0 \leq i \leq m+2)$ and $c_{m+2} = k$, we have $c_{m+1} = k$. Since Γ is bipartite, $b_{m+1} = k - c_{m+1} = 0$, which is a contradiction. Therefore $\Gamma_{1,2,\ldots,m}$ is not distance-regular.

Case 2: $d \geq m+3$

Then $\Gamma'_1(x) = \Gamma_1(x) \cup \Gamma_2(x) \cup \ldots \cup \Gamma_m(x)$ and $\Gamma_{m+1} \cup \Gamma_{m+2}(x) \cup \Gamma_{m+3}(x) \subset$

$\Gamma_2'(x)$. Suppose that $\Gamma_{1,2,\ldots,m}$ is a distance-regular graph. Then b_1' exists. We can calculate b_1' from vertices in $\Gamma_1(x)$ and $\Gamma_3(x)$. Let $y \in \Gamma_1(x)$ and $z \in \Gamma_3(x)$. Since Γ is bipartite, $p_{ij}^h = 0$ whenever $i+j+h$ is odd. So we have $b_1'(x,y) = p^1_{m+1,m}$ and $b_1'(x,z) = p^3_{m+3,m} + p^3_{m+2,m-1} + p^3_{m+1,m-2} + p^3_{m+1,m}$. Thus

$$p^1_{m+1,m} = p^3_{m+3,m} + p^3_{m+2,m-1} + p^3_{m+1,m-2} + p^3_{m+1,m}. \tag{5.5}$$

By Lemma 3.12, we have $p^1_{m+1,m} = \frac{b_m b_{m-1} \cdots b_1}{c_1 c_2 \cdots c_m}$, $p^3_{m+3,m} = \frac{b_{m+2} b_{m+1} \cdots b_3}{c_1 c_2 \cdots c_m}$, $p^3_{m+2,m-1} = \frac{b_{m+1} b_m \cdots b_3}{c_1 c_2 \cdots c_{m-1}}$, and $p^3_{m+1,m-2} = \frac{b_m b_{m-1} \cdots b_3}{c_1 c_2 \cdots c_{m-2}}$. Simplifying equation (5.5) using $b_i = k - c_i$ and Lemma 5.7, we have $c_{m+1} = k$. Since Γ is bipartite, $b_{m+1} = k - c_{m+1} = 0$, which is a contradiction. Therefore $\Gamma_{1,2,\ldots,m}$ is not distance-regular. □

Now it remains to consider the case that Γ is a bipartite distance-regular graph with diameter $d = m+1$ where $m \geq 3$. Since $\Gamma_{1,2,\ldots,m}$ has diameter 2, if it is distance-regular, then it is strongly regular.

We discuss some results that we will use later.

Lemma 5.9. *Let Γ be a bipartite distance-regular graph with diameter d. Then*

$$k_d = \begin{cases} p^i_{d,d-i} + p^i_{d,d-i+2} + p^i_{d,d-i+4} + \ldots + p^i_{d,d-3} + p^i_{d,d-1} & ;\ i \text{ is odd} \\ p^i_{d,d-i} + p^i_{d,d-i+2} + p^i_{d,d-i+4} + \ldots + p^i_{d,d-2} + p^i_{d,d} & ;\ i \text{ is even} \end{cases}$$

for $0 \leq i \leq d-1$.

Proof. Fix $x \in V(\Gamma)$ and let $y_i \in \Gamma_i(x)$ $(0 \leq i \leq d-1)$. Since $d(y_i, z) \geq d-i$ for $z \in \Gamma_d(x)$, we have

$$\begin{aligned} k_d &= |\Gamma_d(x)| \\ &= |\Gamma_d(x) \cap \Gamma_{d-i}(y_i)| + |\Gamma_d(x) \cap \Gamma_{d-i+1}(y_i)| + \ldots + |\Gamma_d(x) \cap \Gamma_d(y_i)| \\ &= p^i_{d,d-i} + p^i_{d,d-i+1} + \ldots + p^i_{d,d}. \end{aligned}$$

By Lemma 3.2, $p^h_{ij} = 0$ for all $h+i+j$ is odd. Thus

$$k_d = \begin{cases} p^i_{d,d-i} + p^i_{d,d-i+2} + p^i_{d,d-i+4} + \ldots + p^i_{d,d-3} + p^i_{d,d-1} & ;\ i \text{ is odd} \\ p^i_{d,d-i} + p^i_{d,d-i+2} + p^i_{d,d-i+4} + \ldots + p^i_{d,d-2} + p^i_{d,d} & ;\ i \text{ is even} \end{cases}$$

The result follows. □

Lemma 5.10. *Let Γ be a bipartite distance-regular graph with diameter d. Then $\Gamma' := \Gamma_{1,2,\ldots,d-1}$ is a strongly regular graph if and only if*

$$k_d = \begin{cases} p^i_{d,d-i} + p^i_{d,d-i+2} + p^i_{d,d-i+4} + \ldots + p^i_{d,d-3} + p^i_{d,d-1} & ;\ i \textit{ is odd} \\ p^i_{d,d-i} + p^i_{d,d-i+2} + p^i_{d,d-i+4} + \ldots + p^i_{d,d-4} + p^i_{d,d-2} & ;\ i \textit{ is even} \end{cases} \tag{5.6}$$

for $0 \leq i \leq d-1$. In this case, $b'_1 = k_d$.

Proof. We first prove the necessity. Assume that Γ' is a strongly regular graph (that is, Γ' is a distance-regular graph with diameter 2). Then b'_1 exists. Let $x \in V(\Gamma)$. Then $\Gamma'_1(x) = \Gamma_1(x) \cup \Gamma_2(x) \cup ... \cup \Gamma_{d-1}(x)$ and $\Gamma'_2(x) = \Gamma_d(x)$. We calculate b'_1 from vertices in $\Gamma_1(x), \Gamma_2(x), \ldots, \Gamma_{d-1}(x)$. Let $y_i \in \Gamma_i(x)$ for $0 \leq i \leq d-1$ and let $z \in \Gamma_d(x)$. By Lemma 3.2, $p^h_{ij} = 0$ whenever $h+i+j$ is odd. So we have

$$b'_1(x, y_i) = \begin{cases} p^i_{d,d-i} + p^i_{d,d-i+2} + p^i_{d,d-i+4} + \ldots + p^i_{d,d-3} + p^i_{d,d-1} & ;\ i \text{ is odd} \\ p^i_{d,d-i} + p^i_{d,d-i+2} + p^i_{d,d-i+4} + \ldots + p^i_{d,d-4} + p^i_{d,d-2} & ;\ i \text{ is even} \end{cases}$$

for $0 \leq i \leq d-1$. Thus

$$\begin{aligned} b'_1 &= p^1_{d,d-1} = p^2_{d,d-2} = p^3_{d,d-3} + p^3_{d,d-1} = p^4_{d,d-4} + p^4_{d,d-2} = \ldots \\ &= \begin{cases} p^{d-1}_{d1} + p^{d-1}_{d3} + p^{d-1}_{d5} + \ldots + p^{d-1}_{d,d-3} + p^{d-1}_{d,d-1} & ;\ d \text{ is even} \\ p^{d-1}_{d1} + p^{d-1}_{d3} + p^{d-1}_{d5} + \ldots + p^{d-1}_{d,d-4} + p^{d-1}_{d,d-2} & ;\ d \text{ is odd} \end{cases} \end{aligned}$$

In particular, $b'_1 = p^1_{d,d-1} = k_d$. The result follows.

For the sufficiency, we assume that equation (5.6) holds. From the above proof, we know that equation (5.6) implies b'_1 exists and $b'_1 = k_d$. Observe that Γ' has diameter 2. We see that $b'_0 = k_1 + k_2 + \ldots + k_{d-1}$, $c'_1 = 1$ and

$$c'_2 = \begin{cases} p^d_{1,d-1} + p^d_{2,d-2} + p^d_{3,d-3} + p^d_{3,d-1} + ... + p^d_{d-1,1} + p^d_{d-1,3} + ... + p^d_{d-1,d-1} & ;\ d \text{ is even} \\ p^d_{1,d-1} + p^d_{2,d-2} + p^d_{3,d-3} + p^d_{3,d-1} + ... + p^d_{d-1,1} + p^d_{d-1,3} + ... + p^d_{d-1,d-2} & ;\ d \text{ is odd} \end{cases}$$

Hence, Γ' is a strongly regular graph. □

We now characterize when merging the first $d-1$ classes in a bipartite distance-regular graph with diameter d produces a distance-regular graph.

Theorem 5.11. *Let Γ be a bipartite distance-regular graph with diameter $d \geq 4$. Then $\Gamma' := \Gamma_{1,2,\ldots,d-1}$ is a strongly regular graph if and only if Γ is antipodal. In this case, Γ' is a strongly regular graph with parameters $(1 + k_1 + k_2 + \ldots + k_d, k_1 + k_2 + \ldots + k_{d-1}, k_1 + k_2 + \ldots + k_{d-1} - k_d - 1, k_1 + k_2 + \ldots + k_{d-1})$ which corresponds to the complete multipartite graph $K_{n\times(1+k_d)}$ where $n = (1 + k_1 + k_2 + \ldots + k_d)/(1 + k_d)$. Moreover, if d is odd, or d is even and $b_{\frac{d}{2}} = c_{\frac{d}{2}}$, then Γ' is an n-cocktail party graph.*

Proof. By Proposition 3.18, Lemma 5.9 and Lemma 5.10, the graph $\Gamma_{1,2,\ldots,d-1}$ is a strongly regular graph if and only if Γ is antipodal. In this case, Γ' is a strongly regular graph with parameters $v' = 1 + k_1 + k_2 + k_3 + \ldots + k_d$, $k' = k_1 + k_2 + k_3 + \ldots + k_{d-1}$, $\lambda' = a'_1 = k_1 + k_2 + k_3 + \ldots + k_{d-1} - b'_1 - c'_1 = k_1 + k_2 + k_3 + \ldots + k_{d-1} - k_d - 1$. Since Γ is antipodal, Γ_d is a disjoint union of complete graphs. Thus for a fixed vertex x in Γ, for each $y \in \Gamma_d(x)$, there is no vertex in $\Gamma_i(x)$ $(1 \leq i \leq d - 1)$ which has distance d from y. That is, $d_\Gamma(y, z) \leq d - 1$ for any $z \in \Gamma_1(x) \cup \Gamma_2(x) \cup \ldots \cup \Gamma_{d-1}(x)$. It follows that $\mu' = c'_2 = k_1 + k_2 + k_3 + \ldots + k_{d-1} = k'$. Hence, Γ' is a strongly regular graph with parameters $(1 + k_1 + k_2 + \ldots + k_d, k_1 + k_2 + \ldots + k_{d-1}, k_1 + k_2 + \ldots + k_{d-1} - k_d - 1, k_1 + k_2 + \ldots + k_{d-1})$ which is a complete multipartite graph $K_{n\times(1+k_d)}$ where $n = (1 + k_1 + k_2 + \ldots + k_d)/(1 + k_d)$. Moreover, if d is odd, or d is even and $b_{\frac{d}{2}} = c_{\frac{d}{2}}$, by Corollary 3.21 and Corollary 3.22, we have $k_d = 1$. It follows that Γ' is an n-cocktail party graph where $n = (1 + k_1 + k_2 + \ldots + k_d)/(1 + k_d)$. □

Corollary 5.12. *Let Γ be a bipartite antipodal distance-regular graph with even diameter d. Then $(1 + k_d) \mid k_{\frac{d}{2}}$.*

Proof. By Theorem 5.11, we have $(1+k_d) \mid (1+k_1+k_2+\ldots+k_d)$. By Corollary 3.22, the result follows. □

Finally, we give some families of distance-regular graphs Γ with diameter d that both are bipartite and antipodal, and the resulting graphs $\Gamma_{1,2,\ldots,d-1}$ obtained by Theorem 5.11.

- Let Γ be a cycle on $2d$ vertices $(d \geq 4)$. Then Γ is a distance-regular graph with parameters $k_0 = k_d = 1$, $k_i = 2$ $(1 \leq i \leq d - 1)$. Thus $\Gamma_{1,2,\ldots,d-1}$ is a strongly regular graph with parameters $(2d, 2d-2, 2d-4, 2d-2)$ which is a d-cocktail party graph.

- Let Γ be a Hamming d-cube $(d \geq 4)$. Then Γ is a distance-regular graph with parameters $k_i = \binom{d}{i}$ $(0 \leq i \leq d)$. Thus $\Gamma_{1,2,\ldots,d-1}$ is a strongly regular graph with parameters $(2^d, 2^d-2, 2^d-4, 2^d-2)$ which is a 2^{d-1}-cocktail party graph.

- Let Γ be a doubled Odd graph on $2m+1$ points $(m \geq 2)$. Then Γ is a distance-regular graph with parameters $k_0 = k_d = 1$, and $k_1+k_2+\ldots+k_{d-1} = 2\binom{2m+1}{m}-2$. Thus $\Gamma_{1,2,\ldots,d-1}$ is a strongly regular graph with parameters $\left(2\binom{2m+1}{m}, 2\binom{2m+1}{m}-2, 2\binom{2m+1}{m}-4, 2\binom{2m+1}{m}-2\right)$ which is a $\binom{2m+1}{m}$-cocktail party graph.

- Let Γ be a graph which has intersection array $\{rc_2, rc_2-1, (r-1)c_2, 1; 1, c_2, rc_2-1, rc_2\}$ where $r \geq 2$ (See [1, p.425]). Then Γ is a distance-regular graph with diameter 4 and parameters $k_0 = 1$, $k_1 = rc_2$, $k_2 = r(rc_2-1)$, $k_3 = rc_2(r-1)$, and $k_4 = r-1$. Thus $\Gamma_{1,2,3}$ is a strongly regular graph with parameters $(2r^2c_2, 2r^2c_2-r, 2r^2c_2-2r, 2r^2c_2-r)$ which is a complete multipartite graph $K_{2rc_2 \times r}$.

Chapter 6

Conclusions

When merging the first and third classes in bipartite distance-regular graphs, we have the following main results.

(1) Let Γ be a bipartite distance-regular graph with diameter 3. Then $\Gamma_{1,3}$ is a distance-regular graph with diameter 2. Moreover $\Gamma_{1,3}$ is a complete bipartite graph $K_{m,m}$ where $m = 1 + k_2 = k_1 + k_3$.

(2) Let Γ be a bipartite distance-regular graph with diameter 4. Then $\Gamma_{1,3}$ is a distance-regular graph with diameter 2. Moreover $\Gamma_{1,3}$ is a complete bipartite graph $K_{m,m}$ where $m = 1 + k_2 + k_4 = k_1 + k_3$.

(3) Let Γ be a bipartite distance-regular graph with diameter 5. Then $\Gamma_{1,3}$ is a distance-regular graph (with diameter 3) if and only if $b_2 = b_4 c_3$. In this case, $\Gamma_{1,3}$ has intersection array $\{k_1 + k_3, k_1 + k_3 - 1, p^2_{53}; 1, k_1 + k_3 - p^2_{53}, k_1 + k_3\}$. Moreover, if $b_4 = 1$, then $\Gamma_{1,3}$ is a distance-regular graph if and only if Γ is antipodal. In this case, $\Gamma_{1,3}$ is the complement of a $2 \times (k_1 + k_3 + 1)$-grid.

(4) If Γ is a bipartite distance-regular graph with diameter $d \geq 6$, then $\Gamma_{1,3}$ is not a distance-regular graph.

When merging the first m classes in bipartite distance-regular graphs where $m \geq 2$, we have the following main results.

(1) Let Γ be a bipartite distance-regular graph with diameter $d \geq 3$. Then $\Gamma_{1,2}$ is distance-regular if and only if Γ is either the complement of a $2 \times (\mu + 2)$-grid, a doubled Odd graph on odd points or a Hamming d-cube. Moreover, the following statements hold.

(i) If Γ is the complement of a $2 \times (\mu + 2)$-grid, then $\Gamma_{1,2}$ is a strongly regular graph with parameters $(2\mu+4, 2\mu+2, 2\mu, 2\mu+2)$ which is a $(\mu+2)$-cocktail party graph.

(ii) If Γ is a doubled Odd graph on d points where d is odd, then $\Gamma_{1,2}$ is a Johnson graph $J(d+1, \frac{d+1}{2})$.

(iii) If Γ is a Hamming d-cube, then $\Gamma_{1,2}$ is a halved $(d+1)$-cube.

(2) For $m \geq 3$, let Γ be a bipartite distance-regular graph with diameter $d \geq m+2$. Then $\Gamma_{1,2,\ldots,m}$ is not distance-regular.

(3) Let Γ be a bipartite distance-regular graph with diameter $d \geq 4$. Then $\Gamma' := \Gamma_{1,2,\ldots,d-1}$ is a strongly regular graph if and only if Γ is antipodal. In this case, Γ' is a strongly regular graph with parameters $(1+k_1+k_2+\ldots+k_d, k_1+k_2+\ldots+k_{d-1}, k_1+k_2+\ldots+k_{d-1}-k_d-1, k_1+k_2+\ldots+k_{d-1})$ which corresponds to the complete multipartite graph $K_{n\times(1+k_d)}$ where $n = (1+k_1+k_2+\ldots+k_d)/(1+k_d)$. Moreover, if d is odd, or d is even and $b_{\frac{d}{2}} = c_{\frac{d}{2}}$, then Γ' is an n-cocktail party graph.

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Disseminations

Publications

1. Siwaporn Mamart, and Chalermpong Worawannotai. "Merging the first and third classes in bipartite distance-regular graphs." **Asian-European Journal of Mathematics** (to appear).

2. Siwaporn Mamart, and Chalermpong Worawannotai. "Merging the first m classes in bipartite distance- regular graphs." (submitted).

Oral presentation

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