# A GALOIS CONNECTION OF ALGEBRAIC RELATIONS AND COLORED RESETS

By

Udom Chotwattakawanit

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree

Doctor of Philosophy Program in Mathematics

International Program

Graduate School, Silpakorn University

Academic Year 2015

# การเชื่อมโยงแบบกาลัวส์ของความสัมพันธ์พีชคณิตและรีเซตซึ่งถูกลงสี

โดย

นายอุดม โชติวรรธกวณิช

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาปรัชญาดุษฎีบัณฑิต

สาขาวิชาคณิตศาสตร์

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ปีการศึกษา 2558

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The Graduate School, Silpakorn University has approved and accredited the Thesis title of "A Galois Connection of Algebraic Relaions and Colored Resets" submitted by Mr. Udom Chotwattakawanit as a partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics.

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ความสัมพันธ์พีชคณิตเหนือพีชคณิต $\underline{M}$ เป็นพีชคณิตย่อยของ $\underline{M}^n$ สำหรับบางจำนวนนับ $n$ สำหรับแต่ละรีเซต $\boldsymbol{M}$ การลงสีบน $\boldsymbol{M}$ คือคู่ของรีเซต $\boldsymbol{H}$ ซึ่งมีแบบเดียวกันกับ $\boldsymbol{M}$ และฟังก์ชันบางส่วน $h$ จาก $H$ ไป $M$

ในวิทยานิพนธ์นี้เราสร้างการเชื่อมโยงแบบกาลัวส์ระหว่างเซตของความสัมพันธ์พีชคณิตและเซตของการลงสีบน $\boldsymbol{M}$ และประยุกต์การเชื่อมโยงนี้สำหรับแก้ปัญหาบางประการในพีชคณิตและทฤษฎีโคลน เราแสดงว่าถ้า $\underline{M} = (M; F)$ เป็นคอนแสตนทีฟและมีโครงสร้างซึ่งทำให้เกิดดูอัลลิตี้แบบจำกัดบน $\mathcal{A} = \mathbb{ISP}(\underline{M})$ แล้วจะมีความสัมพันธ์พีชคณิต $r$ โดยที่ $(M; r, \mathcal{T})$ ทำให้เกิดดูอัลลิตี้บน $\mathcal{A}$ การประยุกต์การเชื่อมโยงแบบกาลัวส์ควบคู่กับทฤษฎีบทเอ็นยู-ดูอัลลิตี้ ทำให้เราสามารถแสดงอัลเตอร์อีโกซึ่งดูอัลไลซ์พีชคณิตพริมอลแบบทอเลอรันซ์เสียงข้างมาก และเราได้จำแนกโคลนใหญ่สุดเฉพาะกลุ่มที่บรรจุ $\langle F \rangle$

ภาควิชาคณิตศาสตร์ บัณฑิตวิทยาลัย มหาวิทยาลัยศิลปากร

ลายมือชื่อนักศึกษา...................................... ปีการศึกษา 2558

ลายมือชื่ออาจารย์ที่ปรึกษาวิทยานิพนธ์ .......................................

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An algebraic relation over an algebra $\underline{M}$ is a subalgebra of $\underline{M}^n$ for some natural number $n$. For each reset $\boldsymbol{M}$, an $\boldsymbol{M}$-colored reset is a pair of reset $\boldsymbol{H}$ of the same type of $\boldsymbol{M}$ and a partial function $h$ from $H$ to $M$.

In this thesis, we formulate a Galois connection between the set of algebraic relations and the set of $\boldsymbol{M}$-colored resets and apply it to solve some problems in algebra and clone theory. We show that if $\underline{M} = (M; F)$ is constantive and there is a structure which yields a duality of finite type on $\mathcal{A} = \mathbb{ISP}(\underline{M})$ then there exists an algebraic relation $r$ such that $(M; r, \mathcal{T})$ yields a duality on $\mathcal{A}$. Applying the Galois connection together with NU-duality Theorem, we show an alter ego which dualise a majority tolerance-primal algebra and then we characterize all maximal clones containing $\langle F \rangle$.

Department of Mathematics Graduate School, Silpakorn University

Student's signature ........................................ Academic Year 2015

Thesis Advisor's signature ........................................

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# Chapter 1

# Introduction

Let $\underline{\mathrm{M}} = (M; F)$ be a finite algebra. An $n$-ary relation $r$ on $M$ is said to be *algebraic* over $\underline{\mathrm{M}}$ if $r$ forms a subalgebra of $\underline{\mathrm{M}}^n$; or equivalently, the smallest clone $\langle F \rangle$ containing $F$ is a subclone of the clone of all operations preserving $r$. An algebraic relation is concerned not only in the concepts of algebra, but also in clone theory. However for a large $n$ or cardinality of $M$, the set $M^n$ is very large; so, it is complicated to study and see some properties of $n$-ary algebraic relation over an algebra $\underline{\mathrm{M}}$. So, it is interesting to find a supportive tool to investigate an algebraic relation.

A *graph* $\mathbf{M}$ is a structure consisting of a set $M$ of *vertices* and a set $\Theta \subseteq M \times M$ of *edges*. If $\Theta$ is *tolerance*, reflexive and symmetric, then $\mathbf{M}$ is called a *reflexive graph*. A graph was extensively studied since it can be represented by a picture. A graph is generalized as a *relational set* (or briefly, *reset*), a structure consisting of a set $M$ and a set of finitary relations on $M$. Some problems about reset $\mathbf{M}$ were solved via an $\mathbf{M}$-colored reset $(\mathbf{H}, h)$, a pair of reset $\mathbf{H}$ of the same type of $\mathbf{M}$ and a partial function $h$ from $H$ to $M$.

A Galois connection is a special connection between two sets of objects (usually) of different kinds. It is a useful tool to study properties of one kind of objects via the properties of the other (normally simpler) kind of objects. In [6], Davey, Haviar and Priestley gave a characterization of algebraic relations over $\underline{\mathrm{M}}$ in terms of morphisms between two resets. We will formulate a Galois connection between the set of $\mathbf{M}$-colored resets and the set of algebraic relations from their characterization and then apply the Galois connection to solve some problems in both algebra and clone theory.

In chapter 2, we summarize some basic concepts about clone theory, Galois connection, duality and relational set which are used in the sequel.

In chapter 3, we show that if $\underline{\mathrm{M}} = (M; F)$ is constantive; that is, $\langle F \rangle$ contains all constants, and dualisable of finite type, then there exists an algebraic relation $r$ such that $(M; r, \mathscr{T})$ yields an optimal duality on $\mathscr{A} = \mathbb{ISP}(\underline{\mathrm{M}})$.

In chapter 4, we apply the Galois connection together with NU-duality Theorem [5] to show a duality for a *majority tolerance-primal algebra* $\underline{\mathrm{M}} = (M; F)$; that is, $\langle F \rangle$ contains a majority operation and all operations in $\langle F \rangle$ preserve a tolerance relation; and then we characterize all maximal clones containing $\langle F \rangle$.

# Chapter 2

# Preliminaries

In this thesis, we study a Galois connection between algebraic relations and colored resets and then we apply it to solve some problems in duality and clone theory. According to unfamiliar concepts which are used in the sequel, we will introduce and review them in this chapter.

## 2.1 Clone Theory

Let $M$ be a finite set and $\mathbb{N}$ be the set of all natural numbers. For each $n \in \mathbb{N}$, a function $f : M^m \to M$ is called an *m-ary operation* on $M$ and is said to have *arity* $m$. Denote $O^m(M)$ the set of all $m$-ary operations on $M$ and let $O(M) \coloneqq \bigcup_{n\in\mathbb{N}} O^n(M)$. For each $m, i \in \mathbb{N}$ with $i \leq m$, the function $e_i^m : M^m \to M$ defined by

$$e_i^m(a_1, \dots, a_m) = a_i$$

for all $a_1, \dots, a_m \in M$ is called a *projection function*. A subset $C$ of $O(M)$ is called a *clone* (*on* $M$) if $C$ contains all projection functions and is *closed under composition*; that is, if $f_1, \dots, f_n$ are $k$-ary functions in $C$ and $g$ is an $n$-ary function in $C$ for some $k, n \in \mathbb{N}$, then $g(f_1, \dots, f_n) \in C$ where $g(f_1, \dots, f_n) : M^k \to M$ is defined by

$$g(f_1, \dots, f_n)(a_1, \dots, a_k) = g(f_1(a_1, \dots, a_k), \dots, f_n(a_1, \dots, a_k))$$

for all $a_1, \dots, a_k \in M$. A clone on a 2-elements set is called the *Boolean clone*. For clones $C_1$ and $C_2$ on $M$, $C_1$ is called a *subclone* of $C_2$ if $C_1 \subseteq C_2$. Note that $O(M)$ is the greatest clone and is called the *full clone* (*on* $M$).

For each $h \in \mathbb{N}$, a subset $\rho$ of $M^h$ is called an *h-ary relation* on $M$. Denote $R^h(M)$ the set of all $h$-ary relations on $M$ and let $R(M) \coloneqq \bigcup_{n\in\mathbb{N}} R^n(M)$. For each $n$-ary operation $f$ and $h$-ary relation $\rho$ on $M$, we say that $f$ *preserves* $\rho$ or $\rho$ *is invariant* under $f$ if

$$(f(x_1^1, \ldots, x_1^n), \ldots, f(x_h^1, \ldots, x_h^n)) \in \rho$$

whenever $(x_1^1, \ldots, x_h^1), \ldots, (x_1^n, \ldots, x_h^n) \in \rho$.

**Example 2.1** *Let* $M = \{a_0, a_1, a_2, a_3\}$ *and*

$$\begin{aligned} \rho &= \{(a_0, a_1), (a_1, a_2), (a_2, a_3), (a_3, a_0)\} \\ &\quad \cup \{(a_1, a_0), (a_2, a_1), (a_3, a_2), (a_0, a_3)\} \cup \{(a_0, a_0), (a_2, a_2)\}. \end{aligned}$$

*Then $M$ and $\rho$ can be shown as a picture such that elements in $M$ are represented as vertices and each element $(x, y)$ in $\rho$ is represented by a line joining $x$ to $y$.*

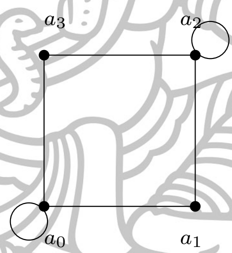


Figure 1. A picture representing the sets $M$ and $\rho$.

*Define functions $f_1$ and $f_2$ from $M$ to $M$ by*

$$f_1(x) = \begin{cases} a_0 & \text{if } x = a_0, \\ a_1 & \text{if } x = a_1 \text{ or } a_3, \\ a_2 & \text{if } x = a_2 \end{cases}$$

*and*

$$f_2(x) = \begin{cases} a_0 & \text{if } x = a_0 \text{ or } a_1, \\ a_2 & \text{if } x = a_2 \text{ or } a_3. \end{cases}$$

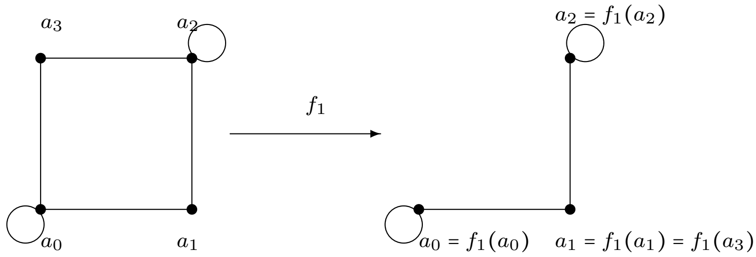


Figure 2. A picture representing the function $f_1$.

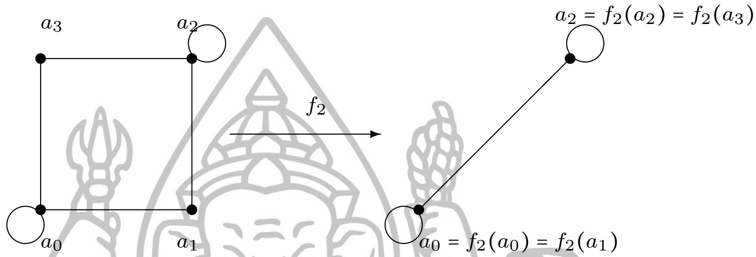


Figure 3. A picture representing the function $f_2$.

*Then* $(f_1(a_0), f_1(a_0)) = (a_0, a_0)$, $(f_1(a_2), f_1(a_2)) = (a_2, a_2)$, $(f_1(a_0), f_1(a_1)) = (a_0, a_1)$, $(f_1(a_1), f_1(a_2)) = (a_1, a_2)$, $(f_1(a_0), f_1(a_3)) = (a_0, a_1)$ *and* $(f_1(a_2), f_1(a_3)) = (a_2, a_1)$, $(f_1(a_1), f_1(a_0)) = (a_1, a_0)$, $(f_1(a_2), f_1(a_1)) = (a_2, a_1)$, $(f_1(a_3), f_1(a_0)) = (a_1, a_0)$ *and* $(f_1(a_3), f_1(a_2)) = (a_1, a_2)$; *and all belong to* $\rho$; *so,* $f_1$ *preserves* $\rho$. *However,* $f_2$ *does not preserve* $\rho$ *since* $(a_1, a_2) \in \rho$ *but* $(f_2(a_1), f_2(a_2)) = (a_0, a_2) \notin \rho$.

*Nevertheless, if* $\rho' = \rho \cup \{(a_0, a_2), (a_2, a_0)\} \cup \{(a_1, a_3), (a_3, a_1)\}$,

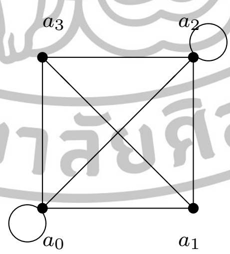


Figure 4. A picture representing the sets $M$ and $\rho'$.

*then* $f_2$ *preserves* $\rho'$; *but,* $f_1$ *does not preserve* $\rho'$ *since* $(a_1, a_3) \in \rho'$ *and* $(f_1(a_1), f_1(a_3)) = (a_1, a_1) \notin \rho'$.

For each $R \subseteq R(M)$ and $F \subseteq O(M)$, we denote the set of all operations preserving all elements in $R$ and the set of all relations which are invariant under all elements in $F$ by $\mathrm{Pol}\,(R)$ and $\mathrm{Inv}\,(F)$, respectively; that is,

$$\mathrm{Pol}\,(R) = \{f \in O(M) \,|\, f \text{ preserves } \rho \text{ for all } \rho \in R\}$$

and

$$\operatorname{Inv}(F) = \{\rho \in R(M) \mid \rho \text{ is invariant under } f \text{ for all } f \in F\}.$$

One can prove that $\operatorname{Pol}(R)$ is a clone and it was proved (e.g. see R. Pöschel and L. A. Kalužnin in [15]) that $C = \operatorname{Pol}(\operatorname{Inv}(C))$ for all clones $C$. Dually, a set $D$ of relations is called *relational clone* if $D = \operatorname{Inv}(\operatorname{Pol}(D))$. It is a well-known fact that the set of all clones on a finite set is an ordered set with respect to inclusion; in fact, it is a complete lattice which is dually isomorphic to the complete lattice of all relational clones.

For each $F \subseteq O(M)$, the *clone generated by F* is the smallest clone containing $F$ and is denoted by $\langle F \rangle$. It is interesting whether a subset $F$ of $O(M)$ generates $O(M)$; this question is known as the *functional completeness problem*. The functional completeness problem can be studied via the *maximal clones*, the co-atoms of the lattice of all clones. E. L. Post [16] proved that $O(M)$ is finitely generated which implies by [13] that every proper subclone of the full clone contains in a maximal one and there are only finitely many maximal clones. Hence, for each $F \subseteq O(M)$, $\langle F \rangle = O(M)$ if and only if $F$ is not contained in one of the maximal clones. Efforts to determine all maximal clones began more than 50 years. I.G. Rosenberg [20, 21] was the first one who succeeded in describing all maximal clones; they are just the clones $\operatorname{Pol}(\rho)$ where $\rho$ is a relation in one of six classes of relations defined as follow:

**Class(1)**: The set of all bounded orders. These are reflexive, transitive and anti-symmetric binary relations $\rho \subseteq M \times M$ with $(0,x) \in \rho$ and $(x,1) \in \rho$ for all $x \in M$ and for some $0, 1 \in M$.

**Class(2)**: The set of all prime permutations. These are permutations on $M$ which all of whose cycles have the same prime length.

**Class(3)**: The class of all prime affine relations. A 4-ary relation $\rho \subseteq M^4$ is *affine* if we can define an abelian group operation, +, on $M$ so that $(a,b,c,d) \in \rho$ if and only if $a + b = c + d$. An affine relation $\rho$ is *prime* if $\langle M; + \rangle$ is an abelian group of prime power order. This class is empty unless $|M|$ is a prime power.

**Class(4)**: The class of all non-trivial equivalence relations. These are reflexive, symmetric and transitive binary relations $\rho \subseteq M \times M$ which are neither the diagonal relation $\triangle_M := \{(a,a) \mid a \in M\}$ nor the universal relation $\nabla_M := M \times M$.

**Class(5)**: The class of all relations which are $k$-regularly generated for some $3 \leq k \leq |M|$. For $3 \leq k \leq |M|$, a set $T = \{\Theta_1, \Theta_2, \ldots, \Theta_m\}\,(m \geq 1)$ of equivalence relations on $M$ is *$k$-regular* if each $\Theta_i$, $(1 \leq i \leq m)$ has exactly $k$ equivalence classes and the intersection $\cap_{i=1}^{m} \varepsilon_i$ of arbitrary equivalence classes $\varepsilon_i$ of $\Theta_i$ is nonempty. A $k$-ary relation $\rho = \{(a_1, \ldots, a_k) | a_i \in A$ for all $i = 1, \ldots, k\}$ is *$k$-regularly generated* by $T$ if for each $i \in \{1, \ldots, m\}$, at least two of the elements $a_1, \ldots, a_k$ are equivalent modulo $\Theta_i$.

**Class(6)**: The class of all central relations. A $k$-ary relation $\rho \subseteq M^k$ for some $k \geq 1$ is *totally reflexive* if $\{(a_1, \ldots, a_k) | a_i = a_j$ for some $i \neq j\} \subseteq \rho$; and is *totally symmetric* if for any permutation $\alpha$ on $\{1, \ldots, k\}$ we have $(a_1, \ldots, a_k) \in \rho$ if and only if $(a_{\alpha(1)}, a_{\alpha(2)}, \ldots, a_{\alpha(k)}) \in \rho$. The center $C_\rho$ of $\rho$ is the set of all $a \in M$ such that $(a, a_2, \ldots, a_k) \in \rho$ for all $a_2, \ldots, a_k \in M$. We say that $\rho$ is *central* if it is totally reflexive, totally symmetric and $\varnothing \neq C_\rho \subsetneq M$.

Describing the lattice of all clones is still well known open problems. Up to now, it is only possible to describe all clones on a finite set is only the set of cardinality 2. The work was first described by E. L. Post [16] in 1941. The lattice of all Boolean clones is also called *Post's lattice*. It is countably infinite and all Boolean clones are finitely generated. The Post's lattice is shown in Figure 5.

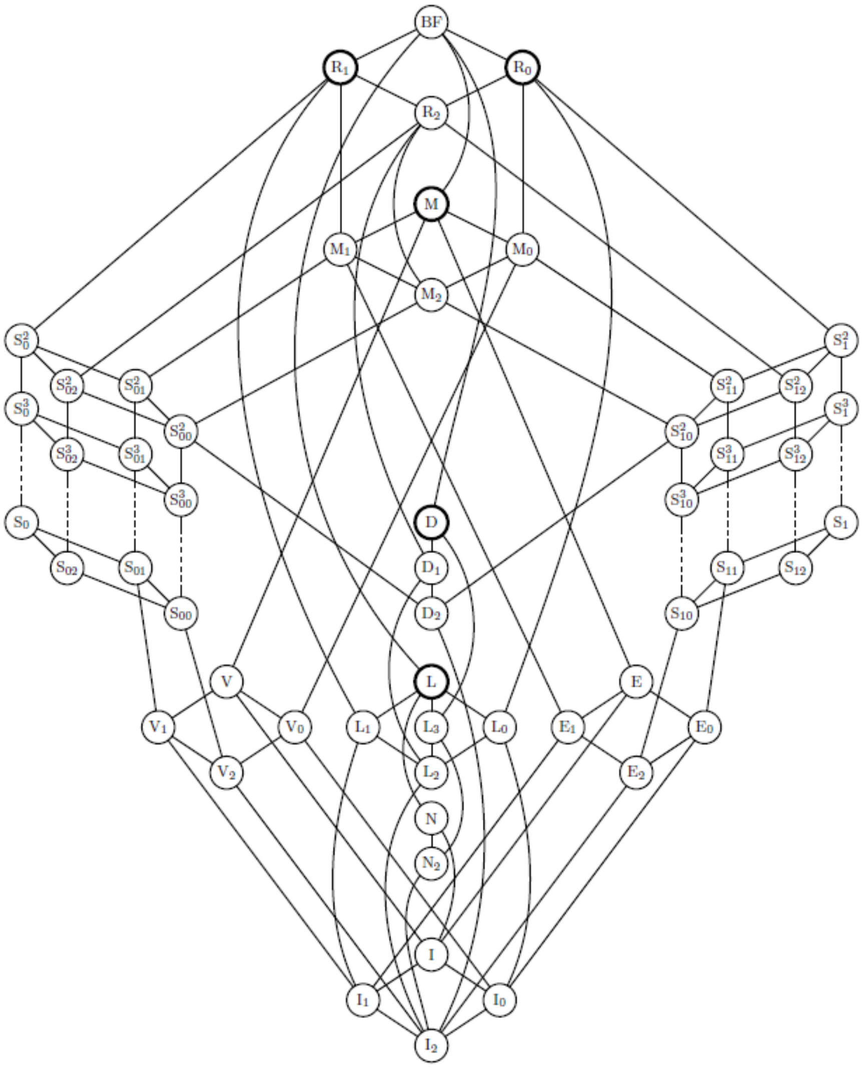


Figure 5. Post's lattice.



However in 1959, Ju. I. Janov and A. A. Muchnik [14] proved that lattice of all clones over a finite set whose cardinality more than 2 is an uncountably infinite; much of the lattice are unknown. Describing of some parts of these lattices is still interesting for studying clone theory; for instance, the clone $Pol(\leq)$ on a set $P$ is the set of all finitary order-preservings with respect to an order $\leq$ on $P$; and we call $Pol(\leq)$ the *monotone clone* of $\mathbf{P} = (P; \leq)$. The monotone clone of a finite ordered set is maximal if and only if the order is bounded. Davey et al. proved in [9] that if a finite ordered set $\mathbf{P}$ is disconnected, then the nontrivial equivalence relation $\Theta$ whose blocks are connected components of $\mathbf{P}$ will give a maximal clone $Pol(\Theta)$ containing the monotone clone of the ordered set $\mathbf{P}$. C. Ratanaprasert [18] has shown that the monotone clone of a finite unbounded connected ordered set is a subclone of a maximal clone preserving only either a $k$-regularly generated relation or a central relation with arity more than 1 and also proved that if the monotone clone of the ordered set contains in a maximal clone preserving a $k$-regularly generated relation, then the monotone clone contains no *near-unanimity functions*, a function $f: P^n \to P$ $(n \geq 3)$ satisfying

$$f(x, x, \ldots, x, y) = f(x, x, \ldots, x, y, x) = \ldots = f(y, x, x, \ldots, x) = x$$

for all $x, y \in P$. If $n = 3$, a near-unanimity function is called a *majority function*. Such function was discovered by K. Baker and A. Pixley [1] in 1975 and then it is extensively studied in many fields of mathematics.

## 2.2 Galois Connection

In 1811-1832, Évariste Galois mentioned a connection between subgroups of the Galois group of an extension $E/F$ and intermediate fields between $E$ and $F$ (see e.g. in [12]). By this connection, properties of permutation groups are applied to study in field theory; so, some problems in field theory can be reduced to simpler problems in group theory. Such connection is generalized to a connection, a so-called Galois connection, between two sets of objects (usually) of different kinds. Galois connection can provide a useful tool for studying properties of one kind of objects via the properties of the other (normally simpler) kind of objects.

A *Galois connection* between the sets $A$ and $B$ is a pair $(\sigma, \tau)$ of functions between the power sets $\mathcal{P}(A)$ and $\mathcal{P}(B)$,

$$\sigma : \mathcal{P}(A) \to \mathcal{P}(B) \text{ and } \tau : \mathcal{P}(B) \to \mathcal{P}(A),$$

such that for all $X, X' \subseteq A$ and all $Y, Y' \subseteq B$ the following conditions are satisfied:

1. $X \subseteq X' \Rightarrow \sigma(X) \supseteq \sigma(X')$, and $Y \subseteq Y' \Rightarrow \tau(Y) \supseteq \tau(Y')$;

2. $X \subseteq \tau\sigma(X)$, and $Y \subseteq \sigma\tau(Y)$.

One of well-known Galois connections is the connection between clones and relational clones.

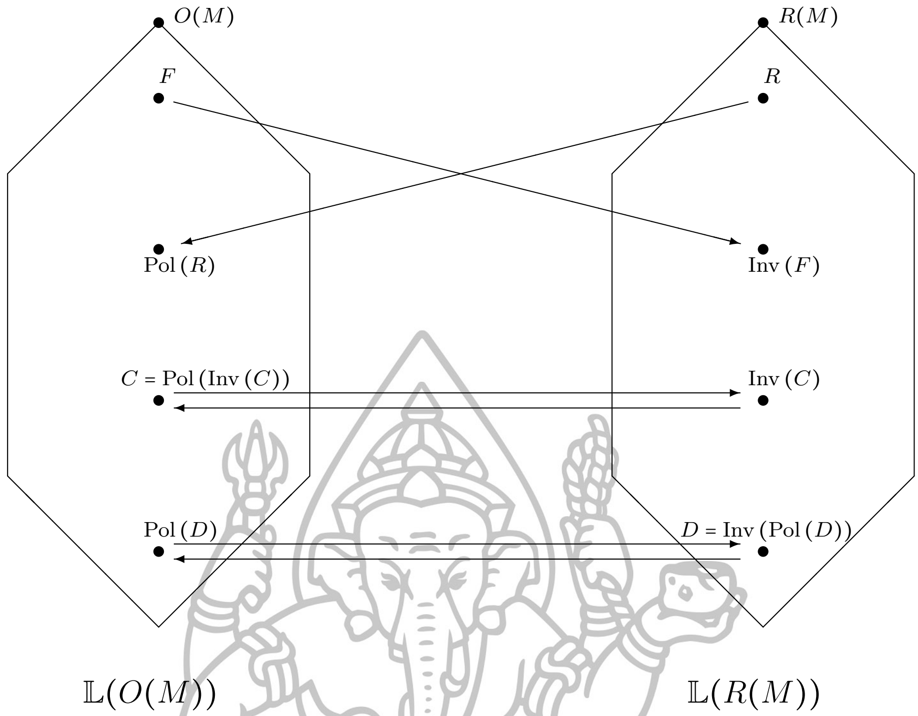


Figure 6. A Galois connection between $\mathbb{L}(O(M))$ and $\mathbb{L}(R(M))$.

In [4], V. G. Bodnarchuk, L. A. Kalužnin, V. N. Kotov and B. A. Romov proved that the function Pol from the lattice $\mathbb{L}(R(M))$ of all relational clones to the lattice $\mathbb{L}(O(M))$ of all clones which maps a relational clone $D$ to the clone $\mathrm{Pol}\,(D)$ and the function Inv from $\mathbb{L}(O(M))$ to $\mathbb{L}(R(M))$ which maps a clone $C$ to the relational clone $\mathrm{Inv}\,(C)$ are bijective reserving the order $\subseteq$; i.e.,

1. for each clones $C$ and $C'$, $C \subseteq C' \Rightarrow \mathrm{Inv}\,(C) \supseteq \mathrm{Inv}\,(C')$;

2. for each relational clones $D$ and $D'$, $D \subseteq D' \Rightarrow \mathrm{Pol}\,(D) \supseteq \mathrm{Pol}\,(D')$.

This is an important tool to understand any clone and so is any *algebra* $\underline{\mathrm{M}} = (M; F)$, a structure consists of a set $M$ and a set $F$ of operations on $M$, since $\underline{\mathrm{M}}$ corresponds to the clone $\langle F \rangle$, the set all *term operations* of $\underline{\mathrm{M}}$. A Galois connection is useful in not only algebras but also many fields of Mathematics; for instance, a Galois connection between subgroups of fundamental groups and covering spaces in field of algebraic topologies. By this connection, algebraic properties about finding all subgroups are used to solve topological problems.

## 2.3 Duality

An algebra is a structure $\underline{\mathrm{M}}$ consisting of a nonempty set $M$, is called the *universe* of $\underline{\mathrm{M}}$ and a set $\{f_i^M\}_{i\in I}$ of operations defined on the universe, are called the *set of fundamental operations* of $\underline{\mathrm{M}}$. The sequence $(n_i)_{i\in I}$ of all arities is called the *type* of the algebra $\underline{\mathrm{M}}$. Some algebraic properties which are commonly studied in general algebras (groups, rings, lattices, etc.) are subalgebras, homomorphisms and direct products.

Let $\underline{\mathrm{M}} = (M; \{f_i^M\}_{i\in I})$ and $\underline{\mathrm{N}} = (N; \{f_i^N\}_{i\in I})$ be algebras of the same type. $\underline{\mathrm{N}}$ is called a *subalgebra* of $\underline{\mathrm{M}}$ if the following conditions are satisfied:

1. $N \subseteq M$;
2. $f_i^N$ is the restriction of the operation $f_i^M$ to the set $N$, denoted by $f_i^M \restriction_N$, for all $i \in I$.

A function $h$ from $M$ to $N$ is called a *homomorphism*, written by $h : \underline{\mathrm{M}} \to \underline{\mathrm{N}}$ if

$$h(f_i^M(a_1, \dots, a_{n_i})) = f_i^N(h(a_1), \dots, h(a_{n_i}))$$

for all $i \in I$. If a homomorphism $h$ is bijective (injective and surjective), then $h$ is called an *isomorphism* from $\underline{\mathrm{M}}$ onto $\underline{\mathrm{N}}$.

For each class $\left\{\underline{\mathrm{M_j}}\right\}_{j\in J}$ of algebras of the same type, the *direct product* $\Pi_{j\in J}\underline{\mathrm{M_j}}$ of $\left\{\underline{\mathrm{M_j}}\right\}_{j\in J}$ is defined as an algebra consisting of the universe

$$P := \Pi_{j\in J} M_j = \left\{a : J \to \bigcup_{j\in J} M_j \mid a(j) \in M_j \text{ for all } j \in J\right\}$$

and each fundamental operation $f_i^P$ defined by

$$(f_i^P(a_1, \dots, a_{n_i}))(j) = f_i^{M_j}(a_1(j), \dots, a_{n_i}(j)),$$

for all $a_1, \dots, a_{n_i} \in P$, $j \in J$ and $i \in I$. If $\underline{\mathrm{M_j}} = \underline{\mathrm{M}}$ for all $j \in J$, then we usually write $\underline{\mathrm{M}}^J$ instead of $\Pi_{j\in J}\underline{\mathrm{M_j}}$.

For each class $\mathcal{M}$ of algebras of the same type, we define:

1. $\mathbb{S}(\mathcal{M})$ is the set of all subalgebras of algebras in $\mathcal{M}$,

2. $\mathbb{H}(\mathcal{M})$ is the set of all homomorphic images of algebras in $\mathcal{M}$,

3. $\mathbb{I}(\mathcal{M})$ is the set of all isomorphic copies of algebras in $\mathcal{M}$,

4. $\mathbb{P}(\mathcal{M})$ is the set of all direct products of algebras in $\mathcal{M}$.

One of well-known studying classes of algebras is *variety*, a class of algebras of the same type which is closed under all homomorphic images, subalgebras and products. A. Tarski proved in [24] that every variety is the class $\mathbb{HSP}(\mathcal{M})$ for some class $\mathcal{M}$ of algebras. G. Birkhoff [3] showed a classical result that every variety can be determined by its subdirectly irreducible algebras. For some algebras having the large universes, they are complicated to study their algebraic properties. In 1970, H.A. Priestley [17] represented bounded distributive lattices by ordered Stone spaces. It is a new branch to use a topology to study an algebra. Moreover, this concept was used to describe homomorphism, congruences and subdirectly irreducible Ockham algebras by A. Urguhart [25]. In 1983, Davey and Werner [10] developed the method to represent every algebra as an algebra of continuous functions. This concept is known as natural duality.

Let $\underline{\mathrm{M}}$ be a finite algebra. An $m$-ary relation on $M$ is said to be *algebraic over* $\underline{\mathrm{M}}$ if it forms a subalgebra of $\underline{\mathrm{M}}^m$. Let $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathbb{S}(\underline{\mathrm{M}}^n)$ be the set of all algebraic relations over $\underline{\mathrm{M}}$. A topological structure $\underset{\sim}{\mathrm{M}} = (M; R, \mathscr{T})$ is called an *alter ego* of $\underline{\mathrm{M}}$ if $R \subseteq \mathcal{B}$ and $\mathscr{T}$ is the discrete topology on $M$. This definition of alter ego is defined in [7, 8]. But in some works, the set of relations $R$ is separated to a set of relations, set of operations and set of partial operations. Let $\mathscr{A} = \mathbb{ISP}(\underline{\mathrm{M}})$ be the category consisting of all isomorphic copies of subalgebras of direct powers of $\underline{\mathrm{M}}$ and let $\mathscr{X} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathrm{M}})$ be the category consisting of all isomorphic copies of closed substructures of non-empty direct powers of $\underset{\sim}{\mathrm{M}}$. For each $\underline{\mathrm{A}} \in \mathscr{A}$ and $\underset{\sim}{\mathrm{X}} \in \mathscr{X}$, we denote

$$D(\underline{\mathrm{A}}) = \{f : A \to M \mid f \text{ is a homomorphism from } \underline{\mathrm{A}} \text{ to } \underline{\mathrm{M}}\}$$

and

$$E(\underset{\sim}{\mathrm{X}}) = \left\{f : X \to M \mid f \text{ is a morphism from } \underset{\sim}{\mathrm{X}} \text{ to } \underset{\sim}{\mathrm{M}}\right\};$$

and call them the *dual* of $\underline{\mathrm{A}}$ and the *dual* of $\underset{\sim}{\mathrm{X}}$, respectively. It was shown in [10] that $D(\underline{\mathrm{A}}) \in \mathscr{X}$ and $E(\underset{\sim}{\mathrm{X}}) \in \mathscr{A}$. For a homomorphism $u : \underline{\mathrm{A}} \to \underline{\mathrm{B}}$, define a morphism

$$D(u) : D(\underline{\mathrm{B}}) \to D(\underline{\mathrm{A}}) \text{ by } D(u)(x) = x \circ u$$

for all $x \in D(\underline{\mathrm{B}})$. Similarly, for a morphism $\varphi : \underset{\sim}{\mathrm{X}} \to \underset{\sim}{\mathrm{Y}}$, define

$$E(\varphi) : E(\underset{\sim}{\mathrm{Y}}) \to E(\underset{\sim}{\mathrm{X}}) \text{ by } E(\varphi)(\alpha) = \alpha \circ \varphi$$

for all $\alpha \in E(\underset{\sim}{\mathrm{Y}})$. For each $\underline{\mathrm{A}} \in \mathscr{A}$ and $\underset{\sim}{\mathrm{X}} \in \mathscr{X}$, we define the *evaluation functions*

$$e_{\underline{\mathrm{A}}} : \underline{\mathrm{A}} \to ED(\underline{\mathrm{A}}) \text{ by } e_{\underline{\mathrm{A}}}(a)(x) = x(a)$$

for all $a \in \underline{\mathrm{A}}$ and $x \in D(\underline{\mathrm{A}})$ and

$$\varepsilon_{\underset{\sim}{\mathrm{X}}} : \underset{\sim}{\mathrm{X}} \to DE(\underset{\sim}{\mathrm{X}}) \text{ by } \varepsilon_{\underset{\sim}{\mathrm{X}}}(x)(\alpha) = \alpha(x)$$

for all $x \in \underset{\sim}{\mathrm{X}}$ and $\alpha \in E(\underset{\sim}{\mathrm{X}})$. We say that $\underset{\sim}{\mathrm{M}}$ (or $R$) *yields a duality* on $\mathscr{A}$ or $\underset{\sim}{\mathrm{M}}$ *dualise* $\underline{\mathrm{M}}$ if $e_{\underline{\mathrm{A}}}$ is an isomorphism for all $\underline{\mathrm{A}} \in \mathscr{A}$. We say that $\underline{\mathrm{M}}$ is *dualisable* if there is a structure $\underset{\sim}{\mathrm{M}}$ which dualise $\underline{\mathrm{M}}$. These mean that every algebra in $\mathscr{A}$ can be represented as a concrete algebra of morphisms from the structure $D(\underline{\mathrm{A}})$ to the structure $\underset{\sim}{\mathrm{M}}$. For further details, see in [5] or [10]. If $R$ is finite, $\underset{\sim}{\mathrm{M}}$ is said to be *finite type*. One of well-known dualisable algebras is an algebra $\underline{\mathrm{M}} = (M; F)$ admitting an $m$-ary near-unanimity function $f$; that is, $f \in \langle F \rangle$. Moreover, $\mathbb{S}(\underline{\mathrm{M}}^{m-1})$ yields a duality on $\mathscr{A}$.

We say that $\underset{\sim}{\mathrm{M}}$ (or $R$) *yields an optimal duality* on $\mathscr{A}$ if $R$ yields a duality on $\mathscr{A}$ but there are no proper subsets of $R$ which yields a duality on $\mathscr{A}$. Optimal dualities are developed by B.A. Davey and H.A. Priestley [7, 8] using the following concepts of entailment.

For each $m$-ary relation $r^M$ on a set $M$ and index set $A$ and $Z \subseteq M^A$, let $r^{M^A}$ be defined componentwise; that is,

$$(x_1, \ldots, x_m) \in r^{M^A} \Leftrightarrow (x_1(a), \ldots, x_m(a)) \in r^M$$

for all $a \in A$ and $r^Z = r^{M^A} \cap Z^m$. A function $\alpha : Z \to M$ is said to *preserve* $r^M$ if

$$(x_1, \ldots, x_m) \in r^Z \Rightarrow (\alpha(x_1), \ldots, \alpha(x_m)) \in r^M.$$

For each $\underline{\mathrm{A}} \in \mathscr{A}$ and $s \in \mathcal{B}$, we say that $R$ *entails* $s$ *on* $D(\underline{\mathrm{A}})$ if

$$\alpha \text{ preserves all elements in } R \Rightarrow \alpha \text{ also preserves } s$$

for all continuous functions $\alpha$ from $D(\underline{\mathrm{A}})$ to $M$. We say that $R$ *entails* $s$, briefly $R \vdash s$, if

$$R \text{ entails } s \text{ on } D(\underline{\mathrm{A}}) \text{ for all } \underline{\mathrm{A}} \in \mathscr{A}.$$

For each $R' \subseteq \mathcal{B}$, we say that $R$ *entails* $R'$, briefly $R \vdash R'$, if

$$R \text{ entails } s \text{ for all } s \in R'.$$

Note by Soundness Theorem [5] that a set $\{r\}$ of an $m$-ary algebraic relation entails $\{r^\sigma \mid \sigma \in S_m\}$ where

$$r^\sigma = \{(a_{\sigma(1)}, \ldots, a_{\sigma(m)}) \mid (a_1, \ldots, a_m) \in r\}$$

for all $\sigma$ in the set $S_m$ of all permutations on $\{1, \ldots, m\}$. One can refine an alter ego via $\underset{\sim}{\mathrm{M}}$-Shift Duality Lemma [5] which is stated that if $R$ entails $R'$ and $R'$ yields a duality on $\mathscr{A}$, then $R$ yields a duality on $\mathscr{A}$.

## 2.4 Relational Set

A binary relation $\leq$ on a set $M$ is an *order* if it satisfies the following conditions for all $x, y, z \in M$,

1. $x \leq x$, (*reflexivity*)
2. $x \leq y$ and $y \leq x$ imply $x = y$, (*anti-symmetry*)
3. $x \leq y$ and $y \leq z$ imply $x \leq z$. (*transitivity*)

A set $M$ equipped with an order relation $\leq$ is said to be an *ordered set* (or *partially ordered set*) and denoted by $(M; \leq)$. Some authors use the shorthand *poset*. A poset permeates mathematics. One of the most attraction of posets is that they are pictural structures. The picture representing a poset is known as a *Hasse diagram*; or shortly *diagram*.

Another well-known pictural structure is a graph. An (*undirected*) *graph* $\mathbf{M} = (M;\Theta)$ is a structure consisting of a finite set $M$ and a symmetric binary relation $\Theta$ on $M$. An element in $M$ and in $\Theta$ is called a *vertex* and an *edge* of $\mathbf{M}$, respectively. A graph $\mathbf{M} = (M;\Theta)$ can be shown as a picture such that elements in $M$ are represented as vertices and elements $(x,y)$ in $\Theta$ are represented as lines from $x$ to $y$. An example of representing a graph as a picture was shown in Example 2.1. If $\Theta$ is *tolerance*; that is, $\Theta$ is reflexive and symmetric, then $(M;\Theta)$ is called a *reflexive* (*undirected*) *graph*. A reflexive graph $\mathbf{M} = (M;\Theta)$ is called a *majority reflexive graph* if $\mathrm{Pol}(\Theta)$ contains a majority operation. H. Bandelt [2] characterized a majority reflexive graph by considering bipartite graphs.

Both ordered sets and graphs are structures consisting of a set and a relation on the carrier set. These structures can be generalized to arbitrary relational sets. A *relational set* $\mathbf{M}$ is a structure consisting of a set $M$ and a set $\{r_i^M\}_{i\in I}$ of finitary relations on $M$; for brevity, a relational set is called *reset*. The sequence $(n_i)_{i\in I}$ of all arities is called the *type* of the reset $\mathbf{M}$. In fact, if all relations in $\{r_i^M\}_{i\in I}$ are functions on $M$, then a reset is an algebra. The concepts in algebras can be investigated in a class of resets.

In 1981, Duffus and Rival [11] defined the notions of an order variety, a representation of a poset and an irreducible poset. In 1992, L. Zádori [26] studied order varieties and in the recent years, he [27] generalized this concept to arbitrary relational set. Let $\mathbf{H} = \left(H;\{r_i^H\}_{i\in I}\right)$ and $\mathbf{M} = \left(M;\{r_i^M\}_{i\in I}\right)$ be resets of the same type. $\mathbf{H}$ is called a *subreset* of $\mathbf{M}$ if the following condition are satisfied:

1. $H \subseteq M$;

2. $r_i^H$ is the restriction of relation $r_i^M$ to $H$ (denoted by $r_i^M \restriction_H$) for all $i \in I$.

A function $\tilde{f}$ from $H$ to $M$ is called a *morphism*, written by $\tilde{f} : \mathbf{H} \to \mathbf{M}$ if $\tilde{f}$ preserves all relations of $\mathbf{H}$; that is,

$$(a_1,\ldots,a_{n_i}) \in r_i^H \Rightarrow (\tilde{f}(a_1),\ldots,\tilde{f}(a_{n_i})) \in r_i^M$$

for all $i \in I$.

For a set $\{\mathbf{M}_j \mid j \in J\}$ of resets of the same type, the *product* $\Pi_{j\in J}\mathbf{M}_j$ is a reset with the base set

$$\Pi_{j\in J}M_j = \left\{a : J \to \bigcup_{j\in J} M_j \mid a(j) \in M_j \text{ for all } j \in J\right\}$$

and the relations defined componentwise; that is,

$$(a_1, \ldots, a_{n_i}) \in r_i^{\Pi_{j\in J}M_j} \Leftrightarrow (a_1(j), \ldots, a_{n_i}(j)) \in r_i^{M_j}$$

for all $j \in J$ and $i \in I$. For each class of resets $\mathcal{K}$, denote the set of all products of resets in $\mathcal{K}$ by $\mathbb{P}(\mathcal{K})$.

For resets $\mathbf{P}$ and $\mathbf{R}$ of the same type, we say that $\mathbf{R}$ is a *retract* of $\mathbf{P}$ if there are morphisms $r : \mathbf{P} \to \mathbf{R}$ and $e : \mathbf{R} \to \mathbf{P}$ such that $r \circ e = \mathrm{id}_R$. The functions $r$ and $e$ are called *retraction* and *coretraction*, respectively. For a class of resets $\mathcal{K}$, denote the set of all retracts of resets in $\mathcal{K}$ by $\mathbb{R}(\mathcal{K})$.

A class of resets of the same type is called a *relation variety* if it is closed under product and retract; or equivalently, a relation variety is the class $\mathbb{RP}(\mathcal{K})$ for some class $\mathcal{K}$ of resets. Some important properties of relation varieties are studied via a colored reset.

A pair $(\mathbf{H}, h)$ is called an $\mathbf{M}$-*colored reset* if $h$ is a partially defined function from $H$ to $M$. The domain of $h$ is denoted by $C(\mathbf{H}, h)$ and an element in $C(\mathbf{H}, h)$ is called a *colored element*. Denote $\mathcal{C}(\mathbf{M})$ be the set of all $\mathbf{M}$-colored resets. If $h$ can be extended to a fully defined morphism $\bar{h} : \mathbf{H} \to \mathbf{M}$ on $H$ then $(\mathbf{H}, h)$ is called an $\mathbf{M}$-*extendable reset*; otherwise, $(\mathbf{H}, h)$ is called an $\mathbf{M}$-*nonextendable reset*. A finite $\mathbf{M}$-nonextendable reset is called an $\mathbf{M}$-*obstruction* if it is minimal under an order defined by $(\mathbf{H_1}, h_1) \subseteq (\mathbf{H_2}, h_2)$ if and only if the following conditions hold:

1. $H_1 \subseteq H_2$ and $h_1 \subseteq h_2$;

2. $r_i^{H_1} \subseteq r_i^{H_2} \downharpoonleft_{H_1}$ for all $i \in I$.

**Example 2.2** *Suppose that* $\mathbf{M}$ *and* $\mathbf{H}$ *are graphs as shown in Figure 7 and Figure 8, respectively.*

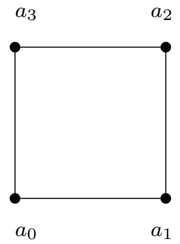


Figure 7. The graph $\mathbf{M}$.

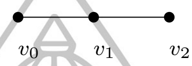


Figure 8. The graph $\mathbf{H}$.

*Define functions $h_1 : \{v_0, v_1\} \to M$ and $h_2 : \{v_0, v_2\} \to M$ by $h_1(v_0) = a_0$, $h_1(v_1) = a_2$, $h_2(v_0) = a_0$ and $h_2(v_2) = a_2$. The colored resets $(\mathbf{H}, h_1)$ and $(\mathbf{H}, h_2)$ can be represented as shown in Figure 9 and Figure 10.*

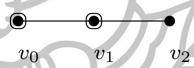


Figure 9. The $\mathbf{M}$-colored reset $(\mathbf{H}, h_1)$.

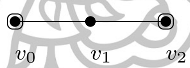


Figure 10. The $\mathbf{M}$-colored reset $(\mathbf{H}, h_2)$.

*Then $(\mathbf{H}, h_2)$ is an $\mathbf{M}$-extendable reset since we can define a morphism $\bar{h_2} : \mathbf{H} \to \mathbf{M}$ by $\bar{h_2}(v_i) = a_i$ for all $i = 1, 2, 3$. In contrast, $(\mathbf{H}, h_1)$ is an $\mathbf{M}$-nonextendable reset since it has no lines from $a_0$ to $a_2$. However, $(\mathbf{H}, h_1)$ is not minimal since there is a subgraph $\mathbf{H_1} = (\{v_0, v_1\}, \{(v_0, v_1), (v_1, v_0)\})$ of $\mathbf{H}$ such that $(\mathbf{H_1}, h_1)$ is an $\mathbf{M}$-nonextendable reset and $(\mathbf{H_1}, h_1) \subseteq (\mathbf{H}, h_1)$. In fact, $(\mathbf{H_1}, h_1)$ is an $\mathbf{M}$-obstruction.*

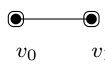


Figure 11. The $\mathbf{M}$-obstruction $(\mathbf{H_1}, h_1)$.

If both $\mathbf{M}$ and $\mathbf{H}$ are ordered sets, an $\mathbf{M}$-obstruction $(\mathbf{H}, h)$ is called a *zigzag*. The concept of zigzag is used to solve many problems about ordered set; for instance, G. Tardos showed a remark in [23] that a finite poset $(M; \leq)$ admits an $n$-ary near unanimity function $f$ (that is, $f : \mathbf{M}^n \to \mathbf{M}$ is a morphism) if and only if the number of colored elements of every $\mathbf{M}$-zigzag is at most $n-1$. L. Zádori [27] generalized G. Tardos's remark by proving that $\mathbf{M}$ admits an $n$-ary near unanimity function $f$ if and only if the number of colored elements in every $\mathbf{M}$-obstruction is at most $n-1$ for all $n \geq 3$ and all finite resets $\mathbf{M}$.

# Chapter 3

# Algebraic Relations and Colored Resets

Let $\underline{\mathrm{M}}$ be a finite algebra. An $m$-ary relation on $M$ is said to be *algebraic over* $\underline{\mathrm{M}}$ if it forms a subalgebra of $\underline{\mathrm{M}}^m$. Let $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathbb{S}(\underline{\mathrm{M}}^n)$ be the set of all algebraic relations over $\underline{\mathrm{M}}$. For each reset $\mathbf{M}$, an $\mathbf{M}$-*colored reset* $(\mathbf{H}, h)$ is a reset $\mathbf{H}$ of the same type equipped with a partial operation $h$ from $H$ to $M$. Let $\mathcal{C}(\mathbf{M})$ be the set of all $\mathbf{M}$-colored resets. In this chapter, we show a connection between the set of algebraic relations over $\underline{\mathrm{M}}$ and the set $\mathbf{M}$-colored resets.

## 3.1 A Galois Connection

One can see that $\mathcal{B} = \mathrm{Inv}\,(\mathrm{Clo}\,(\underline{\mathrm{M}}))$. It is well known that $\mathrm{Clo}\,(\underline{\mathrm{M}}) = \mathrm{Pol}\left(\{r_i^M\}_{i\in I}\right)$ for some sets $\{r_i^M\}_{i\in I}$ of relations on $M$. Hence, $r$ is algebraic over $\underline{\mathrm{M}}$ if and only if $r \in \mathrm{Inv}\left(\mathrm{Pol}\left(\{r_i^M\}_{i\in I}\right)\right)$. Recall for a non-empty set $r$ that $M^r$ is the set of all functions from $r$ to $M$. If $|r| = m$, we substitute $M^r$ by $M^m$. We denote the $i$-projection from $M^m$ to $M$ by $e_i^m$ for all $1 \le i \le m$. The following lemma was proved in [6] by B. A. Davey, M. Haviar and H. A. Priestley.

**Lemma 3.1** [6] *An $m$-ary relation $r$ is algebraic over $\underline{\mathrm{M}}$ if and only if*

$$r = \left\{(\tilde{h}(a_1), \ldots, \tilde{h}(a_m)) \,|\, \tilde{h} : M^r \to M \text{ preserves } r_i^M \text{ for all } i \in I\right\}$$

*where $a_j = e_j^m \restriction_r$ for all $1 \le j \le m$.*

R. Srithus and U. Chotwattakawanit [22] defined $m$-ary relations over $M$ which were used to construct an alter ego dualising an algebra admitted a near-unanimity operation as follows: for a reset $\mathbf{M} = \big(M; \{r_i^M\}_{i\in I}\big)$,

$$\rho^M_{(a_1,\ldots,a_m)} = \big\{(\tilde{h}(a_1),\ldots,\tilde{h}(a_m)) \,|\, \tilde{h} : \mathbf{H} \to \mathbf{M} \text{ is a morphism}\big\} \qquad (*)$$

for all $\mathbf{M}$-color resets $(\mathbf{H}, h)$ with $C(\mathbf{H}, h) = \{a_1,\ldots,a_m\}$. We are now showing that those relations in $(*)$ are algebraic over $\underline{\mathrm{M}}$; and then by Lemma 3.1, all algebraic relations can be represented by these relations as we state in the following theorem.

**Theorem 3.2** *An $m$-ary relation $r$ is algebraic over $\underline{\mathrm{M}}$ if and only if $r = \rho^M_{(a_1,\ldots,a_m)}$ for some $(\mathbf{H}, h) \in \mathcal{C}(\mathbf{M})$ with $C(\mathbf{H}, h) = \{a_1,\ldots,a_m\}$. Moreover, $\mathbf{H} = \mathbf{M}^n$ for some natural number $n$.*

**Proof.** If $r$ is algebraic over $\underline{\mathrm{M}}$, Lemma 3.1 implies that $r = \rho^M_{(a_1,\ldots,a_m)}$ where $\{a_1,\ldots,a_m\} = C(\mathbf{M}^r, h)$. Conversely, suppose that $r = \rho^M_{(a_1,\ldots,a_m)}$ for some $(\mathbf{H}, h) \in \mathcal{C}(\mathbf{M})$ with $C(\mathbf{H}, h) = \{a_1,\ldots,a_m\}$ and $\underline{\mathrm{M}} = \big(M; \{f_j\}_{i\in J}\big)$. Let $j \in J$ and $\tilde{h}_1,\ldots,\tilde{h}_{n_j}$ be morphisms from $\mathbf{H}$ to $\mathbf{M}$. Since $f_j \in \mathrm{Clo}(\underline{\mathrm{M}}) = \mathrm{Pol}\big(\{r_i^M\}_{i\in I}\big)$, we have that $f_j : \mathbf{M}^{n_j} \to \mathbf{M}$ is a morphism. To show that $f_j(\tilde{h}_1,\ldots,\tilde{h}_{n_j}) : \mathbf{H} \to \mathbf{M}$ is a morphism, let $i \in I$ and $(x_1,\ldots,x_{n_i}) \in r_i^H$. Then

$$(h_k(x_1),\ldots,h_k(x_{n_i})) \in r_i^M \text{ for all } 1 \le k \le n_j.$$

Hence,

$$\big(f_j(\tilde{h}_1,\ldots,\tilde{h}_{n_j})(x_1),\ldots,f_j(\tilde{h}_1,\ldots,\tilde{h}_{n_j})(x_{n_i})\big) \in r_i^M$$

which implies that $f_j(\tilde{h}_1,\ldots,\tilde{h}_{n_j}) : \mathbf{H} \to \mathbf{M}$ is a morphism. Therefore,

$$\big(f_j(\tilde{h}_1,\ldots,\tilde{h}_{n_j})(a_1),\ldots,f_j(\tilde{h}_1,\ldots,\tilde{h}_{n_j})(a_m)\big) \in r$$

which implies that $r$ is algebraic over $\underline{\mathrm{M}}$. ■

**Example 3.3** *Suppose that* $\mathbf{M}$ *is a graph as in Figure 12 and* $(\mathbf{H}, h)$ *is an* $\mathbf{M}$*-colored reset as in Figure 13, respectively.*

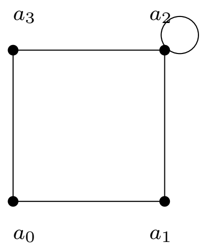


Figure 12. The graph **M**.

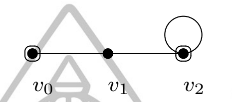


Figure 13. The **M**-colored reset $(\mathbf{H}, h)$.

*Since there is a loop at $v_2$, every morphism $\tilde{h} : \mathbf{H} \to \mathbf{M}$ maps $v_2$ to $a_2$. Theerefore, $\rho^M_{(v_0,v_2)} = \{(a_0, a_2), (a_1, a_2), (a_2, a_2), (a_3, a_2)\} = M \times \{a_2\}$ and $\rho^M_{(v_2,v_0)} = \{a_2\} \times M$.*

Observe that the algebraic relations $\rho^M_{(a_1,\ldots,a_m)}$ and $\rho^M_{(a_m,\ldots,a_1)}$ may be different for all **M**-colored resets $(\mathbf{H}, h)$ with $C(\mathbf{H}, h) = \{a_1, \ldots, a_m\}$. In fact, if we rearrange the $m$-tuple, then there are many algebraic relations corresponding to $(\mathbf{H}, h)$. We will combine these algebraic relations into a class of an equivalence relation on $\mathcal{B}$. For each $r \in \mathcal{B}$ and $\sigma$ in the set $S_m$ of all permutations on $\{1, \ldots, m\}$, let define

$$r^\sigma = \{(a_{\sigma(1)}, \ldots, a_{\sigma(m)}) \,|\, (a_1, \ldots, a_m) \in r\}.$$

**Proposition 3.4** *Let $\sim$ be a binary relation on $\mathcal{B}$ defined by*

$$r_1 \sim r_2 \text{ if and only if } r_2 = r_1^\sigma$$

*for some $\sigma \in S_m$ for all $r_1, r_2 \in \mathcal{B}$. Then $\sim$ is an equivalence relation on $\mathcal{B}$.*

**Proof.** Observe that $r^{id} = r$ for all $r \in \mathcal{B}$ where $id$ denote the identity function on $\{1, \ldots, m\}$. Hence, $\sim$ is reflexive. We will first show that $(r^\sigma)^\varsigma = r^{\varsigma \circ \sigma}$ for all $r \in \mathcal{B}$ and $\sigma, \varsigma \in S_m$. Let $a_1, \ldots, a_m \in M$. Then

$$\begin{aligned}
(a_1, \ldots, a_m) \in (r^\sigma)^\varsigma \quad &\Leftrightarrow \quad (a_1, \ldots, a_m) = (b_{\varsigma(1)}, \ldots, b_{\varsigma(m)}) \\
&\qquad \text{for some } (b_1, \ldots, b_m) \in r^\sigma \\
&\Leftrightarrow \quad (a_1, \ldots, a_m) = (c_{\varsigma(\sigma(1))}, \ldots, c_{\varsigma(\sigma(m))}) \\
&\qquad \text{for some } (c_1, \ldots, c_m) \in r.
\end{aligned}$$

If $r_1, r_2 \in \mathcal{B}$ with $r_1 \sim r_2$, then $r_2 = r_1^{\sigma}$ for some $\sigma \in S_m$; hence,

$$r_2^{\sigma^{-1}} = (r_1^{\sigma})^{\sigma^{-1}} = r_1^{\sigma^{-1}\circ\sigma} = r_1^{id} = r_1$$

which implies that $r_2 \sim r_1$. If $r_1, r_2, r_3 \in \mathcal{B}$ with $r_1 \sim r_2$ and $r_2 \sim r_3$, then $r_2 = r_1^{\sigma}$ and $r_3 = r_2^{\varsigma}$ for some $\sigma, \varsigma \in S_m$; hence, $r_3 = r_2^{\varsigma} = (r_1^{\sigma})^{\varsigma} = r_1^{\varsigma\circ\sigma}$ which implies that $r_1 \sim r_3$. ■

For each $(\mathbf{H}, h) \in \mathcal{C}(\mathbf{M})$ with $C(\mathbf{H}, h) = \{a_1, \ldots, a_m\}$, let $\rho^M_{(\mathbf{H},h)}$ be the equivalence class $[\rho^M_{(a_1,\ldots,a_m)}]_{\sim}$ of algebraic relations containing $\rho^M_{(a_1,\ldots,a_m)}$. Let $\rho^M$ be the function from $\mathcal{C}(\mathbf{M})$ to the set $\mathcal{B}/_{\sim}$ of all equivalence classes under $\sim$ which maps $(\mathbf{H}, h)$ to $\rho^M_{(\mathbf{H},h)}$.

**Proposition 3.5** *The function $\rho^M : \mathcal{C}(\mathbf{M}) \to \mathcal{B}/_{\sim}$ is surjective.*

**Proof.** Let $r \in \mathcal{B}$. By Theorem 3.2, $r = \rho^M_{(a_1,\ldots,a_m)}$ for some $(\mathbf{H}, h) \in \mathcal{C}(\mathbf{M})$ with $C(\mathbf{H}, h) = \{a_1, \ldots, a_m\}$. Hence, $[r]_{\sim} = [\rho^M_{(a_1,\ldots,a_m)}]_{\sim} = \rho^M_{(\mathbf{H},h)}$ ■

**Example 3.6** *Suppose that* $\mathbf{M}$ *is a graph as in Figure 14 and assume that* $(\mathbf{H}, h)$ *and* $(\mathbf{H}', h')$ *are* $\mathbf{M}$*-colored resets as in Figure 15 and Figure 16, respectively.*

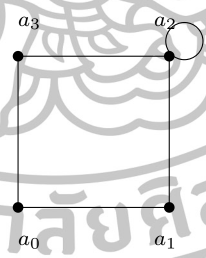


Figure 14. The graph $\mathbf{M}$.

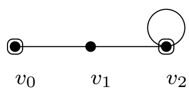


Figure 15. The $\mathbf{M}$-colored reset $(\mathbf{H}, h)$.

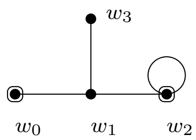


Figure 16. The $\mathbf{M}$-colored reset $(\mathbf{H}', h')$.

*We knew from Example 3.3 that* $\rho^M_{(v_0,v_2)} = M \times \{a_2\}$. *By the same argrument, we have* $\rho^M_{(w_0,w_2)} = M \times \{a_2\} = \rho^M_{(v_0,v_2)}$. *So,* $\rho^M_{(\mathbf{H},h)} = \rho^M_{(\mathbf{H}',h')}$.

Therefore, $\rho^M$ is not injective. We are now showing a condition of all $\mathbf{M}$-colored resets corresponding to the same equivalence class in $\mathcal{B}/_\sim$. Note for each function $f : A \to B$ that the relation $\ker f = \{(x_1, x_2) \in A^2 \mid f(x_1) = f(x_2)\}$ is an equivalence relation. Let $[\mathbf{H}, h]$ denote the equivalence class in $\mathcal{C}(\mathbf{M})/_{\ker \rho^M}$ containing $(\mathbf{H}, h)$. We will identify each equivalence class in the following theorem.

**Theorem 3.7** *For each* $(\mathbf{H}, h), (\mathbf{G}, g) \in \mathcal{C}(\mathbf{M})$, $[\mathbf{H}, h] = [\mathbf{G}, g]$ *if and only if there is a bijection* $\varepsilon : C(\mathbf{H}, h) \to C(\mathbf{G}, g)$ *such that the diagrams in the following Figure 17 and Figure 18 commute for all morphisms* $\tilde{h} : \mathbf{H} \to \mathbf{M}$ *and* $\tilde{g} : \mathbf{G} \to \mathbf{M}$.

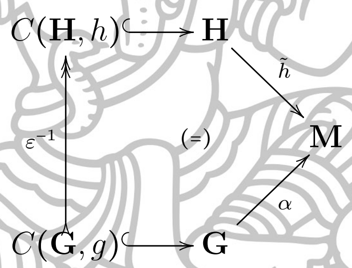


Figure 17. $\tilde{h} \circ \varepsilon^{-1}$ can be extended to a morphism $\alpha$.



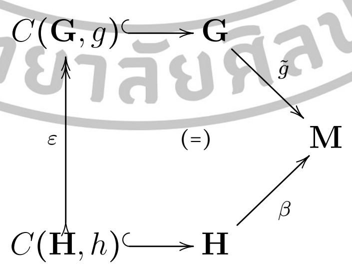


Figure 18. $\tilde{g} \circ \varepsilon$ can be extended to a morphism $\beta$.

**Proof.** Let $C(\mathbf{H}, h) = \{a^1_1, \ldots, a^1_{m_1}\}$ and $C(\mathbf{G}, g) = \{a^2_1, \ldots, a^2_{m_2}\}$. If $[\mathbf{H}, h] = [\mathbf{G}, g]$, then $m_1 = m_2 := m$. Let $\sigma_1, \sigma_2 \in S_m$ with

$$\rho^M_{(a^1_{\sigma_1(1)}, \ldots, a^1_{\sigma_1(m)})} = \rho^M_{(a^2_{\sigma_2(1)}, \ldots, a^2_{\sigma_2(m)})}.$$

So, a function $\varepsilon : a^1_{\sigma_1(k)} \mapsto a^2_{\sigma_2(k)}$ is a bijection from $C(\mathbf{H},h)$ to $C(\mathbf{G},g)$. To show that the diagram in Figure 17 commutes, let $\tilde{h} : \mathbf{H} \to \mathbf{M}$ be a morphism. Then

$$(\tilde{h}(a^1_{\sigma_1(1)}),\dots,g_1(a^1_{\sigma_1(m)})) \in \rho^M_{(a^1_{\sigma_1(1)},\dots,a^1_{\sigma_1(m)})} = \rho^M_{(a^2_{\sigma_2(1)},\dots,a^2_{\sigma_2(m)})}.$$

So, there is a morphism $\alpha : \mathbf{G} \to \mathbf{M}$ such that

$$\alpha(a^2_{\sigma_2(k)}) = \tilde{h}(a^1_{\sigma_1(k)}) = \tilde{h}(\varepsilon^{-1}(a^2_{\sigma_2(k)})$$

for all $1 \le k \le m$. Similarly, the diagram in Figure 18 commutes for all morphisms $\tilde{g} : \mathbf{G} \to \mathbf{H}$. Conversely, if $\varepsilon$ is a bijection, then $m_1 = m_2 := m$. Define $\phi_1 : \{1,\dots,m\} \to C(\mathbf{H},h)$ and $\phi_2 : \{1,\dots,m\} \to C(\mathbf{G},g)$ by $\phi_i(k) = a^i_k$ for all $1 \le k \le m$ and $i \in \{1,2\}$. We will show that $\rho^M_{(\mathbf{H},h)} = \rho^M_{(\mathbf{G},g)}$. Let $\sigma_1 \in S_m$ and

$$\sigma_2 = \phi_2^{-1} \circ \varepsilon \circ \phi_1 \circ \sigma_1.$$

Then $\varepsilon(a^1_{\sigma_1(k)}) = a^2_{\sigma_2(k)}$ for all $1 \le k \le m$. By the assumption,

$$\begin{aligned}
(x_1,\dots,x_m) \in \rho^M_{(a^2_{\sigma_2(1)},\dots,a^2_{\sigma_2(m)})} &\Leftrightarrow \text{there is } \tilde{g} : \mathbf{G} \to \mathbf{M} \text{ with for each } 1 \le k \le m,\\
&\quad\ \tilde{g} \circ \varepsilon(a^1_{\sigma_1(k)}) = \tilde{g}(a^2_{\sigma_2(k)}) = x_k\\
&\Leftrightarrow \text{there is } \tilde{h} : \mathbf{H} \to \mathbf{M} \text{ with for each } 1 \le k \le m,\\
&\quad\ \tilde{h}(a^1_{\sigma_1(k)}) = \tilde{g} \circ \varepsilon(a^1_{\sigma_1(k)}) = x_k\\
&\Leftrightarrow (x_1,\dots,x_m) \in \rho^M_{(a^1_{\sigma_1(1)},\dots,a^1_{\sigma_1(m)})}
\end{aligned}$$

which implies that $\rho^M_{(a^2_{\sigma_2(1)},\dots,a^2_{\sigma_2(m)})} = \rho^M_{(a^1_{\sigma_1(1)},\dots,a^1_{\sigma_1(m)})}$. Hence, $\rho^M_{(\mathbf{H},h)} = \rho^M_{(\mathbf{G},g)}$; that is, $[\mathbf{H},h] = [\mathbf{G},g]$. ■

Naturally, we can map an **M**-colored reset $(\mathbf{H},h)$ to the equivalence class $[\mathbf{H},h]$ in $\mathcal{C}(\mathbf{M})/_{\ker\rho^M}$ via the natural map $\eta$. One can see by the following theorem that $|\mathcal{C}(\mathbf{M})/_{\ker\rho^M}| = |\mathcal{B}/_\sim|$.

**Theorem 3.8** *Let $\varphi : \mathcal{C}(\mathbf{M})/_{\ker\rho^M} \to \mathcal{B}/_\sim$ be defined by $\varphi([\mathbf{H},h]) = \rho^M_{(\mathbf{H},h)}$ for all $(\mathbf{H},h) \in \mathcal{C}(\mathbf{M})$. Then $\varphi$ is the unique bijection with $\varphi \circ \eta = \rho^M$.*

**Proof.** Since $\rho^M$ is surjective, so is $\varphi$. One can see for each $(\mathbf{H},h),(\mathbf{G},g) \in \mathcal{C}(\mathbf{M})$ that if $\rho^M_{(\mathbf{H},h)} = \rho^M_{(\mathbf{G},g)}$, then $[\mathbf{H},h] = [\mathbf{G},g]$ which implies that $\rho^M$ is injective. Suppose that $\varphi' : \mathcal{C}(\mathbf{M})/_{\ker\rho^M} \to \mathcal{B}/_\sim$ with $\varphi' \circ \eta = \rho^M$. Then $\varphi \circ \eta = \varphi' \circ \eta$. Hence,

$$\varphi([\mathbf{H},h]) = \varphi \circ \eta((\mathbf{H},h)) = \varphi' \circ \eta((\mathbf{H},h)) = \varphi'([\mathbf{H},h])$$

for all $[\mathbf{H},h] \in \mathcal{C}(\mathbf{M})/_{\ker\rho^M}$. ■

One can see that $\varphi$ is a Galois connection of the equivalence classes of **M**-colored resets and the equivalence classes of algebraic relations over $\underline{\mathrm{M}}$ under the equivalence relations $\ker \rho^M$and $\sim$, respectively. We may apply this result to solve some problems about algebraic relations over $\underline{\mathrm{M}}$ via **M**-colored resets; for instance, a problem of refining an alter ego of some algebras or a problem about clone theory.

We will define an order on $\mathcal{C}(\mathbf{M})/_{\ker \rho^M}$ analogously to the definition of the retract of resets. Recall that for resets **P** and **R** of the same type, we say that **R** is a *retract* of **P** if there are morphisms $r : \mathbf{P} \to \mathbf{R}$ and $e : \mathbf{R} \to \mathbf{P}$ such that $r \circ e = \mathrm{id}_R$. The functions $r$ and $e$ are called *retraction* and *coretraction*, respectively.

Let define a binary relation $\leq$ on $\mathcal{C}(\mathbf{M})/_{\ker \rho^M}$ as follows:

$$[\mathbf{H_1}, h_1] \leq [\mathbf{H_2}, h_2] \iff \begin{array}{l} \text{either 1. } [\mathbf{H_1}, h_1] = [\mathbf{H_2}, h_2] \text{ or} \\ \text{2. there are morphisms } r : \mathbf{H_2} \to \mathbf{H_1} \text{ and } e : \mathbf{H_1} \to \mathbf{H_2} \\ \text{such that both } r \text{ and } e \text{ preserve colored elements} \\ \text{and } r \circ e \restriction_{C(\mathbf{H_1}, h_1)} = \mathrm{id}_{C(\mathbf{H_1}, h_1)}. \qquad (**) \end{array}$$

For each $i \in \{1, 2\}$, we may write $C(\mathbf{H_i}, h_i) = \{a_1^i, \ldots, a_{m_i}^i\}$ with $e(a_k^1) = a_k^2$ and $r(a_k^2) = a_k^1$ for all $1 \leq k \leq m_1$. We have the following theorem.

**Theorem 3.9** *The relation $\leq$ defined in $(**)$ is an order on $\mathcal{C}(\mathbf{M})/_{\ker \rho^M}$. Furthermore, if $[\mathbf{H_1}, h_1] \leq [\mathbf{H_2}, h_2]$ and $|C(\mathbf{H_1}, h_1)| = |C(\mathbf{H_2}, h_2)|$, then $[\mathbf{H_1}, h_1] = [\mathbf{H_2}, h_2]$.*

**Proof.** It is clear by the definition that $\leq$ is reflexive and transitive. Observe that if $[\mathbf{H_1}, h_1] \leq [\mathbf{H_2}, h_2]$ and $|C(\mathbf{H_1}, h_1)| = |C(\mathbf{H_2}, h_2)|$, then $\varepsilon := e \restriction_{C(\mathbf{H_1}, h_1)}: C(\mathbf{H_1}, h_1) \to C(\mathbf{H_2}, h_2)$ is a bijection such that $\tilde{h_2} \circ e$ and $\tilde{h_1} \circ r$ are morphisms extending $\tilde{h_2} \circ \varepsilon$ and $\tilde{h_1} \circ \varepsilon^{-1}$, respectively for all morphisms $\tilde{h_1} : \mathbf{H_1} \to \mathbf{M}$ and $\tilde{h_2} : \mathbf{H_2} \to \mathbf{M}$; so, $[\mathbf{H_1}, h_1] = [\mathbf{H_2}, h_2]$ follows from Theorem 3.7. Hence, if $[\mathbf{H_1}, h_1] \leq [\mathbf{H_2}, h_2]$ and $[\mathbf{H_2}, h_2] \leq [\mathbf{H_1}, h_1]$, then $|C(\mathbf{H_1}, h_1)| = |C(\mathbf{H_2}, h_2)|$ which implies that $\leq$ is anti-symmetric. ∎

Since the order $\leq$ on $\mathcal{C}(\mathbf{M})/_{\ker \rho^M}$ is defined analogously to the definition of the retract of resets, we will show a relationship between the order and the retract in the next theorem which can be proved directly.

**Theorem 3.10** *Let* $\mathbf{H_1}$ *be a retract of* $\mathbf{H_2}$ *with a retraction* $r$*. If* $C(\mathbf{H_1}, h_1) = r(C(\mathbf{H_2}, h_2))$*, then* $[\mathbf{H_1}, h_1] \leq [\mathbf{H_2}, h_2]$*.*

**Example 3.11** *Suppose that* $\mathbf{M}$*,* $(\mathbf{H}, h)$ *and* $(\mathbf{H}', h')$ *are defined in Example 3.6. Then* $\mathbf{H}$ *is a retract of* $\mathbf{H}'$ *with the retraction* $r : H' \to H$ *defined by*

$$r(x) = \begin{cases} v_i & \text{if } x = w_i \text{ for some } i \in \{0,1,2\}, \\ v_2 & \text{if } x = w_3. \end{cases}$$

*Moreover,* $C(\mathbf{H}, h) = r(C(\mathbf{H}', h'))$*. By Theorem 3.9 and Theorem 3.10,* $[\mathbf{H}, h] = [\mathbf{H}', h']$ *which implies that* $\rho^M_{(\mathbf{H},h)} = \rho^M_{(\mathbf{H}',h')}$*.*

## 3.2 One Type Duality for a Constantive Algebra

For each $m$-ary relation $r^M$ on a set $M$ and index set $A$ and $Z \subseteq M^A$, let $r^{M^A}$ be defined componentwise and $r^Z = r^{M^A} \cap Z^m$; and for a set $S^M$ of finitary relation on $M$, let denote $S^Z = \{r^Z \mid r^M \in S^M\}$. A function $\alpha : Z \to M$ is said to *preserve* $r^M$ if $(\alpha(x_1), \ldots, \alpha(x_m)) \in r^M$ for all $(x_1, \ldots, x_m) \in r^Z$. It is easy to prove that the concept of preserving is precisely a morphism between two resets which we will state in the following proposition.

**Proposition 3.12** *For each set* $S^M$ *of relations on* $M$ *and index set* $A$ *and* $Z \subseteq M^A$*,* $\alpha : Z \to M$ *preserves all elements in* $S^M$ *if and only if* $\alpha : (Z; S^Z) \to (M; S^M)$ *is a morphism.*

For each $\underline{\mathrm{A}} \in \mathscr{A} = \mathbb{ISP}(\underline{\mathrm{M}})$ and for a set $R$ of algebraic relations and algebraic relation $s$, we say that $R$ *entails* $s$ *on* $D(\underline{\mathrm{A}})$ if every continuous function $\alpha$ from $D(\underline{\mathrm{A}})$ to $M$ which preserves all elements in $R$ also preserves $s$; and say that $R$ *entails* $s$, briefly $R \vdash s$, if $R$ entails $s$ on $D(\underline{\mathrm{A}})$ for all $\underline{\mathrm{A}} \in \mathscr{A}$. For each set $R'$ of algebraic relations, we say that $R$ *entails* $R'$, briefly $R \vdash R'$, if $R$ entails $s$ for all $s \in R'$.

**Theorem 3.13** *$R$ entails $R'$ if and only if* $\alpha : (D(\underline{\mathrm{A}}); R'^{D(\underline{\mathrm{A}})}) \to (M; R'^M)$ *is a morphism whenever* $\alpha : (D(\underline{\mathrm{A}}); R^{D(\underline{\mathrm{A}})}) \to (M; R^M)$ *is a continuous morphism for all* $\underline{\mathrm{A}} \in \mathscr{A}$*.*

**Proof.** Suppose that $R$ entails $R'$, $\underline{\mathrm{A}} \in \mathscr{A}$ and $\alpha : (D(\underline{\mathrm{A}}); R^{D(\underline{\mathrm{A}})}) \to (M; R^M)$ is a continuous morphism. By Proposition 3.12, $\alpha$ preserves all elements in $R^M$ which implies that $\alpha$ preserves all elements in $R'^M$. So, $\alpha : (D(\underline{\mathrm{A}}); R'^{D(\underline{\mathrm{A}})}) \to (M; R'^M)$ is a morphism. Conversely, suppose that $\alpha : D(\underline{\mathrm{A}}) \to M$ is a continuous which preserves all elements in $R$. By Proposition 3.12, $\alpha : (D(\underline{\mathrm{A}}); R^{D(\underline{\mathrm{A}})}) \to (M; R^M)$ is a morphism which implies by the assumption that $\alpha : (D(\underline{\mathrm{A}}); R'^{D(\underline{\mathrm{A}})}) \to (M; R'^M)$ is a morphism. So, $\alpha$ preserves all elements in $R'$. ∎

Let $\underset{\sim}{\mathrm{M}} = (M; R, \tau)$ be an alter ego of $\underline{\mathrm{M}}$. Then $R \subseteq \mathcal{B}$; so, $\underset{\sim}{\mathrm{M}}$ corresponds to the subset $\mathcal{B}_{\underset{\sim}{\mathrm{M}}} = \{[r]_\sim \mid r \in R\}$ of $\mathcal{B}/_\sim$. By the Galois connection, we can study $\underset{\sim}{\mathrm{M}}$ via the subset $\varphi^{-1}\left(\mathcal{B}_{\underset{\sim}{\mathrm{M}}}\right)$ of $\mathcal{C}(\mathbf{M})/_{\ker \rho^M}$. We are going to refine an alter ego via the order on $\mathcal{C}(\mathbf{M})/_{\ker \rho^M}$.

**Lemma 3.14** *Let $A$ be an index set. If $r^M = \rho^M_{(a_1,\ldots,a_m)}$ for some $(\mathbf{H}, h) \in \mathcal{C}(\mathbf{M})$ with $C(\mathbf{H}, h) = \{a_1, \ldots, a_m\}$, then $r^{M^A} = \rho^{M^A}_{(a_1,\ldots,a_m)}$ where $\{a_1, \ldots, a_m\} = C(\mathbf{H}, h')$ and $C(\mathbf{H}, h') \in \mathcal{C}(\mathbf{M}^A)$ such that $h'(a_j)(a) = h(a_j)$ for all $a \in A$ and $1 \le j \le m$. Moreover, if $Z \subseteq M^A$, then $r^Z = \rho^{M^A}_{(a_1,\ldots,a_m)} \cap Z^m$.*

**Proof.**

$$
\begin{aligned}
(x_1, \ldots, x_m) \in r^{M^A} &\Leftrightarrow \text{for each } a \in A, (x_1(a), \ldots, x_m(a)) \in r^M \\
&\Leftrightarrow \text{for each } a \in A, \text{ there is a morphism } \tilde{h}_a : \mathbf{H} \to \mathbf{M} \text{ such that} \\
&\qquad \tilde{h}_a(a_j) = x_j(a) \text{ for all } 1 \le j \le m \\
&\Leftrightarrow \text{there is } \hat{h} : \mathbf{H} \to \mathbf{M}^A \text{ (defined by } \hat{h}(x)(a) = h_a(x), x \in H) \\
&\qquad \text{such that } \hat{h}(a_j) = x_j \text{ for all } 1 \le j \le m \\
&\Leftrightarrow (x_1, \ldots, x_m) \in \rho^{M^A}_{(a_1,\ldots,a_m)}
\end{aligned}
$$

∎

**Theorem 3.15** *If $[\mathbf{H_1}, h_1] \le [\mathbf{H_2}, h_2]$, then $\varphi([\mathbf{H_2}, h_2])$ entails $\varphi([\mathbf{H_1}, h_1])$.*

**Proof.** Suppose that $(\mathbf{H_1}, h_1), (\mathbf{H_2}, h_2) \in \mathcal{C}(\mathbf{M})$ with $[\mathbf{H_1}, h_1] \le [\mathbf{H_2}, h_2]$ and $C(\mathbf{H_i}, h_i) = \{a^i_1, \ldots, a^i_{m_i}\}$ for all $i \in \{1, 2\}$. We may assume that there are morphisms $e : \mathbf{H_1} \to \mathbf{H_2}$ and $r : \mathbf{H_2} \to \mathbf{H_1}$ such that $r(C(\mathbf{H_2}, h_2)) = C(\mathbf{H_1}, h_1)$,

$$e(a^1_k) = a^2_k \;\text{ and }\; r(a^2_k) = a^1_k$$

for all $1 \le k \le m_1$. It is left to show that

$$\{\rho^M_{(a^2_1,\dots,a^2_{m_2})}\} \text{ entails } \rho^M_{(a^1_1,\dots,a^1_{m_1})}.$$

Let $\underline{\mathrm{A}} \in \mathscr{A}$ and $\alpha : D(\underline{\mathrm{A}}) \to M$ be a function which preserves $\rho^M_{(a^2_1,\dots,a^2_{m_2})}$ and

$$(x_1,\dots,x_{m_1}) \in \rho^{M^A}_{(a^1_1,\dots,a^1_{m_1})} \cap (D(\underline{\mathrm{A}}))^{m_1}.$$

Then there is a morphism $g_1 : \mathbf{H_1} \to \mathbf{M}^A$ with $g_1(a^1_k) = x_k \in D(\underline{\mathrm{A}})$ for all $1 \le k \le m_1$. From $r(C(\mathbf{H_2}, h_2)) = C(\mathbf{H_1}, h_1)$, we have

$$(g_1 \circ r(a^2_1),\dots,g_1 \circ r(a^2_{m_2})) \in \rho^{M^A}_{(a^2_1,\dots,a^2_{m_2})} \cap (D(\underline{\mathrm{A}}))^{m_2}.$$

Since $\alpha$ preserves $\rho^M_{(a^2_1,\dots,a^2_{m_2})}$, we have

$$(\alpha(g_1 \circ r(a^2_1)),\dots,\alpha(g_1 \circ r(a^2_{m_2}))) \in \rho^M_{(a^2_1,\dots,a^2_{m_2})}.$$

Hence, there is a morphism $g_2 : \mathbf{H_2} \to \mathbf{M}$ such that

$$g_2(a^2_k) = \alpha \circ g_1 \circ r(a^2_k)$$

for all $1 \le k \le m_2$. Therefore, $g_2 \circ e : \mathbf{H_1} \to \mathbf{M}$ is a morphism with

$$g_2(e(a^1_k)) = \alpha \circ g_1 \circ r(e(a^1_k)) = \alpha \circ g_1 \circ (a^1_k) = \alpha(x_k)$$

for all $1 \le k \le m_1$. Thus, $(\alpha(x_1),\dots,\alpha(x_{m_1})) \in \rho^M_{(a^1_1,\dots,a^1_{m_1})}$. ∎

By applying Theorem 3.15, one can improve an alter ego of an algebra; especially for a constantive algebra. We are showing that if $\underline{\mathrm{M}}$ is constantive and $\underset{\sim}{\mathrm{M}}$ yields a duality of finite type on $\mathscr{A}$, then there exists an algebraic relation $r$ such that $(M; r, \tau)$ yields a duality on $\mathscr{A}$. Let $\mathrm{Clo}(\underline{\mathrm{M}}) = \mathrm{Pol}(\{r^M_i\}_{i\in I})$ for some set $\{r^M_i\}_{i\in I}$ of relations on $M$. Then $\underline{\mathrm{M}}$ is constantive if and only if $(a,\dots,a) \in r^M_i$ for all $a \in M$ and $i \in I$. Firstly, if $\underline{\mathrm{M}}$ is constantive we will show that each finite subset of $\mathcal{C}(\mathbf{M})/_{\ker \rho^M}$ has an upper bound where $\mathbf{M} = (M; \{r^M_i\}_{i\in I})$.

**Example 3.16** *Let* $\mathbf{M}$ *be a graph and let* $(\mathbf{H_1}, h_1)$, $(\mathbf{H_2}, h_2)$ *and* $(\mathbf{H_3}, h_3)$ *be* $\mathbf{M}$-*colored resets as in Figure 19, Figure 20 and Figure 21, respectively.*

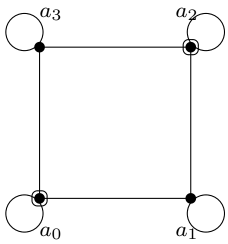


Figure 19. The $\mathbf{M}$-colored reset $(\mathbf{H_1}, h_1)$.

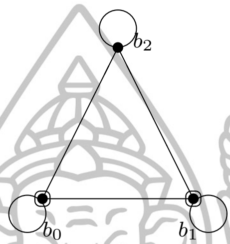


Figure 20. The $\mathbf{M}$-colored reset $(\mathbf{H_2}, h_2)$.

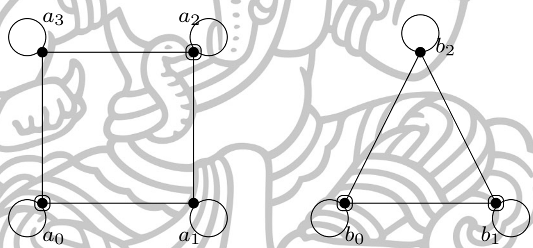


Figure 21. The $\mathbf{M}$-colored reset $(\mathbf{H_3}, h_3)$.

*Then* $[\mathbf{H_1}, h_1] \leq [\mathbf{H_3}, h_3]$ *and* $[\mathbf{H_2}, h_2] \leq [\mathbf{H_3}, h_3]$ *via the morphisms* $r_1 : \mathbf{H_3} \to \mathbf{H_1}$, $e_1 : \mathbf{H_1} \to \mathbf{H_3}$, $r_2 : \mathbf{H_3} \to \mathbf{H_2}$ *and* $e_2 : \mathbf{H_2} \to \mathbf{H_3}$ *defined by*

$$r_1(x) = \begin{cases} a_i & \text{if } x = a_i \text{ for some } i \in \{0,1,2,3\}, \\ a_0 & \text{if } x = b_i \text{ for some } i \in \{0,1,2\}, \end{cases}$$

$$e_1(a_i) = a_i \quad \text{for all } i \in \{0,1,2,3\},$$

$$r_2(x) = \begin{cases} b_0 & \text{if } x = a_i \text{ for some } i \in \{0,1,2,3\}, \\ b_i & \text{if } x = b_i \text{ for some } i \in \{0,1,2\}, \end{cases}$$

*and*

$$e_2(b_i) = b_i \quad \text{for all } i \in \{0,1,2\}.$$

*In fact,* $\mathbf{H_3}$ *is the sum of the graphs* $\mathbf{H_1}$ *and* $\mathbf{H_2}$ *and* $h_3 = h_1 \cup h_2$.

Recall that a *disjoin union* $H_1 \dot{\cup} H_2$ is the union of

$$H_1' = \{(x,1) \mid x \in H_1\}$$

and

$$H_2' = \{(x,2) \mid x \in H_2\}.$$

For each resets $\mathbf{H_1} = \left(H_1; \{r_i^{H_1}\}_{i\in I}\right)$ and $\mathbf{H_2} = \left(H_2; \{r_i^{H_2}\}_{i\in I}\right)$ having the same type, we define a *sum* $\mathbf{H_1} + \mathbf{H_2}$ of resets analogously to the sum of graphs by

$$\left(H_1 \dot{\bigcup} H_2; \{r_i^{H_1 \dot{\cup} H_2}\}_{i\in I}\right)$$

where for each $i \in I$ and $j \in \{1,2\}$,

$$r_i^{H_1 \dot{\cup} H_2} = r_i^{H_1'} \cup r_i^{H_2'}$$

and

$$r_i^{H_j'} = \left\{((x_1,j),\ldots,(x_{n_i},j)) \mid (x_1,\ldots,x_{n_i}) \in r_i^{H_j}\right\}.$$

For each $j \in \{1,2\}$ and $(\mathbf{H_j}, h_j) \in \mathcal{C}(\mathbf{M})$, let define $\mathbf{H_j'} = \left(H_j'; \left\{r_i^{H_j'}\right\}_{i\in I}\right)$ and $h_j' : \{(x,j) \mid x \in C(\mathbf{H_j}, h_j)\} \to M$ by

$$h_j'(x,j) = h_j(x) \text{ for all } x \in C(\mathbf{H_j}, h_j).$$

So, $(\mathbf{H_j'}, h_j') \in \mathcal{C}(\mathbf{M})$. Therefore, Theorem 3.7 and the function $\varepsilon : C(\mathbf{H_j}, h_j) \to C(\mathbf{H_j'}, h_j')$ defined by

$$\varepsilon(x) = (x,j) \text{ for all } x \in C(\mathbf{H_j}, h_j)$$

imply that $[\mathbf{H_j}, h_j] = \left[\mathbf{H_j'}, h_j'\right]$. Now, we define a binary operation + on $\mathcal{C}(\mathbf{M})/_{\ker \rho^M}$ by

$$[\mathbf{H_1}, h_1] + [\mathbf{H_2}, h_2] = [\mathbf{H_1} + \mathbf{H_2}, h_1 \sqcup h_2]$$

where

$$h_1 \sqcup h_2 = h_1' \cup h_2'.$$

We will show that $[\mathbf{H_j}, h_j] \leq [\mathbf{H_1}, h_1] + [\mathbf{H_2}, h_2]$ for all $j \in \{1,2\}$ via the following lemma.

**Lemma 3.17** *Let* $\underline{\mathrm{M}}$ *be constantive. For each* $(\mathbf{H}, h) \in \mathcal{C}(\mathbf{M})$*, there exists* $(\mathbf{G}, g) \in \mathcal{C}(\mathbf{M})$ *such that* $[\mathbf{H}, h] = [\mathbf{G}, g]$ *and* $(a,\ldots,a) \in r_i^G$ *for all* $a \in G$.

**Proof.** Let $(\mathbf{H}, h) \in \mathcal{C}(\mathbf{M})$. The consequence of Theorem 3.2 implies that $\rho^M_{(\mathbf{H},h)}$ is a set of algebraic relations over $\underline{\mathrm{M}}$; it follows that $\rho^M_{(\mathbf{H},h)} = \rho^M_{(\mathbf{G},g)}$ for some reset $(\mathbf{G}, g)$. Therefore, $\mathbf{G} = \mathbf{M}^n$ for some natural number $n$ which implies that $(a, \dots, a) \in r_i^G$ for all $a \in G$. ∎

**Theorem 3.18** *Let* $\underline{\mathrm{M}}$ *be constantive and* $(\mathbf{H_1}, h_1), (\mathbf{H_2}, h_2) \in \mathcal{C}(\mathbf{M})$. *Then* $[\mathbf{H_j}, h_j] \leq [\mathbf{H_1}, h_1] + [\mathbf{H_2}, h_2]$ *for all* $j \in \{1, 2\}$.

**Proof.** Let $\{j, k\} = \{1, 2\}$. By Lemma 3.17, we may assume that $(a, \dots, a) \in r_i^{H_j}$ for all $a \in H_j$ and $i \in I$. Let $b \in C(\mathbf{H_j}, h_j)$. Define $e : H_j \to H_1' \cup H_2'$ and $r : H_1' \cup H_2' \to H_j$ by

$$e(x) = (x, j) \text{ for all } x \in H_j$$

and

$$r(x, j) = x \text{ and } r(y, k) = b$$

for all $x \in H_j$ and $y \in H_k$. Since $(a, \dots, a) \in r_i^{H_j}$ for all $a \in H_j$ and $i \in I$, the both $e$ and $r$ are morphisms which preserve all colored elements and $r \circ e \restriction_{C(\mathbf{H_j}, h_j)} = \mathrm{id}_{C(\mathbf{H_j}, h_j)}$. Hence, $[\mathbf{H_j}, h_j] \leq [\mathbf{H_1}, h_1] + [\mathbf{H_2}, h_2]$. ∎

**Corollary 3.19** *If* $\underline{\mathrm{M}}$ *is constantive, every finite subset of* $\mathcal{C}(\mathbf{M})/_{\ker \rho^M}$ *has an upper bound.*

We will apply these facts to solve a duality-problem. If $\underline{\mathrm{M}}$ is a constantive algebra and $\underset{\sim}{\mathrm{M}} = (M; R, \tau)$ yields a duality of finite type on $\mathscr{A} = \mathbb{ISP}(\underline{\mathrm{M}})$, then by the Galois connection of $\mathbf{M}$-colored resets and algebraic relations over $\underline{\mathrm{M}}$ implies that there is a finite subset of $\mathcal{C}(\mathbf{M})/_{\ker \rho^M}$ corresponding to $\underset{\sim}{\mathrm{M}}$; hence, it is bounded by a class $[\mathbf{H}, h]$ of $\mathbf{M}$-colored reset $(\mathbf{H}, h)$. It follows by Theorem 3.9 that $\rho^M_{(\mathbf{H},h)}$ entails $R$. Let $r \in \rho^M_{(\mathbf{H},h)}$ be fixed. By Soundness Theorem [5], $\{r\}$ entails $\rho^M_{(\mathbf{H},h)}$ which implies by $\underset{\sim}{\mathrm{M}}$-Shift Duality Lemma [5] that $(M; r, \tau)$ yields a duality on $\mathscr{A}$. We conclude the results into the following theorem.

**Theorem 3.20** *If* $\underline{\mathrm{M}}$ *is constantive and dualisable of finite type, then there exists an algebraic relation* $r$ *such that* $(M; r, \tau)$ *yields a duality on* $\mathscr{A}$.

# Chapter 4

# All Maximal Clones of a Majority Reflexive Graph

Recall from Section 2.4 that a binary relation $\Theta$ on a finite set $M$ is called a *tolerance relation* if $\Theta$ is reflexive and symmetric; and the structure $\mathbf{M} = (M; \Theta)$ is called a *reflexive graph*. If $\mathrm{Pol}\,(\Theta)$ contains a majority operation, $\mathbf{M}$ is called a *majority reflexive graph*.

Let $\Theta$ be a tolerance relation on a finite set $M$ and $\mathbf{M} = (M; \Theta)$ be a majority reflexive graph. If $\underline{\mathrm{M}} = (M; F)$ is an algebra whose $\langle F \rangle = \mathrm{Pol}\,(\Theta)$, so-called a *tolerance primal-algebra*, then $\mathbb{S}(\underline{\mathrm{M}}^2)$ is precisely the set of all binary relations $\rho$ such that $\mathrm{Pol}\,(\rho) \supseteq \mathrm{Pol}\,(\Theta)$; moreover, NU-duality Theorem [5] implies that $\mathbb{S}(\underline{\mathrm{M}}^2)$ yields a duality on $\mathbb{ISP}(\underline{\mathrm{M}})$. In this chapter, we will begin with an application of the Galois connection between the set of algebraic relations over $\underline{\mathrm{M}}$ and the set of $\mathbf{M}$-colored resets to describe all elements in $\mathbb{S}(\underline{\mathrm{M}}^2)$; and then we apply these results to characterize all maximal clones of a majority reflexive graph.

## 4.1 A Duality for a Tolerance-primal Algebra Admitting a Majority Operation

For convenience through out this section, we assume that every graph is reflexive and we recall the basic definitions from graph theory. A (*reflexive*) *walk* from

$v_0$ to $v_n$ is a graph $(V, E)$ where

$$V = \{v_0, \ldots, v_n\}$$

and

$$E = \{(v_i, v_{i+1}) \,|\, 0 \le i \le n-1\} \bigcup \{(v_{i+1}, v_i) \,|\, 0 \le i \le n-1\} \bigcup \{(v_i, v_i) \,|\, 0 \le i \le n\}$$

for some $n \in \mathbb{N} \cup \{0\}$. If $v_1, \ldots, v_n$ are all distinct, a walk is called a (*reflexive*) *path* from $v_0$ to $v_n$ with *length* $n$; usually, we denote a path with length $n$ by $\mathbf{P_n}$ or $v_0 v_1 \ldots v_{n-1} v_n$. An example of a path is shown in Figure 22.

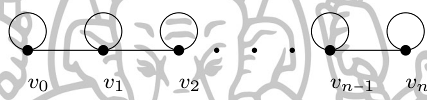


Figure 22. The path $\mathbf{P_n}$.

We denote a graph $(\{v_0, v_\infty\}; \{(v_0, v_0), (v_\infty, v_\infty)\})$ by $\mathbf{P}_\infty$ whose the diagram is shown in Figure 23.


Figure 23. The graph $\mathbf{P}_\infty$.

Let $\mathbf{G} = (G; V)$ be a graph. Recall that if $H \subseteq G$, then $(H; V')$ is a subgraph of $\mathbf{G}$ whenever $V' \subseteq V \downharpoonleft_H$; but, $(H; V \downharpoonleft_H)$ is called an *induced subgraph* of $\mathbf{G}$ (by *the set* $H$). A graph $\mathbf{G}$ is called a *connected graph* if there is a subgraph which is a path from $a$ to $b$ for all two vertices $a$ and $b$ in $G$. A maximal connected subgraph of $\mathbf{G}$ is called a *component* of $\mathbf{G}$. The *distant* between two vertices $a$ and $b$ in $\mathbf{G}$ is the length of the shortest path from $a$ to $b$ and is denoted by $d(a, b)$. We denote

$$d(\mathbf{G}) = \max\{d(a, b) \,|\, a, b \in G \text{ and } d(a, b) \text{ exists}\}.$$

**Example 4.1** *Suppose that* $\mathbf{G}$ *is a graph which is shown in Figure 24.*

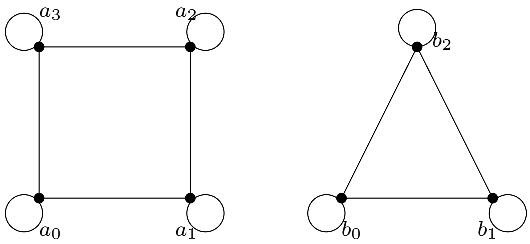


Figure 24. The graph $\mathbf{G}$.

*Then* $\{a_0, a_1, a_2, a_3\}$ *and* $\{b_0, b_1, b_2\}$ *induce subgraphs of* $\mathbf{G}$ *which are components of* $\mathbf{G}$*. Observe that both* $a_0a_1$ *and* $a_0a_3a_2a_1$ *are paths from* $a_0$ *to* $a_1$*; however,* $a_0a_1$ *is the shortest path. So,* $d(a_0, a_1) = 1$*. Moreover,* $d(\mathbf{G}) = 2 = d(a_0, a_2)$*.*

C. Ratanaprasert and U. Chotwattakawanit [19] described all elements in $\mathbb{S}(\underline{\mathrm{P}}^2)$ using the concept of distant function in order-primal algebras $\underline{\mathrm{P}}$. We will apply these concepts to describe all elements in $\mathbb{S}(\underline{\mathrm{M}}^2)$. By the Galois connection, the set $\mathbb{S}(\underline{\mathrm{M}}^2)$ corresponds to the set of $\mathbf{M}$-colored resets $(\mathbf{H}, h)$ with $|C(\mathbf{H}, h)| = 2$. For each $0 \leq n \leq \infty$, let $(\mathbf{P_n}, p_n)$ be the $\mathbf{M}$-colored reset with $C(\mathbf{P_n}, p_n) = \{v_0, v_n\}$.

**Theorem 4.2** *For each* $(\mathbf{H}, h) \in \mathcal{C}(\mathbf{M})$ *with* $|C(\mathbf{H}, h)| = 2$*,* $[\mathbf{H}, h] = [\mathbf{P_n}, p_n]$ *for some* $0 \leq n \leq d(\mathbf{M})$ *or* $n = \infty$*.*

**Proof.** Let $\mathbf{H} = (H; E)$ be a reflexive graph and $(\mathbf{H}, h) \in \mathcal{C}(\mathbf{M})$ with $C(\mathbf{H}, h) = \{a, b\}$ and $a \neq b$.
Case 1: $a$ and $b$ are in the same component of $\mathbf{H}$. Then there exists a shortest path $\mathbf{P_n} \coloneqq v_0v_1 \dots v_n$ from $a$ to $b$ for some $n \in \mathbb{N}$. To show that $[\mathbf{P_n}, p_n] \leq [\mathbf{H}, h]$, define functions $e : P_n \to H$ and $r : H \to P_n$ by

$$e(x) = x \quad \text{for all } x \in P_n$$

and

$$r(x) = \begin{cases} v_i & \text{if } x \in H \text{ with } i = d(a, x) \leq n, \\ v_n & \text{otherwise.} \end{cases}$$

We will show that $r$ is a morphism. Let $(x, y) \in E$. Then $x$ and $y$ are in the same component and $d(x, y) \leq 1$. If $a$ is not in the same component of $x$ and $y$, then $r(x) = v_n = r(y)$.

If $a$ is in the same component of $x$ and $y$, the triangle inequality property of $d$ implies that

$$d(a,x) \leq d(a,y) + d(y,x) \leq d(a,y) + 1$$

and

$$d(a,y) \leq d(a,x) + d(x,y) \leq d(a,x) + 1$$

which also implies that

$$d(a,x) - 1 \leq d(a,y) \leq d(a,x) + 1.$$

If $d(a,x) = i < n$, then $r(x) = v_i$ and $r(y) \in \{v_{i-1}, v_i, v_{i+1}\}$; and if $d(a,x) = i \geq n$, then $r(x) = v_n$ and $r(y) \in \{v_{n-1}, v_n\}$.

In either cases, $(r(x), r(y))$ is an edge of $\mathbf{P_n}$. So, $\mathbf{P_n}$ is a retract of $\mathbf{H}$. Moreover, $r(C(\mathbf{H},h)) = \{a,b\} = C(\mathbf{P_n}, p_n)$. By Theorem 3.10, $[\mathbf{P_n}, p_n] \leq [\mathbf{H}, h]$ which implies by Theorem 3.9 that $[\mathbf{P_n}, p_n] = [\mathbf{H}, h]$.

Case 2: $a \in C$ and $b \notin C$ for some component $\mathbf{C}$ of $\mathbf{H}$. Let $\mathbf{P}_\infty$ be an induced subgraph of $\mathbf{H}$ by $P_\infty = \{a = v_0, b = v_\infty\}$. To show that $[\mathbf{P}_\infty, p_\infty] \leq [\mathbf{H}, h]$, define functions $e : P_n \to H$ and $r : H \to P_\infty$ by

$$e(x) = x \quad \text{for all } x \in P_\infty$$

and

$$r(x) = \begin{cases} a & \text{if } x \in C, \\ b & \text{otherwise.} \end{cases}$$

We will show that $r$ is a morphism. For each $x, y \in H$, if $(x,y) \in E$, then either $x, y \in C$ or $x, y \notin C$ which implies $r(x) = r(y)$. By the reflexivity of $E$, $(r(x), r(y))$ is an edge of $\mathbf{P}_\infty$. So, $\mathbf{P}_\infty$ is a retract of $\mathbf{H}$. By the same argument as in Case 1, $[\mathbf{P}_\infty, p_\infty] = [\mathbf{H}, h]$.

It is left to show that $\left[\mathbf{P_{d(M)}}, p_{d(\mathbf{M})}\right] = [\mathbf{P_n}, p_n]$ for all $n \geq d(\mathbf{M})$. By applying Theorem 3.7, let $d(\mathbf{M}) = m$ and let $P_m = \{v_0, \ldots, v_m\}$ and $P_n = \{v'_0, \ldots, v'_n\}$. Define $\varepsilon : C(\mathbf{P_m}, p_m) \to C(\mathbf{P_n}, p_n)$ by $\varepsilon(v_0) = v'_0$ and $\varepsilon(v_m) = v'_n$. Let $\tilde{h} : \mathbf{P_m} \to \mathbf{M}$ be a morphism. Define $\alpha : \mathbf{P_n} \to \mathbf{M}$ by

$$\alpha(v'_i) = \begin{cases} \tilde{h}(v_i) & \text{if } 1 \leq i \leq m, \\ \tilde{h}(v_m) & \text{if } i \geq m. \end{cases}$$

Then $\alpha$ is a morphism which extends $\tilde{h} \circ \varepsilon^{-1}$.

Let $\tilde{g} : \mathbf{P_n} \to \mathbf{M}$ be a morphism. Then $\tilde{g}(\mathbf{P_n})$ is a connected subgraph of $\mathbf{M}$. Hence, there is a path $\mathbf{P} = v_0''v_1''\ldots v_k''$ from $\tilde{g}(v_0')$ to $\tilde{g}(v_n')$ for some $k \le d(\mathbf{M}) = m$. Define $\beta : \mathbf{P_m} \to \mathbf{M}$ by

$$\beta(v_i) = \begin{cases} v_i'' & \text{if } 1 \le i \le k, \\ v_k'' & \text{if } i \ge k. \end{cases}$$

It is easy to see that $\beta$ is a morphism which extends $\tilde{g} \circ \varepsilon$. By Theorem 3.7, $[\mathbf{P_{d(M)}}, p_{d(\mathbf{M})}] = [\mathbf{P_n}, p_n]$ for all $n \ge d(\mathbf{M})$. ∎

Now, let $\Theta^0 := \triangle_M$ and $\Theta^k = \underbrace{\Theta \circ \ldots \circ \Theta}_{k}$ for all natural numbers $k$.

**Corollary 4.3** *The set of all binary relations whose clones containing* $\mathrm{Pol}(\Theta)$ *is precisely* $\{\Theta^0, \ldots, \Theta^{d(\mathbf{M})}, M \times M\}$. *Moreover,* $\mathbb{S}(\underline{M}^2) = \{\Theta^0, \ldots, \Theta^{d(\mathbf{M})}, M \times M\}$.

**Proof.** By Theorem 3.2, a binary algebraic relation $r$ is $\rho^M_{(a_1,a_2)}$ for some $(\mathbf{H}, h) \in \mathbf{M}$ with $C(\mathbf{H}, h) = \{a_1, a_2\}$ which implies by Theorem 4.2 that $\rho^M_{(a_1,a_2)} = \rho^M_{(v_0,v_n)}$ for some $(\mathbf{P_n}, p_n) \in \mathbf{M}$ with $C(\mathbf{P_n}, p_n) = \{v_0, v_n\}$ and some $0 \le n \le d(\mathbf{M})$ or $n = \infty$. If $0 \le n \le d(\mathbf{M})$, then

$$\begin{aligned} (x,y) \in \rho^M_{(v_0,v_n)} \quad &\Leftrightarrow \quad \text{there is a morphism } \tilde{h} : \mathbf{P_n} \to \mathbf{M} \text{ such that} \\ &\qquad x = \tilde{h}(v_0)\Theta\tilde{h}(v_1)\Theta \ldots \Theta\tilde{h}(v_n) = y \\ &\Leftrightarrow \quad (x,y) \in \Theta^n. \end{aligned}$$

Therefore, $\rho^M_{(v_0,v_\infty)} = M \times M$. ∎

NU-duality Theorem [5] and Theorem 4.3 imply the following corollary.

**Corollary 4.4** *If* $\mathbf{M}$ *is a majority reflexive graph,* $\underset{\sim}{M} = (M; \{\Theta^k \mid 1 \le k \le d(\mathbf{M})\}, \mathscr{T})$ *yields a duality on* $\mathbb{ISP}(\underline{M})$.

## 4.2 All Maximal Clones of a Majority Reflexive Graph

In Section 4.1, we described all binary relations whose clones contain the clone preserving a tolerance relation. It is interesting whether some of them are maximal and the converse is also true. In this section, we study some conditions which prove the questions.

We refer all definitions and notations from Section 4.1. For a set $M$ and $\Theta \subseteq M \times M$, $\mathrm{Pol}(\Theta)$ is the full clone if and only if $\Theta \in \{\triangle_M, M \times M\}$. And also, the lattice of all clones on a singleton set has exactly one element. In this section, we will consider a set $M$ with $|M| \geq 2$ and $\triangle_M \subset \Theta \subset M \times M$. Then $\mathrm{Pol}(\Theta)$ is a subclone of a maximal clone preserving a relation from one of the six classes described by I.G. Rosenberg [20, 21]. If $\Theta$ is a tolerance relation on $M$, the following theorem shows all possible classes of relations whose the clones are a maximal clone containing $\mathrm{Pol}(\Theta)$.

**Theorem 4.5** *Let $\Theta$ be a tolerance relation on $M$. Then* $\mathrm{Pol}(\Theta)$ *is a subclone of a maximal clone* $\mathrm{Pol}(\rho)$ *whose $\rho$ is a non-trivial equivalence relation, a k-regularly generated relation or a central relation.*

**Proof.** If $\rho$ is in the classes (1) or (2), then $\rho$ is binary which implies by Theorem 4.3 that $\rho = M \times M$ or $\rho = \Theta^k$ for some $0 \leq k \leq d(\mathbf{M})$. But, reflexivity and symmetricity of $\Theta^k$ for all $k \geq 0$ imply that $\Theta^k$ is an order or a permutation if and only if $k = 0$. Hence, $\mathrm{Pol}(\rho)$ is the full clone, a contradiction.

Suppose that $\rho$ is an affine relation corresponding to a group $(M; +, -, 0)$. Let $(a, b) \in \Theta$ with $a \neq b$. We may assume that $a \neq 0$. Define $f : M \times M \to M$ by

$$f(x, y) = \begin{cases} a & \text{if } x = y = a, \\ b & \text{otherwise} \end{cases}$$

for all $x, y \in M$. Since $\mathrm{Im} f = \{a, b\}$, we have $f \in \mathrm{Pol}(\Theta) \subseteq \mathrm{Pol}(\rho)$; that is, $f$ preserves $\rho$. So, $(a, 0, a, 0), (a, -a, 0, 0) \in \rho$ implies that

$$(a, b, b, b) = (f(a, a), f(0, -a), f(a, 0), f(0, 0)) \in \rho;$$

thus, $a + b = b + b$ which implies $a = b$, a contradiction. ∎

**Corollary 4.6** *If $\Theta$ is a tolerance relation on $M$ and* $\mathrm{Pol}(\Theta)$ *contains a majority operation, it is a subclone of a maximal clone* $\mathrm{Pol}(\rho)$ *whose $\rho$ is only either a non-trivial equivalence relation or a central relation. Moreover, if $\rho$ is a central relation, then $\rho$ is binary.*

**Proof.** By the result in [19], if $\rho$ is a central relation, then $\rho$ is at most binary. Reflexivity of $\Theta$ implies that all constants are in $\mathrm{Pol}\,(\Theta) \subseteq \mathrm{Pol}\,(\rho)$; so, $\rho$ is not unary. ∎

From now, we consider $\Theta$ is a tolerance relation on $M$ whose $\mathrm{Pol}\,(\Theta)$ contains a majority operation. Theorem 4.3 and Corollary 4.6 imply that all relations $\rho$, whose $\mathrm{Pol}\,(\rho)$ is a maximal clone containing $\mathrm{Pol}\,(\Theta)$, are of the forms $\Theta^k$ for some $1 \le k \le d(\mathbf{M})$. For each $a, b \in M$ with $d(a,b) = d(\mathbf{M})$ and $1 \le k \le d(\mathbf{M})$, if $\Theta^k$ is transitive, then $(a,b) \in \Theta^k$; so, $d(\mathbf{M}) = d(a,b) \le k \le d(\mathbf{M})$.

**Remark 4.7** *For each* $1 \le k \le d(\mathbf{M})$*,* $\Theta^k$ *is an equivalence relation if and only if* $k = d(\mathbf{M})$*.*

**Theorem 4.8** *Suppose that* $\mathrm{Pol}\,(\rho)$ *is a maximal clone containing* $\mathrm{Pol}\,(\Theta)$*.*

1. *If the graph* $\mathbf{M}$ *is connected, then* $\rho$ *is precisely central relations of the form* $\Theta^k$ *for some* $\lceil d(\mathbf{M})/2 \rceil \le k < d(\mathbf{M})$*.*

2. *If the graph* $\mathbf{M}$ *is disconnected, then* $\rho$ *is the non-trivial equivalence relation* $\Theta^{d(\mathbf{M})} = \cup_{1 \le i \le m} C_i \times C_i$ *where* $\mathbf{C_1} \ldots, \mathbf{C_m}$ *are all components of* $\mathbf{M}$*.*

**Proof.** (1). Connectedness of $(M;\Theta)$ implies that $\Theta^{d(\mathbf{M})} = M \times M$. By Remark 4.7, $\rho$ is not non-trivial equivalence relations. So, $\rho$ is a central relation. One can see that $\Theta^k$ is central if and only if $\lceil d(\mathbf{M})/2 \rceil \le k < d(\mathbf{M})$.

(2). It is easily shown that $\Theta^{d(\mathbf{M})} = \cup_{1 \le i \le m} C_i \times C_i$. If $\rho$ is central, the center elements will be related to all elements of $M$; so, one can conclude by Corollary 2.3 that $\rho = M \times M$ which is impossible. Hence, $\rho$ is a non-trivial equivalence relation which implies that $\rho = \Theta^{d(\mathbf{M})}$. ∎

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# APPENDIX

# Publications and Presentation

Publications

U. Chotwattakawanit and C. Ratanaprasert, All Maximal Clones of a Majority Reflexive Graph, Thai Journal of Mathematics 13, no. 1 (2015), 63-68.

U. Chotwattakawanit and C. Ratanaprasert, A Galois Connection of Algebraic Relations and Colored Resets, Far East Journal of Mathematical Sciences 100, no. 3 (2016), 427-437.

Presentation

U. Chotwattakawanit, All Maximal Clones of a Majority Reflexive Graph, International Conference on Discrete Mathematics and Applied Sciences, University of the Thai Chamber of Commerce (UTCC), Bangkok, Thailand, May 21-23, 2014.

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