ALGEBRAIC STRUCTURES OF COMPLEX TWISTULANT MATRICES AND REAL-VALUED FIBONACCI CIRCULANT MATRICES

By

Miss Sirikanya Kittiwut

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree

Master of Science Program in Mathematics Study

Department of Mathematics

Graduate School, Silpakorn University

Academic Year 2015

ALGEBRAIC STRUCTURES OF COMPLEX TWISTULANT MATRICES AND REAL-VALUED FIBONACCI CIRCULANT MATRICES

By

Miss Sirikanya Kittiwut

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree

Master of Science Program in Mathematics Study

Department of Mathematics

Graduate School, Silpakorn University

Academic Year 2015

โครงสร้างพีชคณิตของเมทริกซ์วัฏจักรบิดเชิงซ้อนและเมทริกซ์วัฏจักรฟีโบนักชีค่าจริง

โดย

นางสาวศิริกัญญา กิตติวุฒิ

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

สาขาวิชาคณิตศาสตร์ศึกษา

ภาควิชาคณิตศาสตร์

บัณฑิตวิทยาลัย มหาวิทยาลัยศิลปากร

ปีการศึกษา 2558

วิทยานิพนธ์นี้

ได้รับทุนการศึกษาจากโครงการส่งเสริมการผลิตครูที่มีความสามารถพิเศษ

ทางวิทยาศาสตร์และคณิตศาสตร์ (สควค.)

สถาบันส่งเสริมการสอนทางวิทยาศาสตร์และเทคโนโลยี (สสวท.)

กระทรวงวิทยาศาสตร์และเทคโนโลยี

The Graduate School, Silpakorn University has approved and accredited the Thesis title of “Algebraic Structures of Complex Twistulant Matrices and Real-Valued Fibonacci Circulant Matrices” submitted by MISS Sirikanya Kittiwut as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics Study

....
(Associate Professor Panjai Tantatsanawong, Ph.D.)
Dean of Graduate School
........../..................../..........

The Thesis Advisor
Somphong Jitman, Ph.D.

The Thesis Examination Committee

.... Chairman
(Jittisak Rakbud, Ph.D.)
............/......................../.............

.... Member
(Assistant Professor Sajee Pianskool, Ph.D.)
............/......................../.............

.... Member
(Somphong Jitman, Ph.D.)
............/......................../.............

57316325 : สาขาวิชาคณิตศาสตร์ศึกษา
คำสำคัญ : เมทริกซ์วัฏจักร / เมทริกซ์วัฏจักรบิด / เมทริกซ์วัฏจักรฟีโบนักชีค่าจริง / ดีเทอร์มิแนนต์

ศิริกัญญา กิตติวุฒิ : โครงสร้างพีชคณิตของเมทริกซ์วัฏจักรบิดเชิงซ้อนและเมทริกซ์วัฏจักรฟีโบนักชีค่าจริง. อาจารย์ที่ปรึกษาวิทยานิพนธ์ : อ.ดร. สมพงค์ จิตต์มั่น. 30 หน้า.

เราศึกษาเมทริกซ์วัฏจักรบิดบนจำนวนเชิงซ้อน สำหรับแต่ละ $z \in \mathbb{C} \setminus \{0\}$ และจำนวนนับ n เราเรียกเมทริกซ์ A ขนาด $n \times n$ บน $\mathbb{C}$ ว่า เมทริกซ์วัฏจักรบิด $-z$ ถ้า

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ za_{n-1} & a_0 & a_1 & \dots & a_{n-3} & a_{n-2} \\ za_{n-2} & za_{n-1} & a_0 & \dots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ za_1 & za_2 & za_3 & \dots & za_{n-1} & a_0 \end{bmatrix}$$

สำหรับบางค่า $(a_0, a_1, \dots, a_{n-1}) \in \mathbb{C}^n$ สำหรับกรณี $z = 1$ เมทริกซ์วัฏจักรบิด $-z$ เป็นเมทริกซ์วัฏจักร สำหรับแต่ละจำนวนเต็มบวก n และ $z \in \mathbb{C} \setminus \{0\}$ เราศึกษาโครงสร้างพีชคณิตและสมบัติต่างๆของเมทริกซ์วัฏจักรบิด $-z$ ขนาด $n \times n$ ได้ว่า เมทริกซ์วัฏจักรบิด $-z$ ขนาด $n \times n$ สมสัณฐานกับริงผลหาร $\mathbb{C}[x]/\langle x^n - z \rangle$ แต่ละเมทริกซ์วัฏจักรบิด $-z$ ขนาด $n \times n$ เป็นเมทริกซ์ซึ่งสามารถทำให้เป็นเมทริกซ์ทแยงมุมได้และสามารถหาค่าดีเทอร์มิแนนต์ของเมทริกซ์นี้ได้โดยง่าย ในขณะเดียวกัน เราศึกษาเมทริกซ์วัฏจักรฟีโบนักชีค่าจริงและหาค่าดีเทอร์มิแนนต์ของเมทริกซ์ดังกล่าวด้วย

ภาควิชาคณิตศาสตร์ บัณฑิตวิทยาลัย มหาวิทยาลัยศิลปากร

ลายมือชื่อนักศึกษา.. ปีการศึกษา 2558

ลายมือชื่ออาจารย์ที่ปรึกษาวิทยานิพนธ์ ....

57316325 : MAJOR : MATHEMATICS STUDY

KEY WORD : CIRCULANT MATRIX / TWISTULANT MATRIX / REAL-VALUED FIBONACCI CIRCULANT MATRIX / DETERMINANT

SIRIKANYA KITTIWUT : ALGEBRAIC STRUCTURES OF COMPLEX TWISTULANT MATRICES AND REAL-VALUED FIBONACCI CIRCULANT MATRICES. THESIS ADVISOR : SOMPHONG JITMAN, Ph.D. 30 pp.

A class of z-twistulant matrices over the complex field $\mathbb{C}$ is studied. Given $z \in \mathbb{C} \setminus \{0\}$ and a positive integer n, an $n \times n$ matrix A over $\mathbb{C}$ is said to be $z-$*twistulant* if

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ za_{n-1} & a_0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\ za_{n-2} & za_{n-1} & a_0 & \cdots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ za_1 & za_2 & za_3 & \cdots & za_{n-1} & a_0 \end{bmatrix}$$

for some $(a_0, a_1, \ldots, a_{n-1}) \in \mathbb{C}^n$. It is not difficult to see that a z-twistulant matrix becomes a classical circulant matrix when $z = 1$. Given a positive integer n and a non-zero $z \in \mathbb{C}$, the algebraic structure and properties of the set of all $n \times n$ z-twistulant matrices are studied. The set of $n \times n$ complex z-twistulant matrices is isomorphic to the quotient ring $\mathbb{C}[x]/\langle x^n - z \rangle$. Every $n \times n$ z-twistulant matrix is shown to be diagonalizable and its determinant is determined. Subsequently, a real-valued Fibonacci circulant matrix is studied and the determinant of this matrix is determined.

Department of Mathematics	Graduate School, Silpakorn University
Student's signature	Academic Year 2015
Thesis Advisor's signature	

Acknowledgments

This thesis was supported by the institute for the promotion of teaching science and technology, Thailand and it is achieved by the assistance of many people.

Firstly, I would like to thank my thesis advisor Dr. Somphong Jitman for his invaluable advices, helpfulness, motivation, and immense knowledge.

Secondly, I would like to thank the thesis committee : Dr. Jittisak Rakbud and Assistant Professor Sajee Pianskool, for their insightful comments and suggestions.

Finally, I would like to thank my family for providing me with unfailing support and continuous encouragement throughout my years of study.

Table of Contents

Chapter 1

Introduction

A *circulant matrix* is an $n \times n$ matrix whose rows are composed of cyclically shifted versions of a list $(a_0, a_1, \ldots, a_{n-1})$. Precisely, a circulant matrix is of the form

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_0 \end{bmatrix} =: \mathrm{cir}(a_0, a_1, \ldots, a_{n-1}).$$

Circulant matrices are interesting due to their rich algebraic structures and various applications (see [2], [5], [13], [12], [14] and references therein). Such matrices have been applied to various disciplines such as image processing, communications, signal processing, networked systems and coding theory (see, for examples, [13], [12] and [14]).

Circulant matrices have continuously been studied since their first appearance in the paper by Catalan [1]. In 1994, P. J. Davis [2] published the book "Circulant Matrices" which summarizes the algebraic structures, properties and some applications of circulant matrices. Circulant matrices have been shown to be diagonalized by a discrete Fourier transform. Therefore, a linear system whose coefficient matrix is circulant can be quickly solved using a fast Fourier transform. In cryptography, a circulant matrix is used in the Advanced Encryption Standard (AES). In 2009, M. Grassl and T. A. Gulliver [5] discovered that circulant matrices over finite fields can be applied in constructing good codes and good self-dual codes. In 2015, Y. Zheng and S. Shon [14] studied the inverses of some circulant matrices.

The *Fibonacci sequence* F_r of Fibonacci numbers is defined by the recurrence relation

$$F_r = F_{r-1} + F_{r-2}$$

for all $r \geq 3$ with the initial values $F_1 = 1$ and $F_2 = 1$. A *Fibonacci circulant matrix*, a circulant matrix whose entries are given by Fibonacci numbers, has been studied in [9]. Precisely, an $n \times n$ matrix is called *Fibonacci circulant* if it is of the form $\text{cir}(F_r : n) := \text{cir}(F_r, F_{r+1}, \ldots, F_{r+n-1})$ for some positive integers r and n. In [9], D. Lind determined the determinant of Fibonacci circulant matrix $\text{cir}(F_r : n)$.

In this thesis, we focus on a generalization of circulant matrices and a generalization of Fibonacci circulant matrices.

Given a positive integer n and a nonzero complex number z, an $n \times n$ complex matrix A is called a *z-twistulant* [2] if

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ za_{n-1} & a_0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\ za_{n-2} & za_{n-1} & a_0 & \cdots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ za_1 & za_2 & za_3 & \cdots & za_{n-1} & a_0 \end{bmatrix} =: \text{cir}_z(a_0, a_1, \ldots, a_{n-1})$$

for some $(a_0, a_1, \ldots, a_{n-1}) \in \mathbb{C}^n$. We note that if $z = 1$, a z-twistulant matrix is just a circulant matrix. A z-twistulant matrix is called *negacirculant matrices* when $z = -1$. Denote by $\text{Cir}_{n,z}(\mathbb{C}) := \{\text{cir}_z(\boldsymbol{w}) \mid \boldsymbol{w} \in \mathbb{C}^n\}$ the set of $n \times n$ z-twistulant matrices over $\mathbb{C}$.

As a generalization of the Fibonacci sequence, F.D. Parker and E. Halsey (see [11] and [6]) introduced a *real-valued Fibonacci function* $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = \frac{\alpha^x - (\cos \pi x)\alpha^{-x}}{\sqrt{5}}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ for all real numbers x. Note that the restriction of this real-valued Fibonacci function to the set $\mathbb{N}$ of natural numbers is the Fibonacci sequence. A *real-*

valued Fibonacci circulant matrix is a matrix of the form $\text{cir}(F(r):n) := \text{cir}(F(r), F(r+1), \ldots, F(r+n-1))$ for some positive integer n and real number r.

As discussed above, the algebraic structure and the determinant of circulant matrices are given in [2] and [7], respectively. The determinant of Fibonacci circulant matrices is determined in [9]. However, to the best of my knowledge, properties of z-twistulant matrices and the determinant of a real-valued Fibonacci circulant matrix $\text{cir}(F(r):n)$ have not been well studied. It is therefore of natural interest to determine the determinant of a z-twistulant matrix, the determinant of a real-valued Fibonacci circulant matrix and characterize the algebraic structure of $\text{Cir}_{n,z}(\mathbb{C})$.

The thesis is organized as follows. In Chapter 2, some basic properties of matrices, groups and rings are recalled. In Chapter 3, the algebraic structure of z-twistulant matrices is studied and the characterization of $n \times n$ z-twistulant matrices is established. The explicit formula of the determinant of z-twistulant matrices with $|z| = 1$ is given in Chapter 4. In Chapter 5, the determinant of a real-valued Fibonacci circulant matrix is determined.

Chapter 2

Preliminaries

In this chapter, we recall some basic properties about matrices, groups and rings.

2.1 Matrices

In this section, we recall some special matrices together with their basic properties.

Given a positive integer n, denote by $M_n(\mathbb{C})$ the set of all $n \times n$ complex matrices over $\mathbb{C}$, where $\mathbb{C}$ denotes the set of complex numbers.

Definition 2.1. A *circulant matrix* is an $n \times n$ matrix whose rows are composed of cyclically shifted versions of a list $(a_0, a_1, \ldots, a_{n-1})$. Precisely, a circulant matrix is of the form

$$\begin{bmatrix} a_0 & a_1 & a_2 & \ldots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \ldots & a_{n-3} & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \ldots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1 & a_2 & a_3 & \ldots & a_{n-1} & a_0 \end{bmatrix} =: \mathrm{cir}(a_0, a_1, \ldots, a_{n-1}).$$

Example 2.2. Let $(-3\mathrm{i}, 7, -9, 5+\mathrm{i}) \in \mathbb{C}^4$. Then

$$\mathrm{cir}(-3\mathrm{i}, 7, -9, 5+\mathrm{i}) = \begin{bmatrix} -3\mathrm{i} & 7 & -9 & 5+\mathrm{i} \\ 5+\mathrm{i} & -3\mathrm{i} & 7 & -9 \\ -9 & 5+\mathrm{i} & -3\mathrm{i} & 7 \\ 7 & -9 & 5+\mathrm{i} & -3\mathrm{i} \end{bmatrix}$$

is a 4×4 circulant matrix.

Definition 2.3. For a non-zero $z \in \mathbb{C}$, a matrix $A \in M_n(\mathbb{C})$ is called a *z-twistulant* if

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ za_{n-1} & a_0 & a_1 & \dots & a_{n-3} & a_{n-2} \\ za_{n-2} & za_{n-1} & a_0 & \dots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ za_1 & za_2 & za_3 & \dots & za_{n-1} & a_0 \end{bmatrix}$$
$$=: \mathrm{cir}_z(a_0, a_1, \dots, a_{n-1})$$

for some $(a_0, a_1, \dots, a_{n-1}) \in \mathbb{C}^n$.

Denote by $\mathrm{Cir}_{n,z}(\mathbb{C}) := \{\mathrm{cir}_z(\boldsymbol{w}) \mid \boldsymbol{w} \in \mathbb{C}^n\}$ the set of $n \times n$ z-twistulant matrices over $\mathbb{C}$.

A z-twistulant matrix is called *circulant* and *negacirculant matrices* if $z = 1$ and $z = -1$, respectively.

Example 2.4. Let $(2, 5\mathbf{i}, 3, 2 - \mathbf{i}) \in \mathbb{C}^4$. Then

$$\mathrm{cir}_{\mathbf{i}}(2, 5\mathbf{i}, 3, 2 - \mathbf{i}) = \begin{bmatrix} 2 & 5\mathbf{i} & 3 & 2 - \mathbf{i} \\ 1 + 2\mathbf{i} & 2 & 5\mathbf{i} & 3 \\ 3\mathbf{i} & 1 + 2\mathbf{i} & 2 & 5\mathbf{i} \\ -5 & 3\mathbf{i} & 1 + 2\mathbf{i} & 2 \end{bmatrix}$$

is a 4×4 $\mathbf{i}$-twistulant matrix which is not circulant.

The determinant of a Vandermonde matrix and the following properties of matrices play an important role in determining the determinants of circulant matrices and z-twistulant matrices.

Lemma 2.5 ([10, Chapter 3, Section 3.4, Theorem 2]). *Let $a_0, a_1, a_2, \dots, a_{n-1}$ be complex* *numbers and let $n \geq 2$ be an integer. Then the determinant of a Vandermonde matrix*

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ a_0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ a_0^2 & a_1^2 & a_2^2 & \dots & a_{n-2}^2 & a_{n-1}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_0^{n-1} & a_1^{n-1} & a_2^{n-1} & \dots & a_{n-2}^{n-1} & a_{n-1}^{n-1} \end{bmatrix}$$

is of the form

$$\prod_{1 \leq j < i \leq n} (a_i - a_j).$$

Theorem 2.6 ([8, Chapter 5,Theorem 5])**.** *An $n \times n$ complex matrix A is diagonalizable if and only if the Eigen vectors of A are linearly independent.*

2.2 Groups and Rings

In this section, the definitions and some basic properties of groups and rings are recalled.

Definition 2.7. A *group* is an ordered pair $(G, \star)$ where G is a non-empty set and $\star$ is a binary operation on G satisfying the following axioms:

$i)$ $(a \star b) \star c = a \star (b \star c)$ for all $a, b, c \in G$, i.e., $\star$ is *associative*.

$ii)$ There exists an element e in G, called the *identity* of G, such that $a \star e = e \star a = a$ for all $a \in G$.

$iii)$ For each $a \in G$, there exists an element a^{-1} in G, called the *inverse* of a, such that $a \star a^{-1} = a^{-1} \star a = e$.

Definition 2.8. A group $(G, \star)$ is called *abelian* (or *commutative*) if $a \star b = b \star a$ for all $a, b \in G$.

Example 2.9. The set $M_2(\mathbb{C})$ of 2×2 matrices over $\mathbb{C}$ is an abelian group under the matrix addition.

Definition 2.10. Let $(G, \star)$ be a group. A subset H of G is called a *subgroup* of G if H is a group under $\star|_{H\times H}$. If H is a subgroup of G, we write $H \leq G$.

Example 2.11. Let $H = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \middle| a \in \mathbb{C} \right\}$ and $K = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \middle| a, b \in \mathbb{C} \right\}$. Then $H \leq M_2(\mathbb{C})$ and $K \leq M_2(\mathbb{C})$.

Definition 2.12. A *ring* R is a non-empty set together with two binary operations $+$ and $\cdot$ (called *addition* and *multiplication*, respectively) satisfying the following axioms for all $a, b, c \in R$:

$i)$ $a + b = b + a$.

$ii)$ $(a + b) + c = a + (b + c)$.

$iii)$ There exists an element 0 in R such that $0 + a = a$.

$iv)$ For each a in R, there exists an element $-a \in R$ such that $a + (-a) = 0$.

$v)$ $\cdot$ is associative, i.e., $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

$vi)$ The *distributive laws* hold in R, i.e.,

$$(a + b) \cdot c = (a \cdot c) + (b \cdot c)$$

and

$$c \cdot (a + b) = (c \cdot a) + (c \cdot b).$$

A ring R is said to be *commutative* if the multiplication is commutative and it is said to be *a ring with identity* if there exists an element $1 \in R$ such that

$$1 \cdot a = a = a \cdot 1$$

for all $a \in R$.

Example 2.13. The set $M_2(\mathbb{C})$ is a non-commutative ring with identity.

Definition 2.14. Let S be a subset of a ring R. The set S is said to be a *subring* of R if S is a ring under $+_{|S\times S}$ and $\cdot_{|S\times S}$.

Example 2.15. The sets H and K in Example 2.11 are subrings of $M_2(\mathbb{C})$.

Definition 2.16. Let R and S be rings.

(1) A *ring homomorphism* is a map $\varphi : R \to S$ satisfying the following conditions.

$i)$ $\varphi(a+b) = \varphi(a) + \varphi(b)$ for all $a, b \in R$.

$ii)$ $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in R$.

(2) The *kernel* of a ring homomorphism φ, denoted kerφ, is the set of elements of R mapped to 0 in S.

(3) A ring homomorphism $\varphi : R \to S$ is called a *ring isomorphism* if φ is one-to-one and onto.

Theorem 2.17. *A ring homomorphism $\varphi : R \to S$ is one-to-one if and only if* ker$\varphi = \{0\}$.

Example 2.18. Let $S = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \middle| \, a, b \in \mathbb{C} \right\}$ and let K be defined in Example 2.11. Then S and K are subrings of $M_2(\mathbb{C})$. Let $\varphi : K \to S$ be defined by

$$\varphi\left(\begin{bmatrix} a & b \\ b & a \end{bmatrix}\right) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

It is not difficult to see that

$$\varphi(A+B) = \varphi(A) + \varphi(B)$$

and

$$\varphi(AB) = \varphi(A)\varphi(B) \text{ for all } A, B \in K.$$

For each $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \in S$, we have $\varphi\left(\begin{bmatrix} a & b \\ b & a \end{bmatrix}\right) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, and hence, φ is onto.

Let $\begin{bmatrix} a & b \\ b & a \end{bmatrix} \in \ker \varphi$. Then

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \varphi\left(\begin{bmatrix} a & b \\ b & a \end{bmatrix}\right) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

It follows that $a = 0$ and $b = 0$. Hence, $\ker \varphi = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$. It follows that φ is injective.

Therefore, φ is a ring isomorphism.

Definition 2.19. Let R be a commutative ring and let I be a subset of R. The set I is called an *ideal* of R if

(1) $(I, +)$ is a group.

(2) $ar \in I$ for all $r \in R$ and $a \in I$.

Lemma 2.20. *Let R be a commutative ring and let $a \in R$. Then*

$$aR = \{ar \mid r \in R\} \text{ is an ideal of } R.$$

The ideal aR in Lemma 2.20 is called the *ideal generated* by a, denoted by $\langle a \rangle$.

Example 2.21. Let $\mathbb{C}[x] = \left\{ \sum_{i=0}^{n} a_i x^i \,\middle|\, a_i \in \mathbb{C} \text{ and } n \in \mathbb{N} \cup \{0\} \right\}$. Then $\langle x^2 - 1 \rangle$ is an ideal in $\mathbb{C}[x]$ generated by $x^2 - 1$.

Proposition 2.22 ([4, Chapter 7, Proposition 6])**.** *Let R be a commutative ring and let I be an ideal of R. Then the (additive) quotient group R/I is a ring under the binary operations :*

$$(r + I) + (s + I) := (r + s) + I$$

and

$$(r + I) \times (s + I) := (rs) + I$$

for all $r, s \in R$.

Definition 2.23. When I is an ideal of R the ring R/I with the operations in Proposition 2.22 is called the *quotient ring* of R by I.

Example 2.24. From Example 2.21, it follows that $\mathbb{C}[x]/\langle x^2-1\rangle$ is a quotient ring.

Chapter 3

Characterization of Complex Twistulant Matrices

In this chapter, the algebraic structure of $\text{Cir}_{n,z}(\mathbb{C})$ is determined in terms of polynomials over $\mathbb{C}$ and $\text{Cir}_{n,z}(\mathbb{C})$ is shown to be a commutative ring with identity. Moreover, $\text{Cir}_{n,z}(\mathbb{C})$ and $\text{Cir}_{n,-z}(\mathbb{C})$ are isomorphic as rings.

Proposition 3.1. *Let n be a positive integer and let $z \in \mathbb{C} \setminus \{0\}$. Then $\text{Cir}_{n,z}(\mathbb{C})$ is a vector space over $\mathbb{C}$ with the usual addition and scalar multiplication of matrices.*

Proof. It is not difficult to see that $\text{cir}_z(\boldsymbol{a} + c\boldsymbol{b}) = \text{cir}_z(\boldsymbol{a}) + c\,\text{cir}_z(\boldsymbol{b})$ for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^n$ and $c \in \mathbb{C}$. Hence, $\text{Cir}_{n,z}(\mathbb{C})$ is a subspace of the complex vector space $M_n(\mathbb{C})$. □

Theorem 3.2. *Let n be a positive integer and let $z \in \mathbb{C} \setminus \{0\}$. Then $\text{Cir}_{n,z}(\mathbb{C})$ is a subring (with identity) of $M_n(\mathbb{C})$.*

Proof. Clearly, $I_n = \text{cir}_z(1, 0, 0, \ldots, 0) \in \text{Cir}_{n,z}(\mathbb{C})$. Let $\boldsymbol{a} = (a_0, a_1, \ldots, a_{n-1})$,

$\boldsymbol{b} = (b_0, b_1, \ldots, b_{n-1}) \in \mathbb{C}_n$. Then

$$\begin{aligned}
\mathrm{cir}_z(\boldsymbol{a}) &= \mathrm{cir}_z(a_0, a_1, \ldots, a_{n-1}) \\
&= \begin{bmatrix} a_0 & a_1 & a_2 & \ldots & a_{n-1} \\ za_{n-1} & a_0 & a_1 & \ldots & a_{n-2} \\ za_{n-2} & za_{n-1} & a_0 & \ldots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ za_1 & za_2 & za_3 & \ldots & a_0 \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{cir}_z(\boldsymbol{b}) &= \mathrm{cir}_z(b_0, b_1, \ldots, b_{n-1}) \\
&= \begin{bmatrix} b_0 & b_1 & b_2 & \ldots & b_{n-1} \\ zb_{n-1} & b_0 & b_1 & \ldots & b_{n-2} \\ zb_{n-2} & zb_{n-1} & b_0 & \ldots & b_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ zb_1 & zb_2 & zb_3 & \ldots & b_0 \end{bmatrix}.
\end{aligned}$$

It is not difficult to see that $\mathrm{cir}_z(\boldsymbol{a}) - \mathrm{cir}_z(\boldsymbol{b}) = \mathrm{cir}_z(\boldsymbol{a} - \boldsymbol{b}) \in \mathrm{Cir}_{n,z}(\mathbb{C})$. To show that $\mathrm{cir}_z(\boldsymbol{a})\mathrm{cir}_z(\boldsymbol{b}) \in \mathrm{Cir}_{n,z}(\mathbb{C})$, let $[c_{i,j}]_{n\times n} := \mathrm{cir}_z(\boldsymbol{a})\mathrm{cir}_z(\boldsymbol{b})$. We show that $[c_{i,j}]_{n\times n}$ is a z-twistulant matrix, i.e.,

1. $c_{i,j} = c_{i+1,j+1}$ for all $1 \le i \le n-3$ and $i \le j \le n-3$,

2. $zc_{i,n-1} = c_{i+1,0}$ for all $0 \le i \le n-2$, and $j = n-1$.

3. $c_{i,j} = c_{i+1,j+1}$ for all $1 \le i \le n-2$ and $0 \le j \le i-1$.

We consider the following 3 cases.

Case 1 $1 \le i \le n-3$ and $i \le j \le n-3$. We have

$$\begin{aligned}
c_{i,j} &= \sum_{k=0}^{-i+j} a_k b_{-i+j-k} + z \sum_{k=-i+j+1}^{n-1} a_k b_{-i+j-k} \\
&= c_{i+1,j+1}.
\end{aligned}$$

Case 2 $0 \leq i \leq n-2$ and $j = n-1$. We have

$$c_{i,n-1} = \sum_{k=0}^{n-(i+1)} a_k b_{n-(i+1)-k} + z \sum_{k=n-i}^{n-1} a_k b_{n-(i+1)-k}.$$

Hence,

$$\begin{aligned} zc_{i,n-1} &= z \sum_{k=0}^{n-(i+1)} a_k b_{n-(i+1)-k} + z^2 \sum_{k=n-i}^{n-1} a_k b_{n-(i+1)-k} \\ &= c_{i+1,0}. \end{aligned}$$

Case 3 $1 \leq i \leq n-2$ and $0 \leq j \leq i-1$. We have

$$\begin{aligned} c_{i,j} &= z \sum_{k=0}^{n-i+j} a_k b_{n-i+j-k} + z^2 \sum_{k=n-i+j+1}^{n-1} a_k b_{n-i+i-k} \\ &= c_{i+1,j+1}. \end{aligned}$$

From the 3 cases, $\text{cir}_z(\boldsymbol{a})\text{cir}_z(\boldsymbol{b}) = [c_{i,j}]_{n\times n} \in \text{Cir}_{n,z}(\mathbb{C})$.

Therefore, $\text{Cir}_{n,z}(\mathbb{C})$ is a subring of $M_n(\mathbb{C})$. □

Let $\mathbb{C}[x]$ denote the ring of polynomials over $\mathbb{C}$ and let $\langle f(x) \rangle$ denote the ideal of $\mathbb{C}[x]$ generated by a polynomial $f(x) \in \mathbb{C}[x]$.

Theorem 3.3. *Let n be a positive integer and let $z \in \mathbb{C}\setminus\{0\}$. Then $\text{Cir}_{n,z}(\mathbb{C})$ is isomorphic to $\mathbb{C}[x]/\langle x^n - z\rangle$ as rings.*

Proof. Let $\Psi : \text{Cir}_{n,z}(\mathbb{C}) \to \mathbb{C}[x]/\langle x^n - z\rangle$ be defined by

$$\Psi(\text{cir}_z(a_0, a_1, \ldots, a_{n-1})) = \sum_{i=0}^{n-1} a_i x^i + \langle x^n - z\rangle.$$

Let $\boldsymbol{a} = (a_0, a_1, \ldots, a_{n-1}), \boldsymbol{b} = (b_0, b_1, \ldots, b_{n-1}) \in \mathbb{C}^n$. Then

$$\begin{aligned} \Psi(\text{cir}_z(\boldsymbol{a}) + \text{cir}_z(\boldsymbol{b})) &= \Psi(\text{cir}_z(a_0+b_0, a_1+b_1, \ldots, a_{n-1}+b_{n-1})) \\ &= \sum_{i=0}^{n-1}(a_i+b_i)x^i + \langle x^n - z\rangle \\ &= \left(\sum_{i=0}^{n-1} a_i x^i + \langle x^n - z\rangle\right) + \left(\sum_{i=0}^{n-1} b_i x^i + \langle x^n - z\rangle\right) \\ &= \Psi(\text{cir}_z(\boldsymbol{a})) + \Psi(\text{cir}_z(\boldsymbol{b})). \end{aligned}$$

To show that $\Psi(\text{cir}_z(\boldsymbol{a})\text{cir}_z(\boldsymbol{b})) = \Psi(\text{cir}_z(\boldsymbol{a}))\Psi(\text{cir}_z(\boldsymbol{b}))$, let $[c_{i,j}]_{n\times n} := \text{cir}_z(\boldsymbol{a})\text{cir}_z(\boldsymbol{b})$. By Theorem 3.2, we have $[c_{i,j}]_{n\times n} \in \text{Cir}_{n,z}(\mathbb{C})$. Hence,

$$\begin{aligned}
\Psi(\text{cir}_z(\boldsymbol{a})\text{cir}_z(\boldsymbol{b})) &= \Psi(\text{cir}_z(c_{0,0}, c_{0,1}, \ldots, c_{0,n-1})) \\
&= \sum_{k=0}^{n-1} c_{0,k}x^k + \langle x^n - z\rangle \\
&= \sum_{k=0}^{n-1}\left(\sum_{k=i+j} a_ib_j + z\left(\sum_{k=i+j-n} a_ib_j\right)\right)x^k + \langle x^n - z\rangle \\
&= \sum_{k=0}^{n-1}\left(\sum_{k=i+j} a_ib_j\right)x^k + \sum_{k=0}^{n-1} z\left(\sum_{k=i+j-n} a_ib_j\right)x^k + \langle x^n - z\rangle \\
&= \sum_{k=0}^{n-1}\left(\sum_{k=i+j} a_ib_j\right)x^k + \sum_{k=0}^{n-1}\left(\sum_{k+n=i+j} a_ib_j\right)x^{k+n} + \langle x^n - z\rangle \\
&= \sum_{k=0}^{n-1}\left(\sum_{k=i+j} a_ib_j\right)x^k + \sum_{k=n}^{2n-2}\left(\sum_{k=i+j} a_ib_j\right)x^k + \langle x^n - z\rangle \\
&= \sum_{k=0}^{2n-2}\left(\sum_{k=i+j} a_ib_j\right)x^k + \langle x^n - z\rangle \\
&= \left(\sum_{i=0}^{n-1} a_ix^i + \langle x^n - z\rangle\right)\left(\sum_{i=0}^{n-1} b_ix^i + \langle x^n - z\rangle\right) \\
&= \Psi(\text{cir}_z(\boldsymbol{a}))\Psi(\text{cir}_z(\boldsymbol{b})).
\end{aligned}$$

Therefore, Ψ is a ring homomorphism.

For each $f(x) + \langle x^n - z\rangle \in \mathbb{C}[x]/\langle x^n - z\rangle$, we have

$$f(x) + \langle x^n - z\rangle = \sum_{i=0}^{n-1} a_ix^i + \langle x^n - z\rangle,$$

where $a_i \in \mathbb{C}$ for all $i = 0, 1, \ldots, n-1$, and hence,

$$\Psi(\text{cir}_z(a_0, a_1, \ldots, a_{n-1}) = f(x) + \langle x^n - z\rangle.$$

It follows that Ψ is onto.

To show that Ψ is injective, let $\text{cir}_z(a_0, a_1, \ldots, a_{n-1}) \in \ker \Psi$. Then

$$\langle x^n - z\rangle = \Psi(\text{cir}_z(a_0, a_1, \ldots, a_{n-1})) = \sum_{i=0}^{n-1} a_ix^i + \langle x^n - z\rangle.$$

It follows that $\sum_{i=0}^{n-1} a_i x^i \in \langle x^n - z\rangle$. Since $\deg(\sum_{i=0}^{n-1} a_i x^i) < n$, we have $a_i = 0$ for all $i = 0, 1, \ldots, n-1$. Then

$$\mathrm{cir_z}(\mathrm{a_0}, \mathrm{a_1}, \ldots, \mathrm{a_{n-1}}) = \mathrm{cir_z}(\mathbf{0}) = [0]_{n\times n}$$

which implies that Ψ is an injective.

Therefore, Ψ is a ring isomorphism. Equivalently, we have $\mathrm{Cir}_{n,z}(\mathbb{C})$ is isomorphic to $\mathbb{C}[x]/\langle x^n - z\rangle$ as rings. □

Corollary 3.4. *Let n be a positive integer and let $z \in \mathbb{C} \setminus \{0\}$. Then $\mathrm{Cir}_{n,z}(\mathbb{C})$ is a commutative subring of $M_n(\mathbb{C})$.*

Proof. Note that $\mathrm{Cir}_{n,z}(\mathbb{C})$ is isomorphic to $\mathbb{C}[x]/\langle x^n - z\rangle$ by Theorem 3.3. Since $\mathbb{C}[x]/\langle x^n - z\rangle$ is a commutative ring, we have $\mathrm{Cir}_{n,z}(\mathbb{C})$ is commutative. □

Example 3.5. Let $\boldsymbol{a} = (\mathrm{i}, 2, -5\mathrm{i}, 4), \boldsymbol{b} = (3, \mathrm{i}, -\mathrm{i}, 2\mathrm{i}) \in \mathbb{C}^4$. Then

$$\begin{aligned}\mathrm{cir_i}(\boldsymbol{a}) &= \mathrm{cir_i}(\mathrm{i}, 2, -5\mathrm{i}, 4)\\ &= \begin{bmatrix} \mathrm{i} & 2 & -5\mathrm{i} & 4 \\ 4\mathrm{i} & \mathrm{i} & 2 & -5\mathrm{i} \\ 5 & 4\mathrm{i} & \mathrm{i} & 2 \\ 2\mathrm{i} & 5 & 4\mathrm{i} & \mathrm{i} \end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}\mathrm{cir_i}(\boldsymbol{b}) &= \mathrm{cir_i}(3, \mathrm{i}, -\mathrm{i}, 2\mathrm{i})\\ &= \begin{bmatrix} 3 & \mathrm{i} & -\mathrm{i} & 2\mathrm{i} \\ -2 & 3 & \mathrm{i} & -\mathrm{i} \\ 1 & -2 & 3 & \mathrm{i} \\ -1 & 1 & -2 & 3 \end{bmatrix}.\end{aligned}$$

It follows that

$$\mathrm{cir}_{\mathrm{i}}(\boldsymbol{a})\mathrm{cir}_{i}(\boldsymbol{b}) = \begin{bmatrix} -8-2\mathrm{i} & 9+10\mathrm{i} & -7-13\mathrm{i} & 15-2\mathrm{i} \\ 2+15\mathrm{i} & -8-2\mathrm{i} & 9-10\mathrm{i} & -7-13\mathrm{i} \\ 13-7\mathrm{i} & 2+15\mathrm{i} & -8-2\mathrm{i} & 9-10\mathrm{i} \\ -10+9\mathrm{i} & 13-7\mathrm{i} & 2+15\mathrm{i} & -8-2\mathrm{i} \end{bmatrix} = \mathrm{cir}_{i}(\boldsymbol{b})\mathrm{cir}_{i}(\boldsymbol{a}).$$

We note that the rings $\mathbb{C}[x]/\langle x^n - 1\rangle$ and $\mathbb{C}[x]/\langle x^n + 1\rangle$ are isomorphic as rings [3, Proposition 5.1]. Extending this idea, it can be shown that $\mathbb{C}[x]/\langle x^n - z\rangle$ is isomorphic to $\mathbb{C}[x]/\langle x^n + z\rangle$ as rings, and hence, the following result can be obtained.

Theorem 3.6. *Let n be a positive integer, then $\mathrm{Cir}_{n,z}(\mathbb{C})$ is isomorphic to $\mathrm{Cir}_{n,-z}(\mathbb{C})$ as rings.*

Proof. Let n be a positive integer. By Theorem 3.3, it siffices to show that $\mathbb{C}[x]/\langle x^n - z\rangle$ is isomorphic to $\mathbb{C}[x]/\langle x^n + z\rangle$ as rings. Let α be a primitive nth root of -1.

Let $\varphi : \mathbb{C}[x]/\langle x^n - z\rangle \to \mathbb{C}[x]/\langle x^n + z\rangle$ be defined by

$$\varphi(f(x) + \langle x^n - z\rangle) = f(\alpha x) + \langle x^n + z\rangle.$$

Let $f(x)$ and $g(x)$ be polynomials in $\mathbb{C}[x]$ be such that

$$f(x) + \langle x^n - z\rangle = g(x) + \langle x^n - z\rangle.$$

Hence,

$$f(\alpha x) + \langle (\alpha x)^n - z\rangle = g(\alpha x) + \langle (\alpha x)^n - z\rangle$$

if and only if

$$f(\alpha x) + \langle x^n + z\rangle = g(\alpha x) + \langle x^n + z\rangle$$

Therefore,

$$f(\alpha x) + \langle x^n + z\rangle = g(\alpha x) + \langle x^n + z\rangle.$$

It follows that φ is well defined and injection.

It is not difficult to verify that φ is surjective and it is a ring homomorphism. Therefore, φ is a ring isomorphism. □

Chapter 4

Determinants of Complex Twistulant Matrices

In this chapter, the determinant of z-twistulant matrices over the complex field is studied. A special case where $z = 1$, the determinat of circulant matrices is given in [2] and [7]. Here, we consider a general case where z is an arbitrary non-zero complex number.

Let $z \in \mathbb{C} \setminus \{0\}$. Then $z = r(\cos A\pi + \mathbf{i} \sin A\pi)$ for some $0 \leq A < 2$ and a positive real number r. For each $0 \leq k < n$, let

$$\omega_k = \cos \frac{2k}{n}\pi + \mathbf{i} \sin \frac{2k}{n}\pi$$

and

$$s_k = (\sqrt[n]{r})^k (\cos \frac{Ak}{n}\pi + i \sin \frac{Ak}{n}\pi).$$

Then ω_k's are the nth roots of unity.

Example 4.1. Let $n = 4$ and $z = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$. Then, for each $0 \leq k < 4$, we have

$$\omega_k = \cos \frac{k}{2}\pi + \mathbf{i} \sin \frac{k}{2}\pi$$

and

$$s_k = \cos \frac{k}{6}\pi + \mathbf{i} \sin \frac{k}{6}\pi.$$

The determinant of circulant matrices as given in [7]. Here, we determined the determinant of z-twistulant matrices in Theorem 4.2. The result in [7] can be viewed as a corollary of our result.

Theorem 4.2. *Let* $(a_0, a_1, \ldots, a_{n-1}) \in \mathbb{C}^n$ *and let* $z \in \mathbb{C} \setminus \{0\}$*.Then*

$$\det(\mathrm{cir}_z(a_0, a_1, \ldots, a_{n-1})) = \prod_{k=0}^{n-1} \left(\sum_{j=0}^{n-1} a_j s_j \omega_j^k \right).$$

Proof. For each integer $j \in \{0, 1, \ldots, n-1\}$, let

$$\lambda_j = \sum_{k=0}^{n-1} a_k s_k \omega_k^j$$

and let

$$X_j = \frac{1}{\sqrt{n}} \begin{bmatrix} s_0\omega_0^j \\ s_1\omega_1^j \\ \vdots \\ s_{n-1}\omega_{n-1}^j \end{bmatrix}.$$

Since

$$s_i s_j = \begin{cases} s_{i+j} & \text{if } \; 0 \le i+j < n. \\ z s_{(i+j) \bmod n} & \text{if } \; n \le i+j < 2n. \end{cases}$$

Then

$$\lambda_j X_j = \frac{1}{\sqrt{n}} \begin{bmatrix} \left(\sum_{i=0}^{n-1} a_i s_i \omega_i^j \right) s_0 \omega_0^j \\ \left(\sum_{i=0}^{n-1} a_i s_i \omega_i^j \right) s_1 \omega_1^j \\ \vdots \\ \left(\sum_{i=0}^{n-1} a_i s_i \omega_i^j \right) s_{n-1} \omega_{n-1}^j \end{bmatrix}$$

$$= \frac{1}{\sqrt{n}} \begin{bmatrix} \sum_{i=0}^{n-1} a_i s_i \omega_i^j \\ z a_{n-1} s_0 \omega_0^j + \sum_{i=1}^{n-1} a_{i-1} s_i \omega_i^j \\ \vdots \\ \sum_{i=0}^{n-2} z a_{i+1} s_i \omega_i^j + a_0 s_{n-1} \omega_{n-1}^j \end{bmatrix}$$

and

$$\text{cir}_z(a_0, a_1, \ldots, a_{n-1})X_j = \frac{1}{\sqrt{n}} \begin{bmatrix} \sum_{i=0}^{n-1} a_i s_i \omega_i^j \\ z a_{n-1} s_0 \omega_0^j + \sum_{i=1}^{n-1} a_{i-1} s_i \omega_i^j \\ \vdots \\ \sum_{i=0}^{n-2} z a_{i+1} s_i \omega_i^j + a_0 s_{n-1} \omega_{n-1}^j \end{bmatrix}.$$

Hence,

$$\text{cir}_z(a_0, a_1, \ldots, a_{n-1})X_j = \lambda_j X_j$$

for all $0 \leq j \leq n-1$. Therefore, λ_j is an Eigen value of $\text{cir}_z(a_0, a_1, \ldots, a_{n-1})$ and X_j is an Eigen vector corresponding to λ_j.

Let

$$\begin{aligned} X &= \frac{1}{\sqrt{n}} \begin{bmatrix} X_0 & X_1 & \ldots & X_{n-1} \end{bmatrix} \\ &= \frac{1}{\sqrt{n}} \begin{bmatrix} s_0 & s_0\omega_0 & \ldots & s_0\omega_0^{n-1} \\ s_1 & s_1\omega_1 & \ldots & s_1\omega_1^{n-1} \\ s_2 & s_2\omega_2 & \ldots & s_2\omega_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_{n-1}\omega_{n-1} & \ldots & s_{n-1}\omega_{n-1}^{n-1} \end{bmatrix}. \end{aligned}$$

Then

$$\begin{aligned} \det(X) &= n^{\frac{-n}{2}} s_0 s_1 \ldots s_{n-1} \det \begin{bmatrix} 1 & \omega_0 & \ldots & \omega_0^{n-1} \\ 1 & \omega_1 & \ldots & \omega_1^{n-1} \\ 1 & \omega_2 & \ldots & \omega_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_{n-1} & \ldots & \omega_{n-1}^{n-1} \end{bmatrix} \\ &= n^{\frac{-n}{2}} s_0 s_1 \ldots s_{n-1} \prod_{i>j} (\omega_i - \omega_j). \end{aligned}$$

by Lemma 2.5. Since $n^{\frac{-n}{2}} s_0 s_1 \ldots s_{n-1} \neq 0$ and $\prod_{i>j}(\omega_i - \omega_j) \neq 0$, we have $\det(X) \neq 0$. Hence X is a nonsingular matrix. It follows that $\{X_0, X_1, \ldots, X_{n-1}\}$ is linearly independent. Therefore, $X_0, X_1, \ldots, X_{n-1}$ are linearly independent Eigen vectors of

$\text{cir}_z(a_0, a_1, \ldots, a_{n-1})$. By Theorem 2.6, $\text{cir}_z(a_0, a_1, \ldots, a_{n-1})$ is diagonalizable and $\text{cir}_z(a_0, a_1, \ldots, a_{n-1})$ XDX^{-1}, where $D = diag(\lambda_0, \lambda_1, \ldots, \lambda_{n-1})$ is a diagonal matrix. Hence,

$$\begin{aligned}\det(\text{cir}_z(a_0, a_1, \ldots, a_{n-1}) &= \det(XDX^{-1}) \\ &= \det(X)\det(D)\det(X^{-1}) \\ &= \det(D) \\ &= \lambda_0\lambda_1 \ldots \lambda_{n-1} \\ &= \prod_{k=0}^{n-1}(\lambda_k).\end{aligned}$$

Therefore, we have $\det(\text{cir}_z(a_0, a_1, \ldots, a_{n-1})) = \prod_{k=0}^{n-1}\left(\sum_{j=0}^{n-1} a_j s_j \omega_j^k\right)$ as desired. □

In the case where $z = 1$, we have the following result.

Corollary 4.3 ([9, Equation (3)]). *Let $(a_0, a_1, \ldots, a_{n-1}) \in \mathbb{R}^n$. Then*

$$\det(\text{cir}(a_0, a_1, \ldots, a_{n-1})) = \prod_{k=0}^{n-1}\left(\sum_{j=0}^{n-1} a_j \omega_k^j\right).$$

where the

$$\omega_k = \cos\frac{2k}{n}\pi + \mathbf{i}\sin\frac{2k}{n}\pi$$

are the n^{th} roots of unity for each integer $k \in \{0, 1, \ldots, n-1\}$.

Example 4.4. Let $n = 4$, $z = \cos\frac{2\pi}{3} + \mathbf{i}\sin\frac{2\pi}{3}$ and $\boldsymbol{a} = (\mathbf{i}, 2, -2\mathbf{i}, 4) \in \mathbb{C}^4$. Then $\omega_0 = 1, \omega_1 = \mathbf{i}, \omega_2 = -1, \omega_3 = -\mathbf{i}, s_0 = 1, s_1 = \frac{\sqrt{3}}{2} + \frac{1}{2}\mathbf{i}, s_2 = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}$ and $s_3 = \mathbf{i}$.
By Theorem 4.2, we have

$$\det(\text{cir}_z(\boldsymbol{a})) = \prod_{k=0}^{3}\left(\sum_{j=0}^{3} a_j s_j \omega_j^k\right) = (375 + 90\sqrt{3}) + (180 + 48\sqrt{3})\mathbf{i}.$$

From the proof of Theorem 4.2, we conclude the following corollaries.

Corollary 4.5. *Let $(a_0, a_1, \ldots, a_{n-1}) \in \mathbb{C}^n$ and let $z \in \mathbb{C}\backslash\{0\}$. Then the following statements hold.*

1. *The matrix $\text{cir}_z(a_0, a_1, \ldots, a_{n-1})$ is diagonalizable.*

2. *The Eigen values of* $\mathrm{cir}_z(a_0, a_1, \ldots, a_{n-1})$ *are* $\lambda_k = \sum_{j=0}^{n-1} a_j s_j \omega_j^k$, *where* $k = 0, 1, \ldots, n-1$.

3. *For each* $0 \leq k \leq n-1$, *the Eigen vectors of* $\mathrm{cir}_z(a_0, a_1, \ldots, a_{n-1})$ *corresponding to* λ_k *is*

$$X_k = \frac{1}{\sqrt{n}} \begin{bmatrix} s_0\omega_0^k \\ s_1\omega_1^k \\ \vdots \\ s_{n-1}\omega_{n-1}^k \end{bmatrix}.$$

Corollary 4.6. *Let n be a positive integer and let $z \in \mathbb{C}$ be such that $|z| = 1$. Then every $n \times n$ z-twistulant matrix is diagonalizable.*

Example 4.7. Let $z = \cos\frac{2\pi}{3} + \mathbf{i}\sin\frac{2\pi}{3}$. From Example 4.4 and the proof of Theorem 4.2, the Eigen values of $\mathrm{cir}_z(\mathbf{i}, 2, -2\mathbf{i}, 4)$ are $\lambda_0 = 2\sqrt{3} + 5\mathbf{i}$, $\lambda_1 = (3 - \sqrt{3}) + (2 + \sqrt{3}\mathbf{i})$, $\lambda_2 = 3\mathbf{i}$ and $\lambda_3 = (-3 - \sqrt{3}) + (2 - \sqrt{3}\mathbf{i})$. The Eigen vectors of $\mathrm{cir}_z(\mathbf{i}, 2, -2\mathbf{i}, 4)$ are therefore of the forms

$$X_0 = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{4} + \frac{1}{4}\mathbf{i} \\ \frac{1}{4} + \frac{\sqrt{3}}{4}\mathbf{i} \\ \frac{\mathbf{i}}{2} \end{bmatrix}, X_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{-1}{4} + \frac{\sqrt{3}}{4}\mathbf{i} \\ \frac{-1}{4} - \frac{\sqrt{3}}{4}\mathbf{i} \\ \frac{1}{2} \end{bmatrix}, X_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{-\sqrt{3}}{4} + \frac{-1}{4}\mathbf{i} \\ \frac{1}{4} + \frac{\sqrt{3}}{4}\mathbf{i} \\ \frac{-\mathbf{i}}{2} \end{bmatrix}$$

and

$$X_3 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} + \frac{-\sqrt{3}}{4}\mathbf{i} \\ \frac{-1}{4} + \frac{-\sqrt{3}}{4}\mathbf{i} \\ \frac{-1}{2} \end{bmatrix}.$$

Chapter 5

Determinants of Real-Valued Fibonacci Circulant Matrices

In this chapter, we study real-valued Fibonacci circulant matrices. First, we recall the Fibonacci sequence and the determinant of Fibonacci circulant matrices studied in [9]. Finally, we study a real-valued Fibonacci function and determine the determinant of a real-valued Fibonacci circulant matrix.

5.1 Fibonacci Circulant Matrices

In Chapter 2, a circulant matrix, whose rows are composed of cyclically shifted versions of a list $(a_0, a_1, \ldots, a_{n-1}) \in \mathbb{C}^n$, is introduced. In this section, we recall some circulant matrices whose entries are from special numbers studied in [9].

Definition 5.1. The *Fibonacci sequence* F_r of Fibonacci numbers is defined by the recurrence relation

$$F_r = F_{r-1} + F_{r-2}$$

for all $r \geq 3$ with the initial values $F_1 = 1$ and $F_2 = 1$.

Example 5.2. The following numbers are the first 7 terms of the fibonacci sequence:

$$F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8 \text{ and } F_7 = 13.$$

Definition 5.3. An $n \times n$ *Fibonacci circulant matrix* is a circulant matrix whose entries are given by n consecutive Fibonacci numbers. Precisely, an $n \times n$ matrix is called *Fibonacci circulant* if it is of the form

$$\text{cir}(F_r : n) := \text{cir}(F_r, F_{r+1}, \ldots, F_{r+n-1})$$

for some positive integer r.

Example 5.4. From $F_4 = 3, F_5 = 5, F_6 = 8$ and $F_7 = 13$, we have

$$\text{cir}(F_4 : 4) := \text{cir}(F_4, F_5, F_6, F_7) = \begin{bmatrix} 3 & 5 & 8 & 13 \\ 13 & 3 & 5 & 8 \\ 8 & 13 & 3 & 5 \\ 5 & 8 & 13 & 3 \end{bmatrix}.$$

The determinant of a fibonacci circulant matrix is studied in [9] and the main result is as follows.

Theorem 5.5. *Let n and r be natural numbers. Then*

$$\det(\text{cir}(F_r : n)) = \frac{(F_r - F_{n+r})^n - (F_{n+r-1} - F_{r-1})^n}{1 - L_r + (-1)^n}.$$

where $L_r = F_{r-1} + F_{r+1}$ is the r^{th} Lucas number.

Lemma 5.6 ([9, Equation (4)]). *Let x and y be real numbers. Then*

$$\prod_{k=0}^{n-1} (x - y\omega_k) = x^n - y^n.$$

5.2 Real-Valued Fibonacci Circulant Matrices

F. D. Parker [11] and E. Halsey [6] introduced a *real-valued Fibonacci function* $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = \frac{\alpha^x - (\cos \pi x)\alpha^{-x}}{\sqrt{5}},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ for all real numbers x.

Note that the restriction function of the real-valued Fibonacci function to the set $\mathbb{N}$ of natural numbers is the Fibonacci sequence.

Example 5.7. Some values of the real-valued Fibonacci function are given as follows.

$$F(\frac{1}{2}) = \sqrt{\frac{1+\sqrt{5}}{10}}, F(\frac{3}{2}) = \sqrt{\frac{(1+\sqrt{5})^3}{40}} \text{ and } F(\frac{5}{2}) = \sqrt{\frac{(1+\sqrt{5})^5}{160}}.$$

Definition 5.8. A *real-valued Fibonacci circulant matrix* is a matrix of the form

$$\text{cir}(F(r) : n) := \text{cir}(F(r), F(r+1), \ldots, F(r+n-1))$$

for some positive integer n and real number r.

Example 5.9. From Example 5.7 and Definition 5.8, we have

$$\text{cir}(F(\frac{1}{2}) : 3) = \begin{bmatrix} \sqrt{\frac{1+\sqrt{5}}{10}} & \sqrt{\frac{(1+\sqrt{5})^3}{40}} & \sqrt{\frac{(1+\sqrt{5})^5}{160}} \\ \sqrt{\frac{(1+\sqrt{5})^5}{160}} & \sqrt{\frac{1+\sqrt{5}}{10}} & \sqrt{\frac{(1+\sqrt{5})^3}{40}} \\ \sqrt{\frac{(1+\sqrt{5})^3}{40}} & \sqrt{\frac{(1+\sqrt{5})^5}{160}} & \sqrt{\frac{1+\sqrt{5}}{10}} \end{bmatrix}.$$

The determinant of a real-valued Fibonacci circulant matrix can be determined as follows.

Theorem 5.10. *Let n be a positive integer and let $r \in \mathbb{R}$. Then*

$$\text{det}(\text{cir}(F(r) : n)) = \frac{(F(r) - F(n+r))^n - (F(n+r-1) - F(r-1))^n}{1 - (-\alpha^{-1})^n - \alpha^n + (-1)^n}.$$

Proof. For $0 \leq j < n$, let

$$a_j = F(j+r) = \frac{\alpha^{j+r} - (\cos \pi(j+r))\alpha^{-(j+r)}}{\sqrt{5}}.$$

Then

$$\text{det}(\text{cir}(F(r) : n)) = \text{det}(\text{cir}(a_0, a_1, \ldots, a_{n-1})).$$

By Corollary 4.3, we have Equation (5.1).

$$
\begin{aligned}
&\det(\operatorname{cir}(F(r):n))\\
&= \prod_{k=0}^{n-1}\left(\sum_{j=0}^{n-1} F(j+r)\omega_k^j\right)\\
&= \prod_{k=0}^{n-1}\left(\sum_{j=0}^{n-1}\left(\frac{\alpha^{j+r}-(\cos\pi(j+r))\alpha^{-(j+r)}}{\sqrt{5}}\right)\omega_k^j\right)\\
&= (\sqrt{5})^{-n}\prod_{k=0}^{n-1}\left(\sum_{j=0}^{n-1}\left(\alpha^r(\alpha\omega_k)^j-(\cos\pi(j+r))(\alpha^{-r})(\alpha^{-1}\omega_k)^j\right)\right)\\
&= (\sqrt{5})^{-n}\prod_{k=0}^{n-1}\left(\frac{\alpha^r(1-(\alpha\omega_k)^n)}{1-\alpha\omega_k}-\frac{(\cos\pi r)(\alpha^{-r})(1-((\cos\pi)\alpha^{-1}\omega_k)^n)}{1-(\cos\pi)\alpha^{-1}\omega_k}\right)\\
&= \prod_{k=0}^{n-1}\frac{(\alpha^r-\alpha^{n+r}\omega_k^n)(1-(\cos\pi)\alpha^{-1}\omega_k)-((\cos\pi r)\alpha^{-1}-(\cos\pi(n+r))\alpha^{-(n+r)}\omega_k^n)(1-\alpha\omega_k)}{\sqrt{5}(1-\alpha\omega_k)(1-(\cos\pi)\alpha^{-1}\omega_k)}\\
&= \prod_{k=0}^{n-1}\frac{\alpha^r-(\cos\pi r)\alpha^{-r}-\alpha^{n+r}+(\cos\pi(n+r))\alpha^{-(n+r)}-(((\cos\pi)\alpha^{r-1}-(\cos\pi r)\alpha^{-(r-1)}+\cos\pi(n+r)\alpha^{-(n+r-1)})\omega_k)}{\sqrt{5}(1-\alpha\omega_k)(1-(\cos\pi)\alpha^{-1}\omega_k)}\\
&= \prod_{k=0}^{n-1}\frac{\left(\left(\frac{\alpha^r-(\cos\pi)\alpha^{-r}}{\sqrt{5}}\right)-\left(\frac{\alpha^{n+r}-(\cos\pi(n+r))\alpha^{-(n+r)}}{\sqrt{5}}\right)+\left(\left(\frac{\alpha^{r-1}-(\cos\pi(r-1))\alpha^{-(r-1)}}{\sqrt{5}}\right)-\left(\frac{\alpha^{n+r-1}-(\cos\pi(n+r-1))\alpha^{-(n+r-1)}}{\sqrt{5}}\right)\right)\omega_k\right)}{(1-\alpha\omega_k)(1-(\cos\pi)\alpha^{-1}\omega_k)}\\
&= \prod_{k=0}^{n-1}\frac{F(r)-F(n+r)+(F(r-1)-F(n+r-1))\omega_k}{(1-\alpha\omega_k)(1-(\cos\pi)\alpha^{-1}\omega_k)}. \qquad (5.1)
\end{aligned}
$$

Hence,

$$\det(\text{cir}(F(r):n)) = \prod_{k=0}^{n-1} \frac{F(r) - F(n+r) + (F(r-1) - F(n+r-1))\omega_k}{(1-\alpha\omega_k)(1-(\cos\pi)\alpha^{-1}\omega_k)}.$$

By Lemma 5.6, we have

$$\prod_{k=0}^{n-1}(F(r) - F(n+r) + (F(r-1) - F(n+r-1))\omega_k)$$
$$= (F(r) - F(n+r))^n - (F(n+r-1) - F(r-1))^n,$$

$$\prod_{k=0}^{n-1}(1-\alpha\omega_k) = (1-\alpha^n),$$

and

$$\prod_{k=0}^{n-1}(1-(\cos\pi)\alpha^{-1}\omega_k) = (1-(\cos\pi n)\alpha^{-n}).$$

Hence,

$$\prod_{k=0}^{n-1}(1-\alpha\omega_k)(1-(\cos\pi)\alpha^{-1}\omega_k) = (1-\alpha^n)(1-(\cos\pi n)\alpha^{-n})$$
$$= 1-(\cos\pi n)\alpha^{-n} - \alpha^n + \cos\pi n$$
$$= 1-(-1)^n\alpha^{-n} - \alpha^n + (-1)^n$$
$$= 1-(-\alpha^{-1})^n - \alpha^n + (-1)^n.$$

Therefore, we have

$$\det(\text{cir}(F(r):n)) = \frac{(F(r) - F(n+r))^n - (F(n+r-1) - F(r-1))^n}{1-(-\alpha^{-1})^n - \alpha^n + (-1)^n}$$

as desired. □

Remark 5.11. If r is an integer, we have

$$\det(\text{cir}(F(r):n)) = \frac{(F_r - F_{n+r})^n - (F_{n+r-1} - F_{r-1})^n}{1 - L_r + (-1)^n} = \det(\text{cir}(F_r:n)).$$

Example 5.12. Consider $F(\frac{3}{2}) = \sqrt{\frac{(1+\sqrt{5})^3}{40}}, F(\frac{5}{2}) = \sqrt{\frac{(1+\sqrt{5})^5}{160}}$ and $F(\frac{7}{2}) = \sqrt{\frac{(1+\sqrt{5})^7}{640}}$. Then, by Theorem 5.10. below, we have

$$\det(\text{cir}(F(\frac{3}{2}), F(\frac{5}{2}), F(\frac{7}{2}):3)) \simeq 8.1663.$$

$$
\begin{aligned}
\det(\mathrm{cir}(F(\tfrac{3}{2}):3)) &= \frac{(F(\frac{3}{2})-F(\frac{9}{2}))^3-(F(\frac{7}{2})-F(\frac{1}{2}))^3}{1-(-\alpha^{-1})^3-\alpha^3+(-1)^3} \\
&= \frac{\left(\sqrt{\frac{(1+\sqrt{5})^3}{40}}-\sqrt{\frac{(1+\sqrt{5})^9}{2560}}\right)^3-\left(\sqrt{\frac{(1+\sqrt{5})^7}{640}}-\sqrt{\frac{1+\sqrt{5}}{10}}\right)^3}{-(-\alpha^{-1})^3-\alpha^3} \\
&= \frac{\alpha^3\left(\left(\sqrt{\frac{(1+\sqrt{5})^3}{40}}-\sqrt{\frac{(1+\sqrt{5})^9}{2560}}\right)^3-\left(\sqrt{\frac{(1+\sqrt{5})^7}{640}}-\sqrt{\frac{1+\sqrt{5}}{10}}\right)^3\right)}{1-\alpha^6} \\
&= \frac{(\frac{1+\sqrt{5}}{2})^3\left(\left(\sqrt{\frac{(1+\sqrt{5})^3}{40}}-\sqrt{\frac{(1+\sqrt{5})^9}{2560}}\right)^3-\left(\sqrt{\frac{(1+\sqrt{5})^7}{640}}-\sqrt{\frac{1+\sqrt{5}}{10}}\right)^3\right)}{1-(\frac{1+\sqrt{5}}{2})^6} \\
&= \frac{8(1+\sqrt{5})^3\left(\left(\sqrt{\frac{(1+\sqrt{5})^3}{40}}-\sqrt{\frac{(1+\sqrt{5})^9}{2560}}\right)^3-\left(\sqrt{\frac{(1+\sqrt{5})^7}{640}}-\sqrt{\frac{1+\sqrt{5}}{10}}\right)^3\right)}{64-(1+\sqrt{5})^6} \\
&\simeq 8.1663.
\end{aligned}
\tag{5.2}
$$

References

[1] E. Catalan, “Récherches sur les Déterminants”, *Bulletin de l’Academie Royale de Belgique,* 13, 534–555(1846).

[2] P. J. Davis, *Circulant Matrices*, Chelesa publishing, New York, second edition, 1994.

[3] H. Q. Dinh, S. R. López-Permouth, “Cyclic and negacyclic codes over finite chain rings”, *IEEE Transactions on Information Theory*, 50(8), 1735(2004).

[4] D. S. Dummit, R. M. Foote, *Abstract Algebra*, John Wiley & Sons, United States of America, third edition, 2004.

[5] M. Grassl, T. A. Gulliver, “On circulant self-dual codes over small fields”, *Designs, Codes and Cryptography,* 52, 57–81(2009).

[6] E. Halsey, “The Fibonacci number F_u where u is not an integer”, *Fibonacci Quarterly*, 3(2), 147–152(1965).

[7] I. Kra, S. R. Simanca, “On circulant matrices”, *Notices of the AMS*, 59(3), 368–377(2012).

[8] D. C. Lay, *Linear Algebra and Its Applications*, Addison Wesley Longman, New York, second edition, 1997.

[9] D. Lind, “A Fibonacci circulant”, *Fibonacci Quarterly*, 8, 449–455(1970).

[10] W. K. Nicholson, *Linear Algebra with Applications*, McGraw-Hill, Singapore, fourth edition, 2002.

[11] F. D. Parker, "A Fibonacci function", *Fibonacci Quarterly*, 6(1), 1–2(1968).

[12] D. Rocchesso, J. O. Smith, "Circulant and elliptic feedback delay networks for artificial reverberation", *IEEE Transactions on Speech and Audio Processing,* 5(1), 51–63(1997).

[13] C. Zhang, G. Dangelmayr, I. Oprea, "Storing cycles in Hopfield-type networks with pseudoinverse learning rule: admissibility and network topology", *Neural Networks,* 46, 283–298(2013).

[14] Y. Zheng, S. Shon, "Exact inverse matrices of Fermat and Mersenne circulant matrix", *Abstract and Applied Analysis,* 2015, ID 760823 (10 pages)(2015).

BIOGRAPHY

NAME	Miss Sirikanya Kittiwut
ADDRESS	176/278 Moo 1, Tambon Makhamtia, Amphoe Mueang, Suratthani, 84000
INSTITUTION ATTENDED	
2006	Bachelor of Science in Mathematics, Thaksin University
2007	Graduate Diploma in Teaching Profession, Thaksin University
2016	Master of Science in Mathematics Study, Silpakorn University