PROBLEMS IN LUCAS SEQUENCES OF THE FIRST AND SECOND KINDS.

By

Mr. Kritkhajohn ONPHAENG

A Thesis Proposal Submitted in Partial Fulfillment of the Requirements
for Doctor of Philosophy (MATHEMATICS)
Department of MATHEMATICS
Graduate School, Silpakorn University
Academic Year 2020

ปัญหาเกี่ยวกับลำดับลูคาส์ชนิดที่หนึ่งและชนิดที่สอง

โดย
Mr.Kritkhajohn ONPHAENG

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Title	Problems in Lucas sequences of the first and second kinds.
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In this thesis, we conduct a study on problems in Lucas sequences of the first and second kinds which are the integer sequences $(U_n)_{n\geq 0}$ and $(V_n)_{n\geq 0}$. Both of the sequences depending on parameters $a, b \in \mathbb{Z}$ are defined by the recurrence relations $U_0 = 0$, $U_1 = 1$, and $U_n = aU_{n-1} + bU_{n-2}$ for $n \geq 2$, $V_0 = 2$, $V_1 = a$, and $V_n = aV_{n-1} + bV_{n-2}$ for $n \geq 2$. We obtain exact divisibility results concerning U_n^k and V_n^k for all positive integers n and k. These results complete a long investigation by various authors from 1970 to 2021.

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Kritkhajohn ONPHAENG

Table of contents

Chapter 1

Introduction

Throughout this thesis, let p be a prime, a and b relatively prime integers, k, m, and n positive integers. The exact divisibility denoted by $m^k \| n$ means $m^k | n$ and $m^{k+1} \nmid n$, and for $n \in \mathbb{N}$, the p-adic valuation of n, denoted by $v_p(n)$ is the power of p in the prime factorization of n. The sequence F_n of the Fibonacci numbers is defined by the recurrence relation: $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ and $F_0 = 0$, $F_1 = 1$. The sequence L_n of the Lucas numbers is defined by the recurrence relation: $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$ and $L_0 = 2$, $L_1 = 1$. The sequences $(U_n)_{n\geq 0}$ and $(V_n)_{n\geq 0}$ are the Lucas sequences of the first and second kinds which are defined by the recurrence relations

$$U_0 = 0,\ U_1 = 1,\ U_n = aU_{n-1} + bU_{n-2} \text{ for } n \geq 2,$$

$$V_0 = 2,\ V_1 = a, \text{ and } V_n = aV_{n-1} + bV_{n-2} \text{ for } n \geq 2.$$

To avoid triviality, we always assume that $b \neq 0$ and α/β is not a root of unity where α and β are the roots of the characteristic polynomial $x^2 - ax - b$. In particular, this implies that $\alpha \neq \beta$, $\alpha \neq -\beta$, the discriminant $D = a^2 + 4b \neq 0$, $U_n \neq 0$, and $V_n \neq 0$ for all $n \geq 1$. If $a = b = 1$, then $(U_n)_{n\geq 0}$ reduces to the sequence of Fibonacci numbers F_n and $(V_n)_{n\geq 0}$ reduces to the sequence of Lucas numbers L_n; if $a = 6$ and $b = -1$, then $(U_n)_{n\geq 0}$ becomes the sequence of balancing numbers; if $a = 2$ and $b = 1$, then $(U_n)_{n\geq 0}$ is the sequence of Pell numbers. Many other famous integer sequences are just special cases of the Lucas sequences of the first and second kinds.

The divisibility by powers of the Fibonacci numbers has attracted some attentions because it is applied in Matijasevich's solution to Hilbert's 10th problem [7, 8, 9]. More precisely, Matijasevich showed that

$$F_n^2 \mid F_{nm} \quad \text{if and only if} \quad F_n \mid m. \tag{1.1}$$

In 1977, Hoggatt and Bicknell-Johnson [3] gave another proof of (1.1) and extended it to higher powers. Hoggatt and Bicknell-Johnson [3] proved that

$$\text{if } F_n^k \mid m, \text{ then } F_n^{k+1} \mid F_{nm} \tag{1.2}$$

which is a generalization of the converse statement of (1.1). Some of the different proofs of (1.2) can also be founded in Benjamin and Rouse [1], and Seibert and Trojovský [29]. In 2012, Tangboonduangjit and Wiboonton [31] investigated some properties of the following sequence,

$$F_n, F_{nF_n}, F_{nF_{nF_n}}, F_{nF_{nF_{nF_n}}}, \ldots .$$

Let $G_k(n)$ be the kth term of this sequence. They proved that $F_n^k \mid G_k(n)$ for every positive integers k and n. Furthermore, Panraksa, Tangboonduangjit and Wiboonton [14] proved that $F_n^k \parallel G_k(n)$ for all positive integers k and n with $n > 3$ in 2013. In 2014, Onphaeng and Pongsriiam [12] defined a sequence $(G(k,n,m))_{k\geq 1}$ by $G(1,n,m) = F_n^m$ and $G(k+1,n,m) = F_{nG(k,n,m)}$ for all $k \geq 1$. We showed that $F_n^{k+m-1} \parallel G(k,n,m)$ for all $k,m,n \in \mathbb{N}$. Based on the work studied by Pongsriiam [20] in 2014, a number of general results in this direction are provided in the following three theorems, especially, the property given in (1.2) that is extended to include the divisibility and exact divisibility for both the Fibonacci and Lucas numbers.

Theorem 1.1. [20, Theorem 2] *For $n \geq 3$, we have*

(i) *if $F_n^k \parallel m$ and $n \not\equiv 3 \pmod 6$, then $F_n^{k+1} \parallel F_{nm}$;*

(ii) *if $F_n^k \parallel m$, $n \equiv 3 \pmod 6$, and $\frac{F_n^{k+1}}{2} \nmid m$, then $F_n^{k+1} \parallel F_{nm}$;*

(iii) *if $F_n^k \parallel m$, $n \equiv 3 \pmod 6$, and $\frac{F_n^{k+1}}{2} \mid m$, then $F_n^{k+2} \parallel F_{nm}$.*

Theorem 1.2. [20, Theorem 3] *Let m be an odd integer. Then*

(i) *if $L_n^k \mid m$, then $L_n^{k+1} \mid L_{nm}$;*

(ii) *if $n \geq 2$ and $L_n^k \parallel m$, then $L_n^{k+1} \parallel L_{nm}$.*

Theorem 1.3. [20, Theorem 4] *Let m be even and $n \geq 2$. Then the following statements hold.*

(i) *if* $L_n^k \mid m$, *then* $L_n^{k+1} \mid F_{nm}$;

(ii) *if* $L_n^k \parallel m$ *and* $n \not\equiv 0 \pmod 3$, *then* $L_n^{k+1} \parallel F_{nm}$;

(iii) *if* $L_n^k \parallel m$, $n \equiv 0 \pmod 6$, *and* $\frac{L_n^{k+1}}{2} \nmid m$, *then* $L_n^{k+1} \parallel F_{nm}$;

(iv) *if* $L_n^k \parallel m$, $n \equiv 0 \pmod 6$, *and* $\frac{L_n^{k+1}}{2} \mid m$, *then* $L_n^{k+2} \mid F_{nm}$;

(v) *if* $L_n^k \parallel m$, $n \equiv 3 \pmod 6$, *and* $\frac{L_n^{k+1}}{4} \nmid m$, *then* $L_n^{k+1} \parallel F_{nm}$;

(vi) *if* $L_n^k \parallel m$, $n \equiv 3 \pmod 6$, *and* $\frac{L_n^{k+1}}{4} \mid m$, *then* $L_n^{k+2} \mid 4F_{nm}$.

Recently, Onphaeng and Pongsriiam [13] gave the converse of Theorems 1.1, 1.2, and 1.3 as follows.

Theorem 1.4. [13, Theorem 3.2] *Let k, m, n be positive integers and $n \geq 3$. Then the following statements hold.*

(i) *if* $F_n^{k+1} \parallel F_{nm}$ *and* $n \not\equiv 3 \pmod 6$, *then* $F_n^k \parallel m$;

(ii) *if* $F_n^{k+1} \parallel F_{nm}$, $n \equiv 3 \pmod 6$, *and* $2^k \mid m$, *then* $F_n^k \parallel m$;

(iii) *if* $F_n^{k+1} \parallel F_{nm}$, $n \equiv 3 \pmod 6$, *and* $2^k \nmid m$, *then* $F_n^{k-1} \parallel m$.

Theorem 1.5. [13, Theorem 3.3] *Let k, m, n be positive integers and $n \geq 2$. Then the following statements hold.*

(i) *if* $L_n^{k+1} \mid L_{nm}$, *then* $n \not\equiv 0 \pmod 3$, *m is odd, and* $L_n^k \mid m$;

(ii) *if* $L_n^{k+1} \parallel L_{nm}$, *then* $L_n^k \parallel m$.

Theorem 1.6. [13, Theorem 3.4] *Let k, m, n be positive integers and $n \geq 2$. If $L_n^{k+1} \mid F_{nm}$, then m is even. Moreover, the following statements hold.*

(i) *if* $L_n^{k+1} \mid F_{nm}$ *and* $n \not\equiv 0 \pmod 6$, *then* $L_n^k \mid m$;

(ii) *if* $L_n^{k+1} \parallel F_{nm}$ *and* $n \not\equiv 0 \pmod 6$, *then* $L_n^k \parallel m$;

(iii) if $L_n^{k+1} \mid F_{nm}$ *and* $n \equiv 0 \pmod 6$*, then* $L_n^{min\{v_2(m),k\}} \mid m$*;*

(iv) if $L_n^{k+1} \parallel F_{nm}$ *and* $n \equiv 0 \pmod 6$*, then* $L_n^{min\{v_2(m),k\}} \parallel m$*.*

By applying the results proposed in the articles [13, 20], Onphaeng and Pongsriiam obtained complete answers to this kind of questions for the Fibonacci and Lucas numbers. Then Panraksa and Tangboonduangjit [15] initiated the investigation on a special subsequence of $(U_n)_{n\geq 0}$. Patra, Panda, and Khemaratchatakumthorn [16] also obtained the analogue of those results for the balancing and Lucas-balancing numbers. For other related and recent results on Fibonacci, Lucas, balancing, and Lucas-balancing numbers, see, for example, [2, 4, 5, 6, 17, 18, 19, 21, 22, 23, 24, 25, 26, 27, 30] and references there in.

In this thesis, we extend Theorems 1.1 and 1.4 to the case of U_n, Theorems 1.2 and 1.5 to the case of V_n, and Theorems 1.3 and 1.6 to the case of U_n and V_n mixed in together.

Chapter 2
Preliminaries and Lemmas

In this section, we recall some definitions and well known results, and give some useful lemmas for the reader's convenience. The order (or the rank) of appearance of $n \in \mathbb{N}$ in the Lucas sequence $(U_n)_{n\geq 0}$ is defined as the smallest positive integer m such that $n \mid U_m$ and is denoted by $\tau(n)$. We sometimes write the expression such as $a \mid b \mid c = d$ to mean that $a \mid b$, $b \mid c$, and $c = d$. For each $x \in \mathbb{R}$, we write $\lfloor x \rfloor$ to denote the largest integer less than or equal to x. So $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$. We let $D = a^2 + 4b$ be the discriminant and let α and β be the roots of the characteristic polynomial $x^2 - ax - b$. Then it is well known that if $D \neq 0$, then the Binet formula

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = \alpha^n + \beta^n \text{ holds for all } n \geq 0.$$

Next, we recall Sanna's result [28] on the p-adic valuation of the Lucas sequence of the first kind.

Lemma 2.1. [28, Theorem 1.5] *Let p be a prime number such that $p \nmid b$. Then, for each positive integer n,*

$$v_p(U_n) = \begin{cases} v_p(n) + v_p(U_p) - 1 & \text{if } p \mid D \text{ and } p \mid n, \\ 0 & \text{if } p \mid D \text{ and } p \nmid n, \\ v_p(n) + v_p(U_{p\tau(p)}) - 1 & \text{if } p \nmid D,\ \tau(p) \mid n, \text{ and } p \mid n, \\ v_p(U_{\tau(p)}) & \text{if } p \nmid D,\ \tau(p) \mid n, \text{ and } p \nmid n, \\ 0 & \text{if } p \nmid D \text{ and } \tau(p) \nmid n. \end{cases}$$

In particular, if p is an odd prime such that $p \nmid b$, then, for each positive integer n,

$$v_p(U_n) = \begin{cases} v_p(n) + v_p(U_p) - 1 & \text{if } p \mid D \text{ and } p \mid n, \\ 0 & \text{if } p \mid D \text{ and } p \nmid n, \\ v_p(n) + v_p(U_{\tau(p)}) & \text{if } p \nmid D \text{ and } \tau(p) \mid n, \\ 0 & \text{if } p \nmid D \text{ and } \tau(p) \nmid n. \end{cases}$$

We also recall a result by Panraksa and Tangboonduangjit [15] in their calculation concerning a special subsequence of $(U_n)_{n\geq 0}$.

Lemma 2.2. [15, Lemma 2.3] *Let $m, n \geq 1$ and p a prime factor of U_n such that $p \nmid b$. Then, if* (i) *p is odd, or* (ii) *$p = 2$ and n is even, or* (iii) *$p = 2$ and m is odd, we have*

$$v_p(U_{nm}) = v_p(m) + v_p(U_n).$$

From Lemma 2.1, and the fact that $V_n = U_{2n}/U_n$, we easily obtain the following result.

Lemma 2.3. [11, Lemma 4] *If p is an odd prime and $p \nmid b$. Then, for each positive integer n,*

$$v_p(V_n) = \begin{cases} v_p(n) + v_p(U_{\tau(p)}) & \text{if } p \nmid D,\ \tau(p) \nmid n \text{ and } \tau(p) \mid 2n, \\ 0 & \text{otherwise.} \end{cases}$$

The next two lemmas are also important tools in proving exact divisibility by U_n^k for all $n, k \in \mathbb{N}$.

Lemma 2.4. [12, Lemma 2.3] *Let k, ℓ, m be positive integers, s nonzero integer, and $s^k \mid m$. Then $s^{k+\ell} \mid \binom{m}{j} s^j$ for all $1 \leq j \leq m$ satisfying $2^{j-\ell+1} > j$. In particular, $s^{k+1} \mid \binom{m}{j} s^j$ for all $1 \leq j \leq m$, and $s^{k+2} \mid \binom{m}{j} s^j$ for all $3 \leq j \leq m$.*

Proof. The statement in [12, Lemma 2.3] is given for $s \geq 1$ but it is easy to see that if $s \leq -1$, then we can replace s by $-s$ and every divisibility relation still holds. Therefore this is true for all $s \neq 0$. □

Lemma 2.5. [10, Lemma 5] *Let $m, n \geq 1$ and $r \geq 0$ be integers. Then*

(i) $U_{mn+r} = \sum_{j=0}^{m} \binom{m}{j} U_n^j (bU_{n-1})^{m-j} U_{j+r}$,

(ii) $U_{mn} = \sum_{j=1}^{m} \binom{m}{j} U_n^j (bU_{n-1})^{m-j} U_j$.

Proof. By Binet's formula, we obtain $\alpha^n = \alpha U_n + bU_{n-1}$, $\beta^n = \beta U_n + bU_{n-1}$, and

$$\begin{aligned} U_{mn+r} &= \frac{\alpha^{mn+r} - \beta^{mn+r}}{\alpha - \beta} \\ &= \frac{1}{\alpha-\beta}\left((\alpha U_n + bU_{n-1})^m \alpha^r - (\beta U_n + bU_{n-1})^m \beta^r\right) \\ &= \frac{1}{\alpha-\beta}\left(\sum_{j=0}^{m}\binom{m}{j}(\alpha U_n)^j (bU_{n-1})^{m-j}\alpha^r - \sum_{j=0}^{m}\binom{m}{j}(\beta U_n)^j (bU_{n-1})^{m-j}\beta^r\right) \\ &= \frac{1}{\alpha-\beta}\sum_{j=0}^{m}\left(\binom{m}{j}U_n^j (bU_{n-1})^{m-j}\left(\alpha^{j+r} - \beta^{j+r}\right)\right) \\ &= \sum_{j=0}^{m}\binom{m}{j}U_n^j (bU_{n-1})^{m-j} U_{j+r}. \end{aligned}$$

This proves (i). Since $U_0 = 0$, (ii) follows immediately from (i) by substituting $r = 0$. □

Recall that we assume throughout this article that $(a, b) = 1$. This is necessary for the proof of the following lemmas.

Lemma 2.6. [10, Lemma 6] *Suppose $(a, b) = 1$. Then $(U_m, U_n) = U_{(m,n)}$ and in particular $(U_n, U_{n+1}) = 1$ for each $m, n \in \mathbb{N}$.*

Proof. This is well known. □

Lemma 2.7. [11, Lemma 5] *Let $n \geq 1$ and $(a, b) = 1$. If $p \mid U_n$ or $p \mid V_n$, then $p \nmid b$. Consequently, $(U_n, b) = (V_n, b) = 1$ for all $n \geq 1$.*

Proof. The case for U_n is already given in [10, Lemma 7]. So suppose by way of contradiction that $p \mid V_n$ and $p \mid b$. Since $V_n = aV_{n-1} + bV_{n-2}$ and $(a, b) = 1$, we obtain $p \mid V_{n-1}$. Repeating this argument, we see that $p \mid V_m$ for $1 \leq m \leq n$. In particular, $p \mid V_1 = a$ contradicting $(a, b) = 1$. So if $p \mid V_n$, then $p \nmid b$, and the proof is complete. □

Lemma 2.8. [10, Lemma 8] *Let a and b be odd, $(a, b) = 1$, and $v_2(U_6) \geq v_2(U_3) + 2$. Then $v_2(U_3) = 1$.*

Proof. Since $U_3 = a^2 + b$ is even and $U_6 = a(a^2 + 3b)U_3$, we obtain $v_2(U_3) \geq 1$ and

$$v_2(U_6) = v_2(U_3) + v_2(a^2 + 3b). \tag{2.1}$$

If $v_2(U_3) \geq 2$, then $4 \mid a^2 + b$, and so $b \equiv 3 \pmod 4$ and (2.1) implies $v_2(U_6) = v_2(U_3) + 1$ contradicting $v_2(U_6) \geq v_2(U_3) + 2$. Thus $v_2(U_3) = 1$. □

For convenience, we also calculate the 2-adic valuation of U_n and V_n as follows.

Lemma 2.9. [11, Lemma 7] *Assume that a is odd, b is even, and $n \geq 1$. Then $v_2(U_n) = v_2(V_n) = 0$.*

Proof. Since $U_1 = 1$ and $U_2 = a$ are odd, and $U_r = aU_{r-1} + bU_{r-2} \equiv U_{r-1} \pmod 2$ for $r \geq 3$, it follows by induction that U_n is odd. Since $V_n = \frac{U_{2n}}{U_n}$, V_n is also odd. This proves the lemma. □

Lemma 2.10. [11, Lemma 8] *Assume that a is even, b is odd, and $n \geq 1$. Then*

$$v_2(U_n) = \begin{cases} v_2(n) + v_2(a) - 1 & \text{if } 2 \mid n, \\ 0 & \text{if } 2 \nmid n, \end{cases}$$

$$v_2(V_n) = \begin{cases} 1 & \text{if } 2 \mid n, \\ v_2(a) & \text{if } 2 \nmid n, \end{cases}$$

Proof. Since $2 \mid D$, we obtain by Lemma 2.1 that for each $n \in \mathbb{N}$, $v_2(U_n) = v_2(n) + v_2(U_2) - 1$ if $2 \mid n$ and $v_2(U_n) = 0$ if $2 \nmid n$. Since $U_2 = a$, the formula for $v_2(U_n)$ is verified. Then $v_2(V_n)$ can be obtained from a straightforward calculation and the fact that $V_n = \frac{U_{2n}}{U_n}$. This completes the proof. □

Lemma 2.11. [11, Lemma 9] *Assume that a and b are odd, and $n \geq 1$. Then*

$$v_2(U_n) = \begin{cases} v_2(n) + v_2(U_6) - 1 & \text{if } n \equiv 0 \pmod 6, \\ v_2(U_3) & \text{if } n \equiv 3 \pmod 6, \\ 0 & \text{if } n \not\equiv 0 \pmod 3, \end{cases}$$

$$v_2(V_n) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 6, \\ v_2(U_6) - v_2(U_3) & \text{if } n \equiv 3 \pmod 6, \\ 0 & \text{if } n \not\equiv 0 \pmod 3, \end{cases}$$

Proof. Since U_1 and U_2 are odd, and $U_3 = a^2 + b$ is even, we have $\tau(2) = 3$. In addition, $2 \nmid D$. Furthermore, $3 \mid n$ and $2 \mid n$ if and only if $n \equiv 0 \pmod 6$; $3 \mid n$ and $2 \nmid n$ if and only if $n \equiv 3 \pmod 6$. Then applying Lemma 2.1 and the fact that $V_n = \frac{U_{2n}}{U_n}$, we obtain the desired result. □

Chapter 3
Main Results

In this chapter, we present results of exact divisibility by powers of the integers in the Lucas sequences of the first and second kinds. We begin with the Lucas sequence of the first kind and give some examples. After that we show the result of the Lucas sequence of the second kind and example. Finally, we prove a result of the Lucas sequence in the case of the mix of first and second kinds.

3.1 Exact divisibility by powers of the integers in the Lucas sequence of the first kind

In this section, we extend Theorems 1.1 and 1.4 to the case of U_n and obtain some relevant results.

Theorem 3.1. [10, Theorem 9] *Let k, m, and n be positive integers. If $U_n^k \mid m$, then $U_n^{k+1} \mid U_{nm}$.*

Proof. If $U_n^k \mid m$, then we obtain by Lemma 2.4 that, $U_n^{k+1} \mid \binom{m}{j} U_n^j$ for all $1 \leq j \leq m$, which implies $U_n^{k+1} \mid U_{nm}$, by Lemma 2.5. □

Next, we extend Theorem 3.1 to include exact divisibility. The proof of Theorem 3.2 is much longer than that of Theorem 3.1 since we would like to cover all possible cases. Although many cases can be combined, it is more convenient to state them separately. Recall that for $x \in \mathbb{R}$, the largest integer which is less than or equal to x is denoted by $\lfloor x \rfloor$.

Theorem 3.2. [10, Theorem 10] *Let $k, m, n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $(a, b) = 1$, $n \geq 2$, and $U_n^k \parallel m$. Then*

(i) *if a is odd and b is even, then $U_n^{k+1} \parallel U_{nm}$;*

(ii) *if a is even and b is odd, then* $U_n^{k+1} \parallel U_{nm}$;

(iii) *if a and b are odd and* $n \not\equiv 3 \pmod 6$, *then* $U_n^{k+1} \parallel U_{nm}$;

(iv) *if a and b are odd,* $n \equiv 3 \pmod 6$, *and* $\frac{U_n^{k+1}}{2} \nmid m$, *then* $U_n^{k+1} \parallel U_{nm}$;

(v) *if a and b are odd,* $n \equiv 3 \pmod 6$, $\frac{U_n^{k+1}}{2} \mid m$, *and* $2 \parallel a^2 + 3b$, *then* $U_n^{k+1} \parallel U_{nm}$;

(vi) *if a and b are odd,* $n \equiv 3 \pmod 6$, $\frac{U_n^{k+1}}{2} \mid m$, *and* $4 \mid a^2 + 3b$, *then* $U_n^{k+t+1} \parallel U_{nm}$, *where*

$$t = \min(\{v_2(U_6) - 2\} \cup \{y_p - k \mid p \text{ is an odd prime factor of } U_n\})$$

and $y_p = \left\lfloor \frac{v_p(m)}{v_p(U_n)} \right\rfloor$ *for each odd prime p dividing* U_n.

Proof. By Theorem 3.1, we obtain $U_n^{k+1} \mid U_{nm}$. So for (i) to (v), it is enough to show that $U_n^{k+2} \nmid U_{nm}$. We divide the calculation into several cases.

Case 1 a is odd and b is even. By Lemma 2.9, we obtain U_n is odd. From the assumption $U_n^k \parallel m$, we have $U_n^{k+1} \nmid m$, and so there exists a prime p dividing U_n such that $v_p(U_n^{k+1}) > v_p(m)$. Since U_n is odd, p is also odd. In addition, $p \nmid b$ by Lemma 2.7. So we can apply Lemma 2.2(i) to obtain

$$v_p(U_{nm}) = v_p(m) + v_p(U_n) < v_p(U_n^{k+1}) + v_p(U_n) = v_p(U_n^{k+2}),$$

which implies $U_n^{k+2} \nmid U_{nm}$, as required. This proves (i).

Case 2 a is even and b is odd. Similar to Case 1, we have U_1 is odd, U_2 is even, $U_r \equiv U_{r-2} \pmod 2$ for $r \geq 3$, and so U_n is even if and only if n is even. In addition, there exists a prime p such that $p \mid U_n$, $v_p(U_n^{k+1}) > v_p(m)$, and $p \nmid b$. So if $2 \nmid n$, then U_n is odd, p is odd, and we obtain by Lemma 2.2(i) that

$$v_p(U_{nm}) = v_p(m) + v_p(U_n) < v_p(U_n^{k+1}) + v_p(U_n) = v_p(U_n^{k+2}), \tag{3.1}$$

which implies $U_n^{k+2} \nmid U_{nm}$. If $2 \mid n$, then we can still use either Lemma 2.2(i) or Lemma 2.2(ii) to obtain (3.1), which leads to the same conclusion $U_n^{k+2} \nmid U_{nm}$. This proves (ii).

Case 3 a and b are odd. Similar to Case 1, there is a prime p such that $p \mid U_n$, $v_p(U_n^{k+1}) > v_p(m)$, and $p \nmid b$.

Case 3.1 $n \not\equiv 3 \pmod 6$. If $n \equiv 1, 2, 4, 5 \pmod 6$, then we obtain by Lemmas 2.11 and 2.2, respectively that p is odd and

$$v_p(U_{nm}) = v_p(U_n) + v_p(m) < v_p(U_n) + v_p(U_n^{k+1}) = v_p(U_n^{k+2}). \tag{3.2}$$

If $n \equiv 0 \pmod 6$, then n is even and Lemma 2.2(i) or Lemma 2.2(ii) can still be used to obtain (3.2). In any case, $U_n^{k+2} \nmid U_{nm}$. This proves (iii).

Case 3.2 $n \equiv 3 \pmod 6$ and $\frac{U_n^{k+1}}{2} \nmid m$. Since $U_n^k \parallel m$, we can write $m = cU_n^k$ where $c \geq 1$ and $U_n \nmid c$. By Lemma 2.4, $U_n^{k+2} \mid \binom{m}{j} U_n^j$ for $3 \leq j \leq m$. Then we obtain by Lemma 2.5 that

$$U_{nm} = U_{mn} \equiv mU_n(bU_{n-1})^{m-1} + \frac{m(m-1)}{2} U_n^2 (bU_{n-1})^{m-2} a \pmod{U_n^{k+2}}.$$

By Lemma 2.11, we know that $v_2(U_n) = v_2(U_3) \geq 1$. Since $\frac{U_n^{k+1}}{2} \nmid m$ and $m = cU_n^k$, we see that $\frac{U_n}{2}$ does not c. Let $d = bU_{n-1} + \frac{U_n}{2}(m-1)a$. By Lemmas 2.6 and 2.7, we obtain $\left(\frac{U_n}{2}, d\right) = \left(\frac{U_n}{2}, bU_{n-1}\right) = 1$. Then

$$U_{nm} \equiv mU_n b^{m-2} U_{n-1}^{m-2} \left(bU_{n-1} + \frac{U_n}{2}(m-1)a \right) \equiv cU_n^{k+1} b^{m-2} U_{n-1}^{m-2} d \pmod{U_n^{k+2}}.$$

By Lemmas 2.6 and 2.7, we obtain $U_n^{k+2} \mid U_{nm}$ if and only if $U_n \mid cd$. But if $U_n \mid cd$, then $\frac{U_n}{2} \mid cd$ which implies $\frac{U_n}{2} \mid c$, a contradiction. So $U_n \nmid cd$ and therefore $U_n^{k+2} \nmid U_{nm}$. This proves (iv). To prove (v) and (vi), we first assume that a and b are odd, $n \equiv 3 \pmod 6$, and $\frac{U_n^{k+1}}{2} \mid m$. (The other condition will be assumed later). Then $v_p(U_n^{k+1}) \leq v_p(m)$ for all odd primes p and $v_2(U_n^{k+1}) - 1 \leq v_2(m)$. If $v_2(U_n^{k+1}) - 1 < v_2(m)$, then $v_2(U_n^{k+1}) \leq v_2(m)$, and so $v_p(U_n^{k+1}) \leq v_p(m)$ for all primes p, which implies $U_n^{k+1} \mid m$ contradicting the assumption $U_n^k \parallel m$. Hence

$$v_2(U_n^{k+1}) - 1 = v_2(m) \text{ and } v_p(U_n^{k+1}) \leq v_p(m) \text{ for every odd prime } p \tag{3.3}$$

We now separate the consideration into two cases according to the additional conditions in (v) and (vi). Observe that $v_2(a^2+3b) = 1$ is equivalent to $2 \parallel a^2+3b$.

Case 4 $v_2(a^2+3b) = 1$. Since $U_6 = a(a^2+3b)U_3$, we obtain $v_2(U_6) = v_2(U_3) + 1$. Recall that $n \equiv 3 \pmod 6$ and $U_n^k \mid m$. So n is odd, m is even, and

$nm \equiv 0 \pmod 6$. If $U_n^{k+2} \mid U_{nm}$, then we obtain by Lemma 2.11 and (3.3) that

$$\begin{aligned} v_2(U_n^{k+1}) + v_2(U_n) = v_2(U_n^{k+2}) \le v_2(U_{nm}) &= v_2(n) + v_2(m) + v_2(U_6) - 1 \\ &= v_2(U_n^{k+1}) - 1 + v_2(U_3) \\ &= v_2(U_n^{k+1}) + v_2(U_n) - 1, \end{aligned}$$

which is a contradiction. Therefore $U_n^{k+2} \nmid U_{nm}$. This proves (v).

Case 5 $v_2(a^2+3b) \ge 2$. Then $v_2(U_6) = v_2(U_3) + v_2(a^2+3b) \ge v_2(U_3) + 2$. By Lemma 2.8, $v_2(U_3) = 1$ and so $v_2(U_6) = x + 2$ where $x = v_2(a^2+3b) - 1 \in \mathbb{N}$. For each odd prime p dividing U_n, let $y_p = \left\lfloor \frac{v_p(m)}{v_p(U_n)} \right\rfloor$ be the largest integer which is less than or equal to $\frac{v_p(m)}{v_p(U_n)}$. Since $U_n^k \mid m$, we have $y_p \ge k$ for all odd $p \mid U_n$. Let

$$t = \min(\{x\} \cup \{y_p - k \mid p \text{ is an odd prime factor of } U_n\}).$$

Then $t \ge 0$. By Lemma 2.11 and (3.3), $v_2(m) = (k+1)v_2(U_3) - 1 = k$ and

$$v_2(U_{nm}) = v_2(m) + v_2(U_6) - 1 = k + x + 1 \ge k + t + 1 = v_2(U_n^{k+t+1}). \qquad (3.4)$$

By the definition of y_p, we have $v_p(m) \ge y_p v_p(U_n)$. So by Lemma 2.2, if p is an odd prime dividing U_n, then

$$v_p(U_{nm}) = v_p(m) + v_p(U_n) \ge (y_p+1)v_p(U_n) \ge (k+t+1)v_p(U_n) = v_p(U_n^{k+t+1}). \quad (3.5)$$

By (3.4) and (3.5), $v_p(U_{nm}) \ge v_p(U_n^{k+t+1})$ for all primes p dividing U_n. This show that $U_n^{k+t+1} \mid U_{nm}$. It remains to show that $U_n^{k+t+2} \nmid U_{nm}$. If $t = y_p - k$ for some odd prime p dividing U_n, then we recall the definition of y_p and apply Lemma 2.2 to obtain

$$v_p(U_{nm}) = v_p(m) + v_p(U_n) < (y_p+2)v_p(U_n) = (k+t+2)v_p(U_n) = v_p(U_n^{k+t+2}).$$

If $t = x = v_2(U_6) - 2$, then we use Lemma 2.11 to get

$$v_2(U_{nm}) = v_2(m) + v_2(U_6) - 1 = k + t + 1 < v_2(U_n^{k+t+2}).$$

In any case, $U_n^{k+t+2} \nmid U_{nm}$. This completes the proof. □

Theorem 3.2 is the extension of Theorem 1.1 to the case of U_n. The next example shows that the integer t in Theorem 3.2(vi) can be any odd positive integer.

Example 3.3. Let $M \in \mathbb{N}$ be given. We show that there are positive integers k, m, n, a, b satisfying the conditions in Theorem 3.2(vi) with $t = M$. Choose $a = 1$ and $b = \left(2^{4M} - 1\right)/3$. Then a and b are odd integers, $(a, b) = 1$, and $v_2(a^2 + 3b) = 4M > 2$. Next choose any $k, n \in \mathbb{N}$ such that $n \equiv 3 \pmod 6$. Since $v_2(U_6) = v_2(U_3) + v_2(a^2 + 3b) \geq v_2(U_3) + 2$, we obtain by Lemmas 2.11 and 2.8 that $v_2(U_n) = v_2(U_3) = 1$ and $v_2(U_6) = 4M + 1$. Since $U_n \geq U_3 = a^2 + b > 2$ and $v_2(U_n) = 1$, we can write $U_n = 2p_1^{a_1}p_2^{a_2} \cdots p_s^{a_s}$ where $s \geq 1$, $p_1, p_2, \ldots, p_s$ are distinct odd primes, and $a_1, a_2, \ldots, a_s$ are positive integers. Next, choose $m = 2^k p_1^{a_1(k+M)} p_2^{a_2(k+M)} \cdots p_s^{a_s(k+M)}$. Then $U_n^k \parallel m$ and $\frac{U_n^{k+1}}{2} \mid m$. Therefore k, m, n, a, b satisfy all the conditions in Theorem 3.2(vi). Finally, we have

$$v_2(U_6) - 2 = v_2(a^2 + 3b) - 1 = 4M - 1$$

and $y_p - k = M$ for all $p \in \{p_1, p_2, \ldots, p_s\}$, and therefore $t = \min\{4M - 1, M\} = M$, as desired.

Next, we prove the converse of Theorem 3.2.

Theorem 3.4. [10, Theorem 12] *Let $k, m, n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $(a, b) = 1$, $n \geq 2$, and $U_n^{k+1} \parallel U_{nm}$. Then*

(i) *if a is odd and b is even, then $U_n^k \parallel m$;*

(ii) *if a is even and b is odd, then $U_n^k \parallel m$;*

(iii) *if a and b are odd and $n \not\equiv 3 \pmod 6$, then $U_n^k \parallel m$;*

(iv) *if a and b are odd, $n \equiv 3 \pmod 6$, and $2 \parallel a^2 + 3b$, then $U_n^k \parallel m$;*

(v) *if a and b are odd, $n \equiv 3 \pmod 6$, $4 \mid a^2 + 3b$, and $v_2(m) \geq k$, then $U_n^k \parallel m$;*

(vi) *if a and b are odd, $n \equiv 3 \pmod 6$, $4 \mid a^2 + 3b$, and $v_2(m) < k$, then*

$$m \text{ is even, } v_2(m) \geq k + 1 - v_2(a^2 + 3b), \text{ and } U_n^{v_2(m)} \parallel m.$$

Proof. Some parts of the proof are similar to those of Theorem 3.2, so we skip some details.

Case 1 a is odd and b is even. Similar to Case 1 of Theorem 3.2, we have U_n is odd. For any prime $p \mid U_n$, we obtain by Lemma 2.2 that

$$v_p(U_n^k) + v_p(U_n) = v_p(U_n^{k+1}) \leq v_p(U_{nm}) = v_p(U_n) + v_p(m), \tag{3.6}$$

which implies $U_n^k \mid m$. If $U_n^{k+1} \mid m$, then by Theorem 3.1, we have $U_n^{k+2} \mid U_{nm}$ which contradicts $U_n^{k+1} \parallel U_{nm}$. Therefore $U_n^{k+1} \nmid m$, and thus $U_n^k \parallel m$.

Case 2 a is even and b is odd. Then U_n is even if and only if n is even. So if $2 \nmid n$, then for any prime $p \mid U_n$, we have p is odd, (3.6) holds, and so $U_n^k \mid m$. If $2 \mid n$, then we can still apply Lemma 2.2(i) or Lemma 2.2(ii) to obtain (3.6) and conclude that $U_n^k \mid m$. If $U_n^{k+1} \mid m$, then by Theorem 3.1, we have $U_n^{k+2} \mid U_{nm}$ which contradicts $U_n^{k+1} \parallel U_{nm}$. So $U_n^{k+1} \nmid m$ and therefore $U_n^k \parallel m$.

We now assume throughout that a and b are odd and divide the consideration into four cases according to the additional conditions in (iii) to (vi).

Case 3 $n \not\equiv 3 \pmod 6$. If $n \equiv 1, 2, 4, 5 \pmod 6$, then we apply Lemma 2.11 to obtain $v_2(U_n^k) = 0 \leq v_2(m)$, and use Lemma 2.2 to show that for any odd prime $p \mid U_n$,

$$v_p(U_n) + v_p(U_n^k) = v_p(U_n^{k+1}) \leq v_p(U_{nm}) = v_p(m) + v_p(U_n). \tag{3.7}$$

If $n \equiv 0 \pmod 6$, then n is even and we can apply Lemma 2.2(i) or Lemma 2.2(ii) to obtain (3.7) for any prime $p \mid U_n$. In any case, we have $U_n^k \mid m$. Again, by Theorem 3.1, we have $U_n^{k+1} \nmid m$, and so $U_n^k \parallel m$. This proves (iii).

Case 4 $n \equiv 3 \pmod 6$ and $2 \parallel a^2 + 3b$. Similar to Case 4 in the proof of Theorem 3.2 we have $v_2(U_6) = v_2(U_3) + 1$. If m is odd, then $nm \equiv 3 \pmod 6$ and we obtain by Lemma 2.11 that $v_2(U_{nm}) = v_2(U_3) < (k+1)v_2(U_3) = v_2(U_n^{k+1})$, which contradicts the assumption $U_n^{k+1} \mid U_{nm}$. So m is even, and thus $nm \equiv 0 \pmod 6$. By Lemma 2.11 and the fact that $n \equiv 3 \pmod 6$ is odd, we obtain $v_2(m) + v_2(U_6) - 1 = v_2(U_{nm}) \geq v_2(U_n^{k+1}) = v_2(U_n^k) + v_2(U_n) = v_2(U_n^k) + v_2(U_3) = v_2(U_n^k) + v_2(U_6) - 1$, which implies $v_2(m) \geq v_2(U_n^k)$. If p is odd and $p \mid U_n$, then we apply Lemma 2.2 to obtain (3.7) Therefore $v_p(U_n^k) \leq v_p(m)$ for every prime p dividing U_n. Thus $U_n^k \mid m$. By Theorem 3.1, $U_n^{k+1} \nmid m$. Hence $U_n^k \parallel m$. This proves (iv).

Case 5 $n \equiv 3 \pmod 6$, $4 \mid a^2+3b$, and $v_2(m) \geq k$. Then $U_3 = a^2+b = (a^2+3b)-2b \equiv 2 \pmod 4$, and so $v_2(U_3) = 1$. By Lemma 2.11, we obtain $v_2(m) \geq kv_2(U_3) = kv_2(U_n) = v_2(U_n^k)$. By Lemma 2.2, if p is an odd prime dividing U_n, then (3.6) holds, and so we conclude that $v_p(U_n^k) \leq v_p(m)$ for every prime p dividing U_n. Therefore $U_n^k \mid m$. By Theorem 3.1, $U_n^{k+1} \nmid m$ and so $U_n^k \parallel m$. This proves (v).

Case 6 $n \equiv 3 \pmod 6$, $4 \mid a^2+3b$, and $v_2(m) < k$. For convenience, let $t = v_2(m)$. Similar to Case 4, we have m is even. In addition, $v_2(U_6) = v_2(U_3) + v_2(a^2+3b) = 1+v_2(a^2+3b)$. So $k > t \geq 1$ and $v_2(m) = tv_2(U_3) = tv_2(U_n) = v_2(U_n^t)$. By Lemma 2.2, if p is odd and $p \mid U_n$, then

$$v_p(U_n) + v_p(U_n^t) \leq v_p(U_n) + v_p(U_n^k) = v_p(U_n^{k+1}) \leq v_p(U_{nm}) = v_p(m) + v_p(U_n).$$

From the above inequalities, we obtain that $v_p(U_n^t) \leq v_p(m)$ for every prime p dividing U_n. Therefore $U_n^t \mid m$. If $U_n^{t+1} \mid m$, then we obtain by Lemma 2.11 that $t = v_2(m) \geq v_2(U_n^{t+1}) = t+1$, which is false. So $U_n^{t+1} \nmid m$. Therefore $U_n^t \parallel m$. From $U_n^{k+1} \parallel U_{nm}$, we also obtain $k+1 = v_2(U_n^{k+1}) \leq v_2(U_{nm}) = v_2(m) + v_2(U_6) - 1 = v_2(m) + v_2(a^2+3b)$, which implies $v_2(m) \geq k+1-v_2(a^2+3b)$. This completes the proof. □

Theorem 3.4 is the extension of Theorem 1.4 to the case of U_n. The next example shows that $v_2(m)$ in Theorem 3.4(vi) can be any positive integer in $[1, k)$.

Example 3.5. Let $k \geq 1$ and $1 \leq M < k$ be integers. We show that there are m, n, a, b satisfying the conditions in Theorem 3.4(vi) with $v_2(m) = M$. Choose $n \in \mathbb{N}$ and $n \equiv 3 \pmod 6$.

Case 1 $k - M$ is odd. Choose $a = 1$, $b = \frac{2^{k-M+1}-1}{3}$, and $m = \frac{U_n^k}{2^{k-M}}$. Then a and b are odd integers, $(a, b) = 1$, and $v_2(a^2+3b) = k-M+1 \geq 2$. Since $v_2(U_6) = v_2(U_3) + v_2(a^2+3b) \geq v_2(U_3)+2$, we obtain by Lemma 2.11 and 2.8 that $v_2(U_n) = v_2(U_3) = 1$ and $v_2(U_6) = k-M+2$. By Lemma 2.2, for $p > 2$ and $p \mid U_n$ we obtain

$$v_p(U_{nm}) = v_p(m) + v_p(U_n) = v_p(U_n^k) + v_p(U_n) = v_p(U_n^{k+1}).$$

By Lemma 2.11, we have

$$v_2(m) = v_2(U_n^k) - v_2(2^{k-M}) = k - k + M = M$$

and

$$v_2(U_{nm}) = v_2(m) + v_2(U_6) - 1 = M + k - M + 2 - 1 = v_2(U_n^{k+1}).$$

From these, we obtain $U_n^{k+1} \parallel U_{nm}$ and $U_n^M \parallel m$. Therefore k, m, n, a, b satisfy all the conditions in Theorem 3.4(vi).

Case 2 $k - M$ is even. Choose $a = 1$, $b = \frac{5 \cdot 2^{k-M+1} - 1}{3}$, and $m = \frac{U_n^k}{2^{k-M}}$. The verification is the same as that in Case 1. So we leave the details to the reader.

3.2 Exact divisibility by powers of the integers in the Lucas sequence of the second kind

In this section, we extend Theorems 1.2 and 1.5 to the case of V_n and give an example.

Theorem 3.6. [11, Theorem 10] *Assume that $k, m, n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $(a, b) = 1$, and m is odd. Then*

(i) if $V_n^k \mid m$, then $V_n^{k+1} \mid V_{nm}$;

(ii) if $V_n^k \parallel m$, then $V_n^{k+1} \parallel V_{nm}$;

(iii) if $V_n^k \mid V_{nm}$, then $V_n^{k-1} \mid m$;

(iv) if $V_n^k \parallel V_{nm}$, then $V_n^{k-1} \parallel m$.

Proof. We use Lemma 2.7 without reference. For (i), assume that $V_n^k \mid m$. Since m is odd, V_n is also odd, and so $v_2(V_n^{k+1}) = 0$. If $p > 2$ and $p \mid V_n$, then $p \nmid b$ and we obtain by Lemma 2.3 that

$$\begin{aligned} v_p(V_{nm}) &= v_p(mn) + v_p(U_{\tau(p)}) \\ &= v_p(m) + v_p(n) + v_p(U_{\tau(p)}) \\ &\geq v_p(V_n^k) + v_p(V_n) = v_p(V_n^{k+1}). \end{aligned}$$

Therefore $v_p(V_{nm}) \geq v_p(V_n^{k+1})$ for all primes p dividing V_n. This implies $V_n^{k+1} \mid V_{nm}$.

For (ii), assume that $V_n^k \parallel m$. By (i), it is enough to show that $V_n^{k+2} \nmid V_{nm}$. Since $V_n^{k+1} \nmid m$, there exists a prime p dividing V_n such that $v_p(V_n^{k+1}) > v_p(m)$. Here we remark that the letter p in the proof of (i) and in the proof of (ii) may be different or may be the same. We believe that there is no ambiguity since (i) is already done. Now since $V_n^k \mid m$ and m is odd, V_n is also odd, and so $v_2(V_n^{k+1}) = v_2(m) = 0$. Therefore p is odd. By Lemma 2.3, we obtain

$$\begin{aligned} v_p(V_{nm}) = v_p(nm) + v_p(U_{\tau(p)}) &= v_p(m) + v_p(n) + v_p(U_{\tau(p)}) \\ &= v_p(m) + v_p(V_n) < v_p(V_n^{k+1}) + v_p(V_n) = v_p(V_n^{k+2}). \end{aligned}$$

This shows that $V_n^{k+2} \nmid V_{nm}$, as required.

For (iii), assume that $V_n^k \mid V_{nm}$. We show that $v_p(V_n^{k-1}) \leq v_p(m)$ for all primes p dividing V_n. If p is odd and $p \mid V_n$, then we apply Lemma 2.3 to obtain that

$$\begin{aligned} v_p(V_n) + v_p(V_n^{k-1}) = v_p(V_n^k) \leq v_p(V_{nm}) &= v_p(nm) + v_p(U_{\tau(p)}) \\ &= v_p(m) + v_p(n) + v_p(U_{\tau(p)}) \\ &= v_p(m) + v_p(V_n), \end{aligned}$$

and so $v_p(V_n^{k-1}) \leq v_p(m)$. It remains to show that $v_2(V_n^{k-1}) \leq v_2(m)$. If a is odd and b is even, then it follows from Lemma 2.9 that $v_2(V_n^{k-1}) = 0 \leq v_2(m)$. Recall that $(a, b) = 1$, so a and b cannot be both even. So we have the following two remaining cases: (a is even and b is odd) or (a and b are odd).

Case 1 a is even and b is odd. We will show that k must be 1, and so $v_2(V_n^{k-1}) = 0 \leq v_2(m)$. If $2 \mid n$, then we apply Lemma 2.10 and the assumption that $V_n^k \mid V_{nm}$ to obtain

$$1 \leq k = v_2(V_n^k) \leq v_2(V_{nm}) = 1.$$

Similarly, if $2 \nmid n$, then $2 \nmid nm$ and we can use Lemma 2.10 again to obtain

$$kv_2(a) = v_2(V_n^k) \leq v_2(V_{nm}) = v_2(a).$$

In any case, $k = 1$, as asserted.

Case 2 a and b are odd. We use Lemma 2.11 in this case. If $n \not\equiv 0 \pmod 3$, then $v_2(V_n^{k-1}) = 0 \le v_2(m)$. If $n \equiv 0 \pmod 6$, then $nm \equiv 0 \pmod 6$, and so $k = v_2(V_n^k) \le v_2(V_{nm}) = 1$; thus $v_2(V_n^{k-1}) = 0 \le v_2(m)$. We now suppose $n \equiv 3 \pmod 6$. Since m is odd, $nm \equiv 3 \pmod 6$. Therefore

$$k(v_2(U_6) - v_2(U_3)) = v_2(V_n^k) \le v_2(V_{nm}) = v_2(U_6) - v_2(U_3).$$

So $k = 1$ and thus $v_2(V_n^{k-1}) = 0 \le v_2(m)$. Hence $v_p(V_n^{k-1}) \le v_p(m)$ for all primes p dividing V_n, as desired. This proves (iii).
For (iv), assume that $V_n^k \parallel V_{nm}$. By (iii), we have $V_n^{k-1} \mid m$. If $V_n^k \mid m$, then we obtain by (i) that $V_n^{k+1} \mid V_{nm}$ which contradicts $V_n^k \parallel V_{nm}$. Therefore $V_n^{k-1} \parallel m$. This completes the proof. □

Theorem 3.6 is the extension of Theorems 1.2 and 1.5 to the case of V_n. In the next example, we show that a version of Theorem 3.6 where m is even does not exist.

Example 3.7. Let $k, m, n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $(a, b) = 1$, and m is even. Let p be an odd prime dividing V_n. By Lemma 2.3, we have $p \nmid D$, $\tau(p) \nmid n$ and $\tau(p) \mid 2n$. Since m is even and $\tau(p) \mid 2n$, we obtain $\tau(p) \mid mn$. By Lemma 2.3, we have $p \nmid V_{nm}$, and so $V_n \nmid V_{nm}$. This shows that m in Theorem 3.6 cannot be even.

3.3 Exact divisibility by powers of the integers in the Lucas sequences of the first and second kinds

In this section, we extend Theorems 1.3 and 1.6 to the case of U_n and V_n and obtain some relevant results.

Theorem 3.8. [11, Theorem 13] *Suppose that $k, m, n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $(a, b) = 1$, a is odd, b is even, and m is even. Then*

(i) if $V_n^k \mid m$, then $V_n^{k+1} \mid U_{nm}$;

(ii) if $V_n^k \parallel m$, then $V_n^{k+1} \parallel U_{nm}$;

(iii) if $V_n^{k+1} \mid U_{nm}$, then $V_n^k \mid m$;

(iv) if $V_n^{k+1} \parallel U_{nm}$, then $V_n^k \parallel m$.

Proof. For (i), assume that $V_n^k \mid m$. We show that $v_p(V_n^{k+1}) \le v_p(U_{nm})$ for all primes p dividing V_n. By Lemma 2.9, we have $v_2(V_n) = 0$. So let p be an odd prime dividing V_n. By Lemma 2.3, $p \nmid D$, $\tau(p) \nmid n$, and $\tau(p) \mid 2n$. Then $\tau(p) \mid nm$. By Lemmas 2.1 and 2.3, we obtain

$$\begin{aligned} v_p(U_{nm}) &= v_p(nm) + v_p(U_{\tau(p)}) \\ &= v_p(m) + v_p(n) + v_p(U_{\tau(p)}) \\ &\ge v_p(V_n^k) + v_p(n) + v_p(U_{\tau(p)}) \\ &= v_p(V_n^k) + v_p(V_n) \\ &= v_p(V_n^{k+1}), \text{ as required.} \end{aligned}$$

For (ii), assume that $V_n^k \parallel m$. By (i), it is enough to show that $V_n^{k+2} \nmid U_{nm}$. Since $V_n^{k+1} \nmid m$, there exists a prime p such that $v_p(V_n^{k+1}) > v_p(m)$. By Lemma 2.9, $v_2(V_n^{k+1}) = 0$, and so $p \neq 2$. Since $p \mid V_n$, we know that $p \nmid D$ and $\tau(p) \mid nm$. Therefore we obtain by Lemmas 2.1 and 2.3 that

$$\begin{aligned} v_p(U_{nm}) &= v_p(nm) + v_p(U_{\tau(p)}) \\ &= v_p(m) + v_p(n) + v_p(U_{\tau(p)}) \\ &= v_p(m) + v_p(V_n) \\ &< v_p(V_n^{k+1}) + v_p(V_n) \\ &= v_p(V_n^{k+2}), \text{ as desired.} \end{aligned}$$

For (iii), assume that $V_n^{k+1} \mid U_{nm}$. By Lemma 2.9, $v_2(m) \ge 0 = v_2(V_n^k)$. If p is odd and $p \mid V_n$, then we apply Lemmas 2.1 and 2.3 again to obtain

$$\begin{aligned} v_p(V_n) + v_p(V_n^k) = v_p(V_n^{k+1}) \le v_p(U_{nm}) &= v_p(nm) + v_p(U_{\tau(p)}) \\ &= v_p(m) + v_p(n) + v_p(U_{\tau(p)}) \\ &= v_p(m) + v_p(V_n). \end{aligned}$$

This shows that $v_p(V_n^k) \le v_p(m)$ for every prime p dividing V_n. So $V_n^k \mid m$.
For (iv), suppose $V_n^{k+1} \parallel U_{nm}$. By (iii), it is enough to show that $V_n^{k+1} \nmid m$. If

$V_n^{k+1} \mid m$, we apply (i) to obtain $V_n^{k+2} \mid U_{nm}$ contradicting $V_n^{k+1} \parallel U_{nm}$. Therefore the proof is complete. $\square$

Theorem 3.9. [11, Theorem 14] *Assume that $k, m, n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $(a, b) = 1$, a is even, b is odd and m is even. Let*

$$t = \min(\{v_2(n) + v_2(a) - 2\} \cup \{y_p - k \mid p \text{ is an odd prime factor of } V_n\}) \text{ and}$$

$$y_p = \left\lfloor \frac{v_p(m)}{v_p(V_n)} \right\rfloor \text{ for each odd prime } p \text{ dividing } V_n.$$

Then

(i) if $V_n^k \mid m$ and $2 \mid n$, then $V_n^{k+1} \mid U_{nm}$;
if $V_n^k \mid m$ and $2 \nmid n$, then $\frac{V_n^{k+1}}{2} \mid U_{nm}$;
if $V_n^k \mid m$, $2 \nmid n$, and $v_2(m) \geq v_2(V_n^k) + 1$, then $V_n^{k+1} \mid U_{nm}$;
if $V_n^k \mid m$, $2 \mid n$, and $\frac{V_n^{k+1}}{2} \mid m$, then $t \geq 0$, $v_2(m) \geq k$, and $V_n^{k+t+1} \mid U_{nm}$;

(ii) if $V_n^k \parallel m$, $2 \mid n$ and $\frac{V_n^{k+1}}{2} \nmid m$, then $V_n^{k+1} \parallel U_{nm}$;

(iii) if $V_n^k \parallel m$, $2 \mid n$ and $\frac{V_n^{k+1}}{2} \mid m$, then $V_n^{k+t+1} \parallel U_{nm}$;

(iv) if $V_n^k \parallel m$, $2 \nmid n$ and $v_2(m) = v_2(V_n^k)$, then $V_n^k \parallel U_{nm}$;

(v) if $V_n^k \parallel m$, $2 \nmid n$ and $v_2(m) \geq v_2(V_n^k) + 1$, then $V_n^{k+1} \parallel U_{nm}$;

Proof. For (i), assume that $V_n^k \mid m$. If p is an odd prime and $p \mid V_n$, then $p \nmid D$, $\tau(p) \mid nm$, and we can apply Lemmas 2.1 and 2.3, to obtain

$$\begin{aligned} v_p(U_{nm}) &= v_p(nm) + v_p(U_{\tau(p)}) \\ &= v_p(m) + v_p(n) + v_p(U_{\tau(p)}) \\ &\geq v_p(V_n^k) + v_p(V_n) = v_p(V^{k+1}). \end{aligned}$$

From this point on, we sometimes use Lemmas 2.1 and 2.3 without reference. Next, we consider $v_2(V_n^{k+1})$ and $v_2(U_{nm})$. If $2 \mid n$, then we apply Lemma 2.10 to obtain

$$\begin{aligned} v_2(U_{nm}) = v_2(nm) + v_2(a) - 1 &= v_2(m) + v_2(n) + v_2(a) - 1 \\ &\geq v_2(V_n^k) + v_2(n) + v_2(a) - 1 \\ &\geq v_2(V_n^k) + 1 = v_2(V_n^k) + v_2(V_n) = v_2(V_n^{k+1}). \end{aligned}$$

This implies the first part of (i). Since m is even, $2 \mid nm$. So if $2 \nmid n$, then we can still apply Lemma 2.10 to obtain

$$\begin{aligned} v_2(U_{nm}) &= v_2(nm) + v_2(a) - 1 \\ &= v_2(m) + v_2(a) - 1 \qquad (3.8) \\ &\geq v_2(V_n^k) + v_2(a) - 1 = v_2(V_n^k) + v_2(V_n) - 1 = v_2\left(\frac{V_n^{k+1}}{2}\right). \end{aligned}$$

This implies the second part of (i). For the third part of (i), we assume that $2 \nmid n$ and $v_2(m) \geq v_2(V_n^k) + 1$, and then we repeat the argument used in the second part to obtain

$$v_2(U_{nm}) = v_2(m) + v_2(a) - 1 \geq v_2(V_n^k) + v_2(a) = v_2(V_n^{k+1}).$$

Therefore $v_p(U_{nm}) \geq v_p(V_n^{k+1})$ for all primes p, which implies the desired result. Next, we prove the last part of (i). Assume that $V_n^k \mid m$, $2 \mid n$, and $\frac{V_n^{k+1}}{2} \mid m$. Since a and n are even, $v_2(n) + v_2(a) - 2 \geq 0$. In addition, $v_p(m) \geq v_p(V_n^k) = kv_p(V_n)$, and so $y_p \geq k$. Therefore $t \geq 0$ and $t + 1 \leq v_2(n) + v_2(a) - 1$. By Lemma 2.10, we have $v_p(V_n) = 1$, and therefore $v_p(m) \geq k$ and

$$\begin{aligned} v_2(U_{nm}) = v_2(nm) + v_2(a) - 1 &= v_2(m) + v_2(n) + v_2(a) - 1 \\ &\geq k + t + 1 = v_2(V_n^{k+t+1}). \end{aligned}$$

If p is an odd prime and $p \mid V_n$, then

$$\begin{aligned} v_p(U_{nm}) &= v_p(m) + v_p(n) + v_p(U_{\tau(p)}) \\ &= v_p(m) + v_p(V_n) \\ &\geq y_p v_p(V_n) + v_p(V_n) \\ &= (y_p + 1) v_p(V_n) \\ &\geq (k + t + 1) v_p(V_n) = v_p(V_n^{k+t+1}). \end{aligned}$$

Hence $v_p(U_{nm}) \geq v_p(V_n^{k+t+1})$ for all primes p dividing V_n. Thus $V_n^{k+t+1} \mid U_{nm}$, as desired.

Next, we prove (ii). Assume that $V_n^k \parallel m$, $2 \mid n$ and $\frac{V_n^{k+1}}{2} \nmid m$. By (i), it is enough to show that $V_n^{k+2} \nmid U_{nm}$. By Lemma 2.10, we know that $v_2(V_n) = 1$. Then

$v_2(m) \geq v_2(V_n^k) = v_2\left(\frac{V_n^{k+1}}{2}\right)$. Since $\frac{V_n^{k+1}}{2} \nmid m$, there exists an odd prime p dividing V_n such that $v_p(V_n^{k+1}) > v_p(m)$. Then $p \nmid D$, $\tau(p) \mid nm$, and

$$\begin{aligned} v_p(V_n^{k+2}) = v_p(V_n^{k+1}) + v_p(V_n) &> v_p(m) + v_p(V_n) \\ &= v_p(m) + v_p(n) + v_p(U_{\tau(p)}) \\ &= v_p(U_{nm}). \end{aligned}$$

This implies $V_n^{k+2} \nmid U_{nm}$.

For (iii), assume that $V_n^k \parallel m$, $2 \mid n$, and $\frac{V_n^{k+1}}{2} \mid m$. By (i), we obtain $t \geq 0$, $v_2(m) \geq k$, and $V_n^{k+t+1} \mid U_{nm}$. So it remains to show that $V_n^{k+t+2} \nmid U_{nm}$. We first observe that since $\frac{V_n^{k+1}}{2} \mid m$, we obtain $v_p(V_n^{k+1}) \leq v_p(m)$ for every odd prime p. If $v_2(m) \geq k+1$, then $v_2(m) \geq v_2(V_n^{k+1})$ which implies $V_n^{k+1} \mid m$ contradicting the assumption $V_n^k \parallel m$. Therefore $v_2(m) = k$. Next, we show that $V_n^{k+t+2} \nmid U_{nm}$. If $t = y_p - k$ for some odd prime p dividing V_n, then we apply Lemmas 2.1 and 2.3 to obtain

$$\begin{aligned} v_p(U_{nm}) &= v_p(nm) + v_p(U_{\tau(p)}) \\ &= v_p(m) + v_p(V_n) = \left(\frac{v_p(m)}{v_p(V_n)} + 1\right) v_p(V_n) \\ &< (y_p + 2) v_p(V_n) = (k+t+2) v_p(V_n) = v_p(V_n^{k+t+2}), \end{aligned}$$

and so $V_n^{k+t+2} \nmid U_{nm}$. If $t = v_2(n) + v_2(a) - 2$, then we obtain by Lemma 2.10 that

$$\begin{aligned} v_2(U_{nm}) &= v_2(nm) + v_2(a) - 1 \\ &= v_2(m) + v_2(n) + v_2(a) - 1 \\ &= k + t + 1 < v_2(V_n^{k+t+2}), \end{aligned}$$

and so $V_n^{k+t+2} \nmid U_{nm}$. This proves (iii).

Next, we prove (iv). Assume that $V_n^k \parallel m$, $2 \nmid n$ and $v_2(m) = v_2(V_n^k)$. By (i), we have $\frac{V_n^{k+1}}{2} \mid U_{nm}$. To show that $V_n^k \mid U_{nm}$, it suffices to prove that $v_2(V_n^k) \leq v_2(U_{nm})$. Recall from (3.8) in the proof of the second part of (i) that

$$v_2(U_{nm}) = v_2(m) + v_2(a) - 1 = v_2(V_n^k) + v_2(a) - 1 \geq v_2(V_n^k),$$

and

$$v_2(U_{nm}) = v_2(m) + v_2(a) - 1 = v_2(V_n^k) + v_2(V_n) - 1 < v_2(V_n^{k+1}).$$

So $V_n^k \mid U_{nm}$ and $V_n^{k+1} \nmid U_{nm}$. Thus $V_n^k \parallel U_{nm}$.

For (v), assume that $V_n^k \parallel m$, $2 \nmid n$, and $v_2(m) \geq v_2(V_n^k) + 1$. By (i), it suffices to show that $V_n^{k+2} \nmid U_{nm}$. Since $V_n^{k+1} \nmid m$, there exists a prime p dividing V_n such that $v_p(V_n^{k+1}) > v_p(m)$. If $p = 2$, then we obtain by Lemma 2.10 that

$$v_2(U_{nm}) = v_2(m) + v_2(a) - 1 < v_2(V_n^{k+1}) + v_2(V_n) - 1 < v_2(V_n^{k+2}),$$

and so $V_n^{k+2} \nmid U_{nm}$. If $p > 2$, then we obtain

$$v_p(U_{nm}) = v_p(nm) + v_p(U_{\tau(p)}) = v_p(m) + v_p(V_n) < v_p(V_n^{k+1}) + v_p(V_n) = v_p(V_n^{k+2}),$$

which implies $V_n^{k+2} \nmid U_{nm}$. This completes the proof. □

Next, we prove the converse of Theorem 3.9. From this point on, we apply Lemmas 2.1, 2.3, 2.7, and 2.10 without reference.

Theorem 3.10. [11, Theorem 15] *Suppose that $k, m, n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $(a, b) = 1$, a is even, b is odd, and m is even. Then*

(i) for all odd primes p, if $v_p(V_n^{k+1}) \leq v_p(U_{nm})$, then $v_p(V_n^k) \leq v_p(m)$;

(ii) if $V_n^{k+1} \mid U_{nm}$ and $2 \mid n$, then $V_n^{\min(k, v_2(m))} \mid m$;
if $V_n^{k+1} \parallel U_{nm}$ and $2 \mid n$, then $V_n^{\min(k, v_2(m))} \parallel m$;

(iii) if $V_n^{k+1} \mid U_{nm}$ and $2 \nmid n$, then $V_n^k \mid m$;

(iv) if $V_n^{k+1} \parallel U_{nm}$, $2 \nmid n$ and $\frac{V_n^{k+2}}{2} \nmid U_{nm}$, then $V_n^k \parallel m$;

(v) if $V_n^{k+1} \parallel U_{nm}$, $2 \nmid n$, and $\frac{V_n^{k+2}}{2} \mid U_{nm}$, then $V_n^{k+1} \parallel m$.

Proof. For (i), assume that p is an odd prime and $v_p(V_n^{k+1}) \leq v_p(U_{nm})$. If $p \mid V_n$, then

$$\begin{aligned} v_p(V_n) + v_p(V_n^k) = v_p(V_n^{k+1}) \leq v_p(U_{nm}) &= v_p(nm) + v_p(U_{\tau(p)}) \\ &= v_p(m) + v_p(n) + v_p(U_{\tau(p)}) \\ &= v_p(m) + v_p(V_n), \end{aligned}$$

which implies (i). By (i), we only need to consider the 2-adic valuation in the proofs of (ii), (iii), (iv), and (v).

For (ii), assume that $V_n^{k+1} \mid U_{nm}$ and $2 \mid n$. For convenience, let $c = \min(k, v_2(m))$. If $v_2(m) \geq k$, then $v_2(V_n^k) = k \leq v_2(m)$, and so $V_n^k \mid m$. If $v_2(m) < k$, then $v_2(V_n^{v_2(m)}) = v_2(m)$ and $v_p(V_n^{v_2(m)}) \leq v_p(V_n^k) \leq v_p(m)$ for all odd primes p, and therefore $V_n^{v_2(m)} \mid m$. In any case, we obtain $V_n^c \mid m$. This proves the first part of (ii). Suppose further that $V_n^{k+1} \parallel U_{nm}$ but $V_n^{c+1} \mid m$. Then

$$v_2(m) \geq v_2(V_n^{c+1}) = \min(k, v_2(m)) + 1,$$

which implies $c = k$. Then $V_n^{k+1} = V_n^{c+1} \mid m$. By (i) of Theorem 3.9, we obtain $V_n^{k+2} \mid U_{nm}$ contradicting $V_n^{k+1} \parallel U_{nm}$. This completes the proof of (ii).

For (iii), assume that $V_n^{k+1} \mid U_{nm}$ and $2 \nmid n$. Then

$$\begin{aligned} v_2(a) + v_2(V_n^k) = v_2(V_n^{k+1}) \leq v_2(U_{nm}) &= v_2(nm) + v_2(a) - 1 \\ &= v_2(m) + v_2(a) - 1. \end{aligned}$$

Therefore $v_2(V_n^k) < v_2(m)$, and so $V_n^k \mid m$.

For (iv), assume that $V_n^{k+1} \parallel U_{nm}$, $2 \nmid n$, and $\frac{V_n^{k+2}}{2} \nmid U_{nm}$. By (iii), $V_n^k \mid m$. If $V_n^{k+1} \mid m$, then we obtain from (i) of Theorem 3.9 that $\frac{V_n^{k+2}}{2} \mid U_{nm}$, a contradiction. So $V_n^k \parallel m$.

For (v), assume that $V_n^{k+1} \parallel U_{nm}$, $2 \nmid n$, and $\frac{V_n^{k+2}}{2} \mid U_{nm}$. If p is odd, then $v_p(V_n^{k+2}) \leq v_p(U_{nm})$, and so we obtain by (i) that $v_p(V_n^{k+1}) \leq v_p(m)$. In addition,

$$\begin{aligned} v_2(V_n^{k+1}) + v_2(a) - 1 = v_2(V_n^{k+2}) - 1 \leq v_2(U_{nm}) &= v_2(nm) + v_2(a) - 1 \\ &= v_2(m) + v_2(a) - 1, \end{aligned}$$

and so $v_2(V_n^{k+1}) \leq v_2(m)$. Therefore $V_n^{k+1} \mid m$. If $V_n^{k+2} \mid m$, we obtain from (i) of Theorem 3.9 that $\frac{V_n^{k+3}}{2} \mid U_{nm}$, which implies $V_n^{k+2} \mid U_{nm}$ contradicting $V_n^{k+1} \parallel U_{nm}$. Therefore $V_n^{k+1} \parallel m$ and the proof is complete. □

The following Theorem is an extension of Theorem 1.3.

Theorem 3.11. [11, Theorem 16] *Suppose that $k, m, n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $(a, b) = 1$, a and b are odd, and m is even. Let $c = v_2(U_6) - 1$,*

$$t = \min(\{v_2(n) + c - 1\} \cup \{y_p - k \mid p \text{ is an odd prime factor of } V_n\}),$$

$$s = \min(\{c-1\} \cup \{y_p - k \mid p \textit{ is an odd prime factor of } V_n\}), \textit{ and}$$

$$y_p = \left\lfloor \frac{v_p(m)}{v_p(V_n)} \right\rfloor \textit{ for each odd prime } p \textit{ dividing } V_n.$$

Then

(i) *if* $V_n^k \mid m$, *then* $V_n^{k+1} \mid U_{nm}$;

(ii) *if* $V_n^k \parallel m$ *and* $n \not\equiv 0 \pmod 3$, *then* $V_n^{k+1} \parallel U_{nm}$;

(iii) *if* $V_n^k \parallel m$, $n \equiv 0 \pmod 6$ *and* $\frac{V_n^{k+1}}{2} \nmid m$, *then* $V_n^{k+1} \parallel U_{nm}$;

(iv) *if* $V_n^k \mid m$, $n \equiv 0 \pmod 6$, *and* $\frac{V_n^{k+1}}{2} \mid m$, *then* $t \geq 0$ *and* $V_n^{k+t+1} \mid U_{nm}$;
if $V_n^k \parallel m, n \equiv 0 \pmod 6$ *and* $\frac{v_n^{k+1}}{2} \mid m$, *then* $V_n^{k+t+1} \parallel U_{nm}$;

(v) *if* $V_n^k \parallel m$, $n \equiv 3 \pmod 6$, $2 \parallel a^2 + 3b$ *and* $\frac{V_n^{k+1}}{2} \nmid m$, *then* $V_n^{k+1} \parallel U_{nm}$;

(vi) *if* $V_n^k \mid m$, $n \equiv 3 \pmod 6$, $2 \parallel a^2 + 3b$, *and* $\frac{V_n^{k+1}}{2} \mid m$, *then* $s \geq 0$ *and* $V_n^{k+s+1} \mid U_{nm}$;
if $V_n^k \parallel m$, $n \equiv 3 \pmod 6$, $2 \parallel a^2 + 3b$ *and* $\frac{V_n^{k+1}}{2} \mid m$, *then* $V_n^{k+s+1} \parallel U_{nm}$;

(vii) *if* $V_n^k \parallel m$, $n \equiv 3 \pmod 6$, $4 \mid a^2 + 3b$ *and* $\frac{V_n^{k+1}}{2^c} \nmid m$, *then* $V_n^{k+1} \parallel U_{nm}$;

(viii) *if* $V_n^k \mid m$, $n \equiv 3 \pmod 6$, $4 \mid a^2 + 3b$ *and* $\frac{V_n^{k+1}}{2^c} \mid m$, *then* $V_n^{k+2} \mid 2^c U_{nm}$;
if $V_n^k \parallel m$, $n \equiv 3 \pmod 6$, $4 \mid a^2 + 3b$ *and* $\frac{V_n^{k+1}}{2^c} \mid m$, *then* $V_n^{k+2} \parallel 2^c U_{nm}$.

Proof. As usual, to prove that $V_n^d \mid U_{nm}$, we show that $v_p(V_n^d) \leq v_p(U_{nm})$ for all primes p dividing V_n. Similarly, if we would like to prove that $V_n^d \nmid U_{nm}$, then we show that $v_p(V_n^d) > v_p(U_{nm})$ for some prime p. If p is odd, then we apply Lemmas 2.1 and 2.3; if $p = 2$, then we use Lemma 2.11; and we will do this without further reference. For (i), assume that $V_n^k \mid m$. If p is odd and $p \mid V_n$, then

$$\begin{aligned} v_p(U_{nm}) = v_p(nm) + v_p(U_{\tau(p)}) &= v_p(m) + v_p(n) + v_p(U_{\tau(p)}) \\ &\geq v_p(V_n^k) + v_p(V_n) = v_p(V^{k+1}). \end{aligned}$$

So it remains to show that $v_2(U_{nm}) \geq v_2(V_n^{k+1})$. If $n \not\equiv 0 \pmod 3$, then $v_2(V_n^{k+1}) = 0 \leq v_2(U_{nm})$. So suppose that $n \equiv 0 \pmod 3$. Then $nm \equiv 0 \pmod 6$ and so

$$v_2(U_{nm}) = v_2(nm) + v_2(U_6) - 1 \geq v_2(V_n^k) + v_2(n) + v_2(U_6) - 1. \tag{3.9}$$

Since $U_3 = a^2 + b$ is even and $U_6 = a(a^2 + 3b)U_3$, we know that $v_2(U_3) \geq 1$ and $v_2(U_6) \geq 1$. So if $n \equiv 0 \pmod 6$, then $v_2(n) \geq 1$ and (3.9) implies that

$$v_2(U_{nm}) \geq v_2(V_n^k) + v_2(U_6) \geq v_2(V_n^k) + v_2(V_n) = v_2(V_n^{k+1}).$$

If $n \equiv 3 \pmod 6$, then (3.9) implies

$$v_2(U_{nm}) \geq v_2(V_n^k) + v_2(U_6) - 1 \geq v_2(V_n^k) + v_2(U_6) - v_2(U_3) = v_2(V_n^{k+1}).$$

In any case, $v_2(U_{nm}) \geq v_2(V_n^{k+1})$. This proves (i).
For (ii), assume that $V_n^k \parallel m$ and $n \not\equiv 0 \pmod 3$. By (i), it is enough to show that $V_n^{k+2} \nmid U_{nm}$. Since $V_n^{k+1} \nmid m$, there exists a prime p dividing V_n such that $v_p(V_n^{k+1}) > v_p(m)$. Since $v_2(V_n^{k+1}) = 0$, we see that $p \neq 2$. Then

$$\begin{aligned} v_p(U_{nm}) = v_p(nm) + v_p(U_{\tau(p)}) &= v_p(m) + v_p(n) + v_p(U_{\tau(p)}) \\ &< v_p(V_n^{k+1}) + v_p(V_n) = v_p(V_n^{k+2}), \text{ as desired.} \end{aligned}$$

For (iii), assume that $V_n^k \parallel m$, $n \equiv 0 \pmod 6$, and $\frac{V_n^{k+1}}{2} \nmid m$. By (i), it is enough to show that $V_n^{k+2} \nmid U_{nm}$. Since $\frac{V_n^{k+1}}{2} \nmid m$ and $v_2(\frac{V_n^{k+1}}{2}) = v_2(V_n^k) \leq v_2(m)$, we see that there exists an odd prime p dividing V_n such that $v_p(V_n^{k+1}) > v_p(m)$. Then

$$v_p(U_{nm}) = v_p(nm) + v_p(U_{\tau(p)}) = v_p(m) + v_p(V_n) < v_p(V_n^{k+1}) + v_p(V_n) = v_p(V_n^{k+2}).$$

Therefore $V_n^{k+2} \nmid U_{nm}$, as required.
For (iv), we first assume that $V_n^k \mid m$, $n \equiv 0 \pmod 6$, and $\frac{V_n^{k+1}}{2} \mid m$. Since $v_2(n) \geq 1$ and $v_2(U_6) \geq v_2(U_3) \geq 1$, it is not difficult to see that $t \geq 0$. If p is an odd prime dividing V_n, then

$$\begin{aligned} v_p(U_{nm}) &= v_p(nm) + v_p(U_{\tau(p)}) = v_p(m) + v_p(V_n) \\ &\geq y_p v_p(V_n) + v_p(V_n) = (y_p + 1) v_p(V_n) \\ &\geq (k + t + 1) v_p(V_n) = v_p(V_n^{k+t+1}). \end{aligned}$$

In addition,

$$\begin{aligned} v_2(U_{nm}) = v_2(nm) + v_2(U_6) - 1 &= v_2(m) + v_2(n) + v_2(U_6) - 1 \\ &\geq v_2(V_n^k) + t + 1 = k + t + 1 = v_2(V_n^{k+t+1}). \end{aligned}$$

Therefore $V_n^{k+t+1} \mid U_{nm}$. This proves the first part of (iv). Next, assume further that $V_n^k \parallel m$. It is enough to show that $V_n^{k+t+2} \nmid U_{nm}$. Recall that $y_p = \left\lfloor \frac{v_p(m)}{v_p(V_n)} \right\rfloor$, so $v_p(m) < (y_p+1)v_p(V_n)$. So if $t = y_p - k$ for some odd prime p dividing V_n, then

$$\begin{aligned} v_p(U_{nm}) &= v_p(nm) + v_p(U_{\tau(p)}) = v_p(m) + v_p(V_n) \\ &< (y_p+2)v_p(V_n) = (k+t+2)v_p(V_n) = v_p(V_n^{k+t+2}), \end{aligned}$$

which implies $V_n^{k+t+2} \nmid U_{nm}$. So suppose $t = v_2(n) + v_2(U_6) - 2$. Since $\frac{V_n^{k+1}}{2} \mid m$, we see that $v_p(m) \geq v_p(V_n^{k+1})$ for all odd primes p. If $v_2(m) \geq k+1$, then $v_2(m) \geq v_2(V_n^{k+1})$,which implies $V_n^{k+1} \mid m$ contradicting the assumption $V_n^k \parallel m$. Therefore $v_2(m) \leq k$. Then

$$v_2(U_{nm}) = v_2(nm)+v_2(U_6)-1 = v_2(m)+v_2(n)+v_2(U_6)-1 \leq k+t+1 < v_2(V_n^{k+t+2}).$$

Therefore, $V_n^{k+t+2} \nmid U_{nm}$ as required.

For (v), assume that $V_n^k \parallel m$, $n \equiv 3 \pmod 6$, $2 \parallel a^2+3b$, and $\frac{V_n^{k+1}}{2} \nmid m$. By (i), it suffices to show that $V_n^{k+2} \nmid U_{nm}$. Since $U_6 = a(a^2+3b)U_3$ and $2 \parallel a^2+3b$, we obtain $v_2(V_n) = v_2(U_6) - v_2(U_3) = 1$. Since $\frac{V_n^{k+1}}{2} \nmid m$ and $v_2\left(\frac{V_n^{k+1}}{2}\right) = v_2(V_n^k) \leq v_2(m)$, there exists an odd prime p dividing V_n such that $v_p(V_n^{k+1}) > v_p(m)$. Therefore

$$\begin{aligned} v_p(U_{nm}) = v_p(nm) + v_p(U_{\tau(p)}) = v_p(m) + v_p(V_n) &< v_p(V_n^{k+1}) + v_p(V_n) \\ &= v_p(V_n^{k+2}), \text{ as desired.} \end{aligned}$$

For (vi), assume that $V_n^k \mid m$, $n \equiv 3 \pmod 6$, $2 \parallel a^2+3b$, and $\frac{V_n^{k+1}}{2} \mid m$. Since a^2+3b and U_3 are even, and $U_6 = a(a^2+3b)U_3$, we have $v_2(U_6) - 2 \geq 0$. Since $V_n^k \mid m$, we have $y_p \geq k$ for all odd primes p dividing V_n. Therefore $s \geq 0$. By the same argument as in the proof of (v), we obtain $v_2(V_n) = 1$. In addition, $v_2(m) \geq v_2(V_n^k) = k$ and $v_p(V_n^{k+1}) = v_p(\frac{V_n^{k+1}}{2}) \leq v_p(m)$ for every odd prime p. If $V_n^k \parallel m$ and $v_2(m) \geq k+1 = v_2(V_n^{k+1})$, then $V_n^{k+1} \mid m$ which is a contradiction. Therefore,

$$\text{if } V_n^k \parallel m, \text{ then } v_2(m) = k. \tag{3.10}$$

We will apply (3.10) later. For now, we only need to apply $v_2(m) \geq k$. We obtain

$$\begin{aligned} v_2(U_{nm}) = v_2(nm) + v_2(U_6) - 1 = v_2(m) + v_2(U_6) - 1 &\geq k + v_2(U_6) - 1 \\ &\geq k+s+1 = v_2(V_n^{k+s+1}). \end{aligned}$$

If $p > 2$ and $p \mid V_n$, then

$$\begin{aligned} v_p(U_{nm}) = v_p(nm) + v_p(U_{\tau(p)}) &= v_p(m) + v_p(V_n) \geq (y_p + 1)v_p(V_n) \\ &\geq (k + s + 1)v_p(V_n) = v_p(V_n^{k+s+1}). \end{aligned}$$

This implies $V_n^{k+s+1} \mid U_{nm}$. Next, assume further that $V_n^k \parallel m$. It remains to show that $V_n^{k+s+2} \nmid U_{nm}$. By the definition of y_p, we know that $(y_p + 1)v_p(V_n) > v_p(m)$. So if $s = y_p - k$ for some odd prime p dividing V_n, then

$$\begin{aligned} v_p(U_{nm}) = v_p(nm) + v_p(U_{\tau(p)}) &= v_p(m) + v_p(V_n) < (y_p + 2)v_p(V_n) \\ &= (k + s + 2)v_p(V_n) = v_p(V_n^{k+s+2}), \end{aligned}$$

which implies $V_n^{k+s+2} \nmid U_{nm}$. By (3.10), we know that $v_2(m) = k$. So if $s = v_2(U_6) - 2$, then

$$v_2(U_{nm}) = v_2(nm) + v_2(U_6) - 1 = v_2(m) + v_2(U_6) - 1 = k + s + 1 < v_2(V_n^{k+s+2}).$$

So in any case, $V_n^{k+s+2} \nmid U_{nm}$, as required.

For (vii), we let $c = v_2(U_6) - 1$ and assume that $V_n^k \parallel m$, $n \equiv 3 \pmod 6$, $4 \mid a^2 + 3b$, and $\frac{V_n^{k+1}}{2^c} \nmid m$. By (i), it is enough to show that $V_n^{k+2} \nmid U_{nm}$. Since $4 \mid a^2 + 3b$ and $U_6 = a(a^2 + 3b)U_3$, we have $v_2(U_6) \geq v_2(U_3) + 2$. By Lemma 2.8, we obtain $v_2(U_3) = 1$, and so $v_2(V_n) = v_2(U_6) - v_2(U_3) = v_2(U_6) - 1 = c$. Since $\frac{V_n^{k+1}}{2^c} \nmid m$ and

$$v_2\left(\frac{V_n^{k+1}}{2^c}\right) = (k + 1)v_2(V_n) - v_2(V_n) = v_2(V_n^k) \leq v_2(m),$$

there exists an odd prime p dividing V_n such that $v_p(V_n^{k+1}) > v_p(m)$. Then

$$v_p(U_{nm}) = v_p(nm) + v_p(U_{\tau(p)}) = v_p(m) + v_p(V_n) < v_p(V_n^{k+1}) + v_p(V_n) = V_p(V_n^{k+2}).$$

Therefore $V_n^{k+2} \nmid U_{nm}$.

For (viii), assume that $V_n^k \mid m$, $n \equiv 3 \pmod 6$, $4 \mid a^2 + 3b$, and $\frac{V_n^{k+1}}{2^c} \mid m$. Then for each odd prime p dividing V_n, we have

$$v_p(V_n^{k+1}) = v_p\left(\frac{V_n^{k+1}}{2^c}\right) \leq v_p(m). \tag{3.11}$$

Since $4 \mid a^2 + 3b$ and $U_6 = a(a^2 + 3b)U_3$, we obtain $v_2(U_6) \geq v_2(U_3) + 2$. By the same argument as in the proof of (vii), we obtain $v_2(V_n) = v_2(U_6) - 1 = c$. Since $V_n^k \mid m$, we see that $v_2(m) \geq v_2(V_n^k) = kv_2(V_n)$. If $V_n^k \parallel m$ and $v_2(m) \geq (k+1)v_2(V_n)$, then $v_p(m) \geq v_p(V_n^{k+1})$ for all primes p, and so $V_n^{k+1} \mid m$, a contradiction. Therefore

$$v_2(m) \geq kv_2(V_n), \tag{3.12}$$

and

$$\text{if } V_n^k \parallel m, \text{ then } kv_2(V_n) \leq v_2(m) < (k+1)v_2(V_n). \tag{3.13}$$

We will apply (3.13) later. For now (3.12) is good enough. We obtain

$$\begin{aligned}
v_2(2^c U_{nm}) &= v_2(U_6) - 1 + v_2(U_{nm}) = v_2(U_6) - 1 + v_2(nm) + v_2(U_6) - 1 \\
&= 2(v_2(U_6) - 1) + v_2(m) \\
&\geq 2(v_2(U_6) - 1) + kv_2(V_n) \\
&= 2(v_2(U_6) - 1) + k(v_2(U_6) - 1) \\
&= (k+2)(v_2(U_6) - 1) = v_2(V_n^{k+2}).
\end{aligned}$$

If $p > 2$ and $p \mid V_n$, then

$$\begin{aligned}
v_p(2^c U_{nm}) = v_p(U_{nm}) = v_p(nm) + v_p(U_{\tau(p)}) &= v_p(m) + v_p(V_n) \\
&\geq v_p(V_n^{k+1}) + v_p(V_n) = v_p(V_n^{k+2}),
\end{aligned}$$

where the last inequality is obtained from (3.11). This implies that $V_n^{k+2} \mid 2^c U_{nm}$. So the first part of (viii) is proved. Next, assume further that $V_n^k \parallel m$. To prove the second part, it now suffices to show that $V_n^{k+3} \nmid 2^c U_{nm}$. We have

$$\begin{aligned}
v_2(2^c U_{nm}) &= v_2(U_6) - 1 + v_2(U_{nm}) \\
&= v_2(U_6) - 1 + v_2(nm) + v_2(U_6) - 1 \\
&= 2(v_2(U_6) - 1) + v_2(m) \\
&< 2(v_2(U_6) - 1) + (k+1)(v_2(U_6) - 1) \\
&= (k+3)(v_2(U_6) - 1) = v_2(V_n^{k+3}),
\end{aligned}$$

where the inequality is obtained form (3.13) and the fact that $v_2(V_n) = v_2(U_6) - 1$. This completes the proof. □

The next Theorem is the converse of Theorem 3.11 and also the extension of Theorem 1.6.

Theorem 3.12. [11, Theorem 17] *Suppose that* $k, m, n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $(a, b) = 1$, *a and b are odd and m is even. Then*

(i) for every odd prime p dividing V_n, if $v_p(V_n^{k+1}) \le v_p(U_{nm})$, then $v_p(V_n^k) \le v_p(m)$;

(ii) if $V_n^{k+1} \mid U_{nm}$ and $n \not\equiv 0 \pmod 3$, then $V_n^k \mid m$;
if $V_n^{k+1} \parallel U_{nm}$ and $n \not\equiv 0 \pmod 3$, then $V_n^k \parallel m$;

(iii) if $V_n^{k+1} \mid U_{nm}$, $n \equiv 0 \pmod 6$, and $v_2(m) \ge k$, then $V_n^k \mid m$;
if $V_n^{k+1} \parallel U_{nm}$, $n \equiv 0 \pmod 6$, and $v_2(m) \ge k$, then $V_n^k \parallel m$;
if $V_n^{k+1} \mid U_{nm}$, $n \equiv 0 \pmod 6$, and $v_2(m) < k$, then $V_n^{v_2(m)} \parallel m$;

(iv) if $V_n^{k+1} \mid U_{nm}$, $n \equiv 3 \pmod 6$, $2 \parallel a^2 + 3b$, and $v_2(m) \ge k$, then $V_n^k \mid m$;
if $V_n^{k+1} \parallel U_{nm}$, $n \equiv 3 \pmod 6$, $2 \parallel a^2 + 3b$, and $v_2(m) \ge k$, then $V_n^k \parallel m$;
if $V_n^{k+1} \mid U_{nm}$, $n \equiv 3 \pmod 6$, $2 \parallel a^2 + 3b$, and $v_2(m) < k$, then $V_2^{v_2(m)} \parallel m$;

(v) if $V_n^{k+1} \mid U_{nm}$, $n \equiv 3 \pmod 6$, and $4 \mid a^2 + 3b$, then $V_n^k \mid m$;
if $V_n^{k+1} \parallel U_{nm}$, $n \equiv 3 \pmod 6$, and $4 \mid a^2 + 3b$, then $V_n^k \parallel m$.

Proof. We apply Lemmas 2.1, 2.3, and 2.11 throughout the proof without reference. For (i), assume that p is an odd prime dividing V_n and $v_p(V_n^{k+1}) \le v_p(U_{nm})$. Then

$$v_p(V_n) + v_p(V_n^k) = v_p(V_n^{k+1}) \le v_p(U_{nm}) \le v_p(nm) + v_p(U_{\tau(p)}) = v_p(m) + v_p(V_n),$$

which implies (i). Therefore we only need to consider the 2-adic valuation in the proof of (ii) to (v).

For (ii), assume that $V_n^{k+1} \mid U_{nm}$ and $n \not\equiv 0 \pmod 3$. Since $v_2(V_n^k) = 0 \le v_2(m)$, we obtain by (i) that $V_n^k \mid m$. Suppose futher that $V_n^{k+1} \parallel U_{nm}$. If $V_n^{k+1} \mid m$, then (i) of Theorem 3.11 implies $V_n^{k+2} \mid U_{nm}$, which contradicts $V_n^{k+1} \parallel U_{nm}$, and so $V_n^k \parallel m$.

For (iii), assume that $V_n^{k+1} \mid U_{nm}$ and $n \equiv 0 \pmod 6$.

Case 1 $v_2(m) \ge k$. Then $v_2(V_n^k) = k \le v_2(m)$. So we obtain by (i) that $V_n^k \mid m$. If $V_n^{k+1} \parallel U_{nm}$, then we obtain by (i) of Theorem 3.11 that $V_n^{k+1} \nmid m$, and so $V_n^k \parallel m$. This proves (iii) in the case $v_2(m) \ge k$.

Case 2 $v_2(m) < k$. For convenience, let $d = v_2(m)$. Since $v_2(V_n^d) = d = v_2(m)$ and $v_p(V_n^d) \le v_p(V_n^k) \le v_p(m)$ for every odd prime p dividing V_n, we obtain $V_n^d \mid m$. If $V_n^{d+1} \mid m$, then $d + 1 = v_2(V_n^{d+1}) \le v_2(m) = d$, a contradiction. So $V_n^d \parallel m$.

For (iv), assume that $V_n^{k+1} \mid U_{nm}$, $n \equiv 3 \pmod 6$, and $2 \parallel a^2 + 3b$. Since $U_6 = a(a^2 + 3b)U_3$ and $2 \parallel a^2 + 3b$, we obtain $v_2(V_n) = v_2(U_6) - v_2(U_3) = 1$.

Case 1 $v_2(m) \ge k$. Then $v_2(V_n^k) = k \le v_2(m)$, and so we obtain by (i) that $V_n^k \mid m$. If $V_n^{k+1} \parallel U_{nm}$, then we obtain by (i) of Theorem 3.11 that $V_n^k \parallel m$. This proves (iv) in the case $v_2(m) \ge k$.

Case 2 $v_2(m) < k$. For convenience, let $d = v_2(m)$. Then $v_2(V_n^d) = d = v_2(m)$ and $v_p(V_n^d) \le v_p(V_n^k) \le v_p(m)$. Therefore $V_n^d \mid m$. If $V_n^{d+1} \mid m$, then $d + 1 = v_2(V_n^{d+1}) \le v_2(m) = d$, a contradiction. Therefore $V_n^d \parallel m$.

For (v), assume that $V_n^{k+1} \mid U_{nm}$, $n \equiv 3 \pmod 6$, and $4 \mid a^2 + 3b$. Since $U_6 = a(a^2 + 3b)U_3$ and $4 \mid a^2 + 3b$, we obtain $v_2(U_6) \ge v_2(U_3) + 2$. By Lemma 2.8, we have $v_2(U_3) = 1$. Then $v_2(V_n) = v_2(U_6) - v_2(U_3) = v_2(U_6) - 1$ and

$$v_2(V_n^k) + v_2(V_n) = v_2(V_n^{k+1}) \le v_2(U_{nm}) = v_2(nm) + v_2(U_6) - 1 = v_2(m) + v_2(V_n).$$

So $v_2(V_n^k) \le v_2(m)$. By (i), we obtain $V_n^k \mid m$. If $V_n^{k+1} \parallel U_{nm}$, then we obtain by (i) of Theorem 3.11 that $V_n^{k+1} \nmid m$, and so $V_n^k \parallel m$. This completes the proof. $\square$

The next example shows that m in Theorems 3.8 to 3.12 is necessarily even.

Example 3.13. Let $k, m, n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $(a, b) = 1$ and m is odd. Let p be an odd prime dividing V_n. By Lemma 2.3, we have $p \nmid D$, $\tau(p) \nmid n$ and $\tau(p) \mid 2n$. Therefore $\tau(p)$ is even and $v_2(\tau(p)) = v_2(n) + 1$. So $\tau(p) \nmid nm$. By Lemma 2.1, $v_p(U_{nm}) = 0$. Therefore $V_n \nmid U_{nm}$. This shows that m in Theorems 3.8 to 3.12 cannot be odd.

Example 3.14. Let $k, m, n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $(a, b) = 1$. Let $p > 2$ and $p \mid U_n$. By Lemma 2.1, we have (i) $v_p(U_n) = v_p(n) + v_p(U_p) - 1$ if $p \mid D$ and $p \mid n$, and (ii) $v_p(U_n) = v_p(n) + v_p(U_{\tau(p)})$ if $p \nmid D$ and $\tau(p) \mid n$. For (i), we have $p \mid D$ and so $v_p(V_{nm}) = 0$ and $U_n \nmid V_{nm}$. For (ii), we have $\tau(p) \mid nm$ and so $v_p(V_{nm}) = 0$

and $U_n \nmid V_{nm}$. This shows that there is no interesting divisibility relation such as $U_n^k \mid V_{nm}$.

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Appendix

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Research article

Exact divisibility by powers of the integers in the Lucas sequence of the first kind

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Abstract: Lucas sequence of the first kind is an integer sequence $(U_n)_{n\geq 0}$ which depends on parameters $a, b \in \mathbb{Z}$ and is defined by the recurrence relation $U_0 = 0$, $U_1 = 1$, and $U_n = aU_{n-1} + bU_{n-2}$ for $n \geq 2$. In this article, we obtain exact divisibility results concerning U_n^k for all positive integers n and k. This extends many results in the literature from 1970 to 2020 which dealt only with the classical Fibonacci and Lucas numbers ($a = b = 1$) and the balancing and Lucas-balancing numbers ($a = 6, b = -1$).

Keywords: Lucas sequence; Lucas number; Fibonacci number; exact divisibility; p-adic valuation
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1. Introduction

Throughout this article, let a and b be relatively prime integers and let $(U_n)_{n\geq 0}$ be the Lucas sequence of the first kind which is defined by the recurrence relation $U_0 = 0$, $U_1 = 1$, $U_n = aU_{n-1} + bU_{n-2}$ for $n \geq 2$. To avoid triviality, we also assume that $b \neq 0$ and α/β is not a root of unity where α and β are the roots of the characteristic polynomial $x^2 - ax - b$. In particular, this implies that $\alpha \neq \beta$ and the discriminant $D = a^2 + 4b \neq 0$. If $a = b = 1$, then $(U_n)_{n\geq 0}$ reduces to the sequence of Fibonacci numbers F_n; if $a = 6$ and $b = -1$, then $(U_n)_{n\geq 0}$ becomes the sequence of balancing numbers; if $a = 2$ and $b = 1$, then $(U_n)_{n\geq 0}$ is the sequence of Pell numbers; and many other famous integer sequences are just special cases of the Lucas sequence of the first kind.

The divisibility by powers of the Fibonacci numbers has attracted some attentions because it is used in Matijasevich's solution to Hilbert's 10th problem [7–9]. More precisely, Matijasevich shows that

$$F_n^2 \mid F_{nm} \quad \text{if and only if} \quad F_n \mid m. \tag{1.1}$$

Hoggatt and Bicknell-Johnson [3] give a generalization of (1.1) by replacing F_n^2 by F_n^3, and for a general k, they prove that

$$\text{if } F_n^k \mid m, \text{ then } F_n^{k+1} \mid F_{nm}. \tag{1.2}$$

Benjamin and Rouse [1], and Seibert and Trojovský [27] also provide a different proof of (1.2). Then the investigation on the exact divisibility results for a subsequence of $(F_n)_{n\geq 1}$ begin with the work of Tangboonduangjit et. al [12, 29] and is generalized by Onphaeng and Pongsriiam [10]. The most general results in this direction are obtained by Pongsriiam [18] where (1.2) is extended to include the divisibility and exact divisibility for both the Fibonacci and Lucas numbers. Finally, Onphaeng and Pongsriiam [11] have recently given the converse of the results in [18] which completely answers this kind of questions for the Fibonacci and Lucas numbers. Then Panraksa and Tangboonduangjit [13] initiate the investigation on a special subsequence of $(U_n)_{n\geq 0}$. Patra, Panda, and Khemaratchatakumthorn [14] also obtain the analogue of those results for the balancing and Lucas-balancing numbers. For other related and recent results on Fibonacci, Lucas, balancing, and Lucas-balancing numbers, see for example in [2, 4–6, 15–17, 19–25, 28] and references there in.

In this article, we extend all results in the literature to the Lucas sequence of the first kind. We organize this article as follows. In Section 2, we give some auxiliary results which are needed later. In Section 3, we give main theorems and some related examples. Remark that the corresponding results for other generalizations of the Fibonacci sequence have not been discovered. For example, the question on exact divisibility by powers of the Tribonacci numbers T_n is wide open, where T_n is given by $T_0 = 0$, $T_1 = T_2 = 1$, and $T_k = T_{k-1} + T_{k-2} + T_{k-3}$ for $k \geq 3$. We leave this problem to the interested readers.

2. Preliminaries and Lemmas

In this section, we recall some well-known results and give some useful lemmas for the reader's convenience. The order (or the rank) of appearance of $n \in \mathbb{N}$ in the Lucas sequence $(U_n)_{n\geq 0}$ is defined as the smallest positive integer m such that $n \mid U_m$ and is denoted by $\tau(n)$. The exact divisibility $m^k \parallel n$ means that $m^k \mid n$ and $m^{k+1} \nmid n$. For a prime p and $n \in \mathbb{N}$, the p-adic valuation of n, denoted by $v_p(n)$ is the power of p in the prime factorization of n. We sometimes write the expression such as $a \mid b \mid c = d$ to mean that $a \mid b$, $b \mid c$, and $c = d$. We let $D = a^2 + 4b$ be the discriminant and let α and β be the roots of the characteristic polynomial $x^2 - ax - b$. It is well known that if $D \neq 0$, then the Binet formula $U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ holds for all $n \geq 0$. Next, we recall Sanna's result [26] on the p-adic valuation of Lucas sequence of the first kind.

Lemma 1. [26, Theorem 1.5] *Let p be a prime number such that $p \nmid b$. Then, for each positive integer n,*

$$v_p(U_n) = \begin{cases} v_p(n) + v_p(U_p) - 1 & \text{if } p \mid D \text{ and } p \mid n, \\ 0 & \text{if } p \mid D \text{ and } p \nmid n, \\ v_p(n) + v_p(U_{p\tau(p)}) - 1 & \text{if } p \nmid D,\ \tau(p) \mid n, \text{ and } p \mid n, \\ v_p(U_{\tau(p)}) & \text{if } p \nmid D,\ \tau(p) \mid n, \text{ and } p \nmid n, \\ 0 & \text{if } p \nmid D \text{ and } \tau(p) \nmid n. \end{cases}$$

In fact, we use Lemma 1 only for $p = 2$, because there is a more suitable version of Lemma 1 when

p is odd as given by Panraksa and Tangboonduangjit [13] in their calculation concerning a special subsequence of $(U_n)_{n\geq 0}$. We state it in the next lemma.

Lemma 2. [13, Lemma 2.3] *Let $m, n \geq 1$ and p a prime factor of U_n such that $p \nmid b$. Then, if* (i) *p is odd, or* (ii) *$p = 2$ and n is even, or* (iii) *$p = 2$ and m is odd, we have*

$$v_p(U_{nm}) = v_p(m) + v_p(U_n).$$

Lemma 3. *Let a and b be odd integers. Then, for each positive integer n,*

$$v_2(U_n) = \begin{cases} v_2(n) + v_2(U_6) - 1 & \text{if } n \equiv 0 \pmod 6, \\ v_2(U_3) & \text{if } n \equiv 3 \pmod 6, \\ 0 & \text{if } n \equiv 1, 2, 4, 5 \pmod 6. \end{cases}$$

Proof. Since $U_1 = 1$, $U_2 = a$ are odd and $U_3 = a^2 + b$ is even, we have $\tau(2) = 3$. Applying Lemma 1 for $p = 2$, we obtain the desired result. □

The next two lemmas are also important tools in proving exact divisibility by U_n^k for all $n, k \in \mathbb{N}$.

Lemma 4. [10, Lemma 2.3] *Let k, ℓ, m be positive integers, s nonzero integer, and $s^k \mid m$. Then $s^{k+\ell} \mid \binom{m}{j} s^j$ for all $1 \leq j \leq m$ satisfying $2^{j-\ell+1} > j$. In particular, $s^{k+1} \mid \binom{m}{j} s^j$ for all $1 \leq j \leq m$, and $s^{k+2} \mid \binom{m}{j} s^j$ for all $3 \leq j \leq m$.*

Proof. The statement in [10, Lemma 2.3] is given for $s \geq 1$ but it is easy to see that if $s \leq -1$, then we can replace s by $-s$ and every divisibility relation still holds. Therefore this is true for all $s \neq 0$. □

Lemma 5. *Let $m, n \geq 1$ and $r \geq 0$ be integers. Then*

(i) $U_{mn+r} = \sum_{j=0}^{m} \binom{m}{j} U_n^j (bU_{n-1})^{m-j} U_{j+r}$,

(ii) $U_{mn} = \sum_{j=1}^{m} \binom{m}{j} U_n^j (bU_{n-1})^{m-j} U_j$.

Proof. By Binet's formula, we obtain $\alpha^n = \alpha U_n + bU_{n-1}$, $\beta^n = \beta U_n + bU_{n-1}$, and

$$\begin{aligned} U_{mn+r} &= \frac{\alpha^{mn+r} - \beta^{mn+r}}{\alpha - \beta} \\ &= \frac{1}{\alpha - \beta} \left((\alpha U_n + bU_{n-1})^m \alpha^r - (\beta U_n + bU_{n-1})^m \beta^r \right) \\ &= \frac{1}{\alpha - \beta} \left(\sum_{j=0}^{m} \binom{m}{j} (\alpha U_n)^j (bU_{n-1})^{m-j} \alpha^r - \sum_{j=0}^{m} \binom{m}{j} (\beta U_n)^j (bU_{n-1})^{m-j} \beta^r \right) \\ &= \frac{1}{\alpha - \beta} \sum_{j=o}^{m} \left(\binom{m}{j} U_n^j (bU_{n-1})^{m-j} \left(\alpha^{j+r} - \beta^{j+r} \right) \right) \\ &= \sum_{j=0}^{m} \binom{m}{j} U_n^j (bU_{n-1})^{m-j} U_{j+r}. \end{aligned}$$

This proves (i). Since $U_0 = 0$, (ii) follows immediately from (i) by substituting $r = 0$. □

Recall that we assume throughout this article that $(a, b) = 1$. This is necessary for the proof of the following lemmas.

Lemma 6. *Suppose* $(a, b) = 1$. *Then* $(U_m, U_n) = U_{(m,n)}$ *and in particular* $(U_n, U_{n+1}) = 1$ *for each* $m, n \in \mathbb{N}$.

Proof. This is well known. □

Lemma 7. *Let* $n \geq 1$ *and* $(a, b) = 1$. *If* p *is a prime factor of* U_n, *then* $p \nmid b$. *Consequently,* $(U_n, b) = 1$ *for all* $n \geq 1$.

Proof. Suppose for a contradiction that $(a, b) = 1$, $n \geq 1$, $p \mid U_n$, and $p \mid b$. We can choose n to be the smallest such integer. Since $U_1 = 1$, $U_2 = a$, we see that $n \geq 3$. Since $p \mid U_n = aU_{n-1} + bU_{n-2}$ and $p \mid b$, we have $p \mid aU_{n-1}$. By the choice of n, $p \nmid U_{n-1}$. So $p \mid a$. Therefore $p \mid (a, b) = 1$, a contradiction. □

Lemma 8. *Let* a *and* b *be odd,* $(a, b) = 1$, *and* $v_2(U_6) \geq v_2(U_3) + 2$. *Then* $v_2(U_3) = 1$.

Proof. Since $U_3 = a^2 + b$ is even and $U_6 = a(a^2 + 3b)U_3$, we obtain $v_2(U_3) \geq 1$ and

$$v_2(U_6) = v_2(U_3) + v_2(a^2 + 3b). \tag{2.1}$$

If $v_2(U_3) \geq 2$, then $4 \mid a^2 + b$, and so $b \equiv 3 \pmod 4$ and (2.1) implies $v_2(U_6) = v_2(U_3) + 1$ contradicting $v_2(U_6) \geq v_2(U_3) + 2$. Thus $v_2(U_3) = 1$.

□

3. Main results

We begin with the simplest main result of this paper.

Theorem 9. *Let* k, m, *and* n *be positive integers. If* $U_n^k \mid m$, *then* $U_n^{k+1} \mid U_{nm}$.

Proof. If $U_n^k \mid m$, then we obtain by Lemma 4 that, $U_n^{k+1} \mid \binom{m}{j} U_n^j$ for all $1 \leq j \leq m$, which implies $U_n^{k+1} \mid U_{nm}$, by Lemma 5. □

Next, we extend Theorem 9 to include exact divisibility. The proof of Theorem 10 is much longer than that of Theorem 9 since we would like to cover all possible cases. Although many cases can be combined, it is more convenient to state them separately. Recall that for $x \in \mathbb{R}$, the largest integer which is less than or equal to x is denoted by $\lfloor x \rfloor$.

Theorem 10. *Let* $k, m, n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $(a, b) = 1$, $n \geq 2$, *and* $U_n^k \parallel m$. *Then*

(i) *if* a *is odd and* b *is even, then* $U_n^{k+1} \parallel U_{nm}$;
(ii) *if* a *is even and* b *is odd, then* $U_n^{k+1} \parallel U_{nm}$;
(iii) *if* a *and* b *are odd and* $n \not\equiv 3 \pmod 6$, *then* $U_n^{k+1} \parallel U_{nm}$;
(iv) *if* a *and* b *are odd,* $n \equiv 3 \pmod 6$, *and* $\frac{U_n^{k+1}}{2} \nmid m$, *then* $U_n^{k+1} \parallel U_{nm}$;
(v) *if* a *and* b *are odd,* $n \equiv 3 \pmod 6$, $\frac{U_n^{k+1}}{2} \mid m$, *and* $2 \parallel a^2 + 3b$, *then* $U_n^{k+1} \parallel U_{nm}$;
(vi) *if* a *and* b *are odd,* $n \equiv 3 \pmod 6$, $\frac{U_n^{k+1}}{2} \mid m$, *and* $4 \mid a^2 + 3b$, *then* $U_n^{k+t+1} \parallel U_{nm}$, *where*

$$t = \min(\{v_2(U_6) - 2\} \cup \{y_p - k \mid p \text{ is an odd prime factor of } U_n\})$$

and $y_p = \left\lfloor \frac{v_p(m)}{v_p(U_n)} \right\rfloor$ *for each odd prime* p *dividing* U_n.

Proof. By Theorem 9, we obtain $U_n^{k+1} \mid U_{nm}$. So for (i) to (v), it is enough to show that $U_n^{k+2} \nmid U_{nm}$. We divide the calculation into several cases.

Case 1. a is odd and b is even. Since U_1 and U_2 are odd and $U_r = aU_{r-1} + bU_{r-2} \equiv U_{r-1} \pmod 2$ for $r \geq 3$, it follows by induction that U_n is odd. From the assumption $U_n^k \parallel m$, we have $U_n^{k+1} \nmid m$, and so there exists a prime p dividing U_n such that $v_p(U_n^{k+1}) > v_p(m)$. Since U_n is odd, p is also odd. In addition, $p \nmid b$ by Lemma 7. So we can apply Lemma 2(i) to obtain

$$v_p(U_{nm}) = v_p(m) + v_p(U_n) < v_p(U_n^{k+1}) + v_p(U_n) = v_p(U_n^{k+2}),$$

which implies $U_n^{k+2} \nmid U_{nm}$, as required. This proves (i).

Case 2. a is even and b is odd. Similar to Case 1, we have U_1 is odd, U_2 is even, $U_r \equiv U_{r-2} \pmod 2$ for $r \geq 3$, and so U_n is even if and only if n is even. In addition, there exists a prime p such that $p \mid U_n$, $v_p(U_n^{k+1}) > v_p(m)$, and $p \nmid b$. So if $2 \nmid n$, then U_n is odd, p is odd, and we obtain by Lemma 2(i) that

$$v_p(U_{nm}) = v_p(m) + v_p(U_n) < v_p(U_n^{k+1}) + v_p(U_n) = v_p(U_n^{k+2}), \tag{3.1}$$

which implies $U_n^{k+2} \nmid U_{nm}$. If $2 \mid n$, then we can still use either Lemma 2(i) or Lemma 2(ii) to obtain (3.1), which leads to the same conclusion $U_n^{k+2} \nmid U_{nm}$. This proves (ii).

Case 3. a and b are odd. Similar to Case 1, there is a prime p such that $p \mid U_n$, $v_p(U_n^{k+1}) > v_p(m)$, and $p \nmid b$.

Case 3.1 $n \not\equiv 3 \pmod 6$. If $n \equiv 1, 2, 4, 5 \pmod 6$, then we obtain by Lemmas 3 and 2, respectively that p is odd and

$$v_p(U_{nm}) = v_p(U_n) + v_p(m) < v_p(U_n) + v_p(U_n^{k+1}) = v_p(U_n^{k+2}). \tag{3.2}$$

If $n \equiv 0 \pmod 6$, then n is even and Lemma 2(i) or Lemma 2(ii) can still be used to obtain (3.2). In any case, $U_n^{k+2} \nmid U_{nm}$. This proves (iii).

Case 3.2 $n \equiv 3 \pmod 6$ and $\frac{U_n^{k+1}}{2} \nmid m$. Since $U_n^k \parallel m$, we can write $m = cU_n^k$ where $c \geq 1$ and $U_n \nmid c$. By Lemma 4, $U_n^{k+2} \mid \binom{m}{j} U_n^j$ for $3 \leq j \leq m$. Then we obtain by Lemma 5 that

$$U_{nm} = U_{mn} \equiv mU_n(bU_{n-1})^{m-1} + \frac{m(m-1)}{2} U_n^2 (bU_{n-1})^{m-2} a \pmod{U_n^{k+2}}.$$

By Lemma 3, we know that $v_2(U_n) = v_2(U_3) \geq 1$. Since $\frac{U_n^{k+1}}{2} \nmid m$ and $m = cU_n^k$, we see that $\frac{U_n}{2}$ does not c. Let $d = bU_{n-1} + \frac{U_n}{2}(m-1)a$. By Lemmas 6 and 7, we obtain $\left(\frac{U_n}{2}, d\right) = \left(\frac{U_n}{2}, bU_{n-1}\right) = 1$. Then

$$U_{nm} \equiv mU_n b^{m-2} U_{n-1}^{m-2} \left(bU_{n-1} + \frac{U_n}{2}(m-1)a\right) \equiv cU_n^{k+1} b^{m-2} U_{n-1}^{m-2} d \pmod{U_n^{k+2}}.$$

By Lemmas 6 and 7, we obtain $U_n^{k+2} \mid U_{nm}$ if and only if $U_n \mid cd$. But if $U_n \mid cd$, then $\frac{U_n}{2} \mid cd$ which implies $\frac{U_n}{2} \mid c$, a contradiction. So $U_n \nmid cd$ and therefore $U_n^{k+2} \nmid U_{nm}$. This proves (iv). To prove (v) and (vi), we first assume that a and b are odd, $n \equiv 3 \pmod 6$, and $\frac{U_n^{k+1}}{2} \mid m$. (The other condition will be assumed later). Then $v_p(U_n^{k+1}) \leq v_p(m)$ for all odd primes p and $v_2(U_n^{k+1}) - 1 \leq v_2(m)$. If $v_2(U_n^{k+1}) - 1 < v_2(m)$, then $v_2(U_n^{k+1}) \leq v_2(m)$, and so $v_p(U_n^{k+1}) \leq v_p(m)$ for all primes p, which implies $U_n^{k+1} \mid m$ contradicting the assumption $U_n^k \parallel m$. Hence

$$v_2(U_n^{k+1}) - 1 = v_2(m) \text{ and } v_p(U_n^{k+1}) \leq v_p(m) \text{ for every odd prime } p \tag{3.3}$$

We now separate the consideration into two cases according to the additional conditions in (v) and (vi). Observe that $v_2(a^2 + 3b) = 1$ is equivalent to $2 \parallel a^2 + 3b$.

Case 4. $v_2(a^2 + 3b) = 1$. Since $U_6 = a(a^2 + 3b)U_3$, we obtain $v_2(U_6) = v_2(U_3) + 1$. Recall that $n \equiv 3 \pmod 6$ and $U_n^k \mid m$. So n is odd, m is even, and $nm \equiv 0 \pmod 6$. If $U_n^{k+2} \mid U_{nm}$, then we obtain by Lemma 3 and (3.3) that

$$\begin{aligned} v_2(U_n^{k+1}) + v_2(U_n) = v_2(U_n^{k+2}) \le v_2(U_{nm}) &= v_2(n) + v_2(m) + v_2(U_6) - 1 \\ &= v_2(U_n^{k+1}) - 1 + v_2(U_3) \\ &= v_2(U_n^{k+1}) + v_2(U_n) - 1, \end{aligned}$$

which is a contradiction. Therefore $U_n^{k+2} \nmid U_{nm}$. This proves (v).

Case 5. $v_2(a^2+3b) \ge 2$. Then $v_2(U_6) = v_2(U_3)+v_2(a^2+3b) \ge v_2(U_3)+2$. By Lemma 8, $v_2(U_3) = 1$ and so $v_2(U_6) = x+2$ where $x = v_2(a^2 + 3b) - 1 \in \mathbb{N}$. For each odd prime p dividing U_n, let $y_p = \left\lfloor \frac{v_p(m)}{v_p(U_n)} \right\rfloor$ be the largest integer which is less than or equal to $\frac{v_p(m)}{v_p(U_n)}$. Since $U_n^k \mid m$, we have $y_p \ge k$ for all odd $p \mid U_n$. Let

$$t = \min(\{x\} \cup \{y_p - k \mid p \text{ is an odd prime factor of } U_n\}).$$

Then $t \ge 0$. By Lemma 3 and (3.3), $v_2(m) = (k + 1)v_2(U_3) - 1 = k$ and

$$v_2(U_{nm}) = v_2(m) + v_2(U_6) - 1 = k + x + 1 \ge k + t + 1 = v_2(U_n^{k+t+1}). \tag{3.4}$$

By the definition of y_p, we have $v_p(m) \ge y_p v_p(U_n)$. So by Lemma 2, if p is an odd prime dividing U_n, then

$$v_p(U_{nm}) = v_p(m) + v_p(U_n) \ge (y_p + 1)v_p(U_n) \ge (k + t + 1)v_p(U_n) = v_p(U_n^{k+t+1}). \tag{3.5}$$

By (3.4) and (3.5), $v_p(U_{nm}) \ge v_p(U_n^{k+t+1})$ for all primes p dividing U_n. This show that $U_n^{k+t+1} \mid U_{nm}$. It remains to show that $U_n^{k+t+2} \nmid U_{nm}$. If $t = y_p - k$ for some odd prime p dividing U_n, then we recall the definition of y_p and apply Lemma 2 to obtain

$$v_p(U_{nm}) = v_p(m) + v_p(U_n) < (y_p + 2)v_p(U_n) = (k + t + 2)v_p(U_n) = v_p(U_n^{k+t+2}).$$

If $t = x = v_2(U_6) - 2$, then we use Lemma 3 to get

$$v_2(U_{nm}) = v_2(m) + v_2(U_6) - 1 = k + t + 1 < v_2(U_n^{k+t+2}).$$

In any case, $U_n^{k+t+2} \nmid U_{nm}$. This completes the proof. □

The next example shows that the integer t in Theorem 10(vi) can be any odd positive integer.

Example 11. Let $M \in \mathbb{N}$ be given. We show that there are positive integers k, m, n, a, b satisfying the conditions in Theorem 10(vi) with $t = M$. Choose $a = 1$ and $b = \left(2^{4M} - 1\right)/3$. Then a and b are odd integers, $(a, b) = 1$, and $v_2(a^2 + 3b) = 4M > 2$. Next choose any $k, n \in \mathbb{N}$ such that $n \equiv 3 \pmod 6$. Since $v_2(U_6) = v_2(U_3) + v_2(a^2 + 3b) \ge v_2(U_3) + 2$, we obtain by Lemmas 3 and 8 that $v_2(U_n) = v_2(U_3) = 1$ and $v_2(U_6) = 4M + 1$. Since $U_n \ge U_3 = a^2 + b > 2$ and $v_2(U_n) = 1$, we can write $U_n = 2p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$ where $s \ge 1$, $p_1, p_2, \ldots, p_s$ are distinct odd primes, and $a_1, a_2, \ldots, a_s$ are positive integers. Next, choose $m = 2^k p_1^{a_1(k+M)} p_2^{a_2(k+M)} \cdots p_s^{a_s(k+M)}$. Then $U_n^k \parallel m$ and $\frac{U_n^{k+1}}{2} \mid m$. Therefore k, m, n, a, b satisfy all the conditions in Theorem 10(vi). Finally, we have

$$v_2(U_6) - 2 = v_2(a^2 + 3b) - 1 = 4M - 1$$

and $y_p - k = M$ for all $p \in \{p_1, p_2, \ldots, p_s\}$, and therefore $t = \min\{4M - 1, M\} = M$, as desired.

Next, we prove the converse of Theorem 10.

Theorem 12. *$k, m, n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $(a, b) = 1$, $n \geq 2$, and $U_n^{k+1} \parallel U_{nm}$. Then*

(i) *if a is odd and b is even, then $U_n^k \parallel m$;*
(ii) *if a is even and b is odd, then $U_n^k \parallel m$;*
(iii) *if a and b are odd and $n \not\equiv 3 \pmod 6$, then $U_n^k \parallel m$;*
(iv) *if a and b are odd, $n \equiv 3 \pmod 6$, and $2 \parallel a^2 + 3b$, then $U_n^k \parallel m$;*
(v) *if a and b are odd, $n \equiv 3 \pmod 6$, $4 \mid a^2 + 3b$, and $v_2(m) \geq k$, then $U_n^k \parallel m$;*
(vi) *if a and b are odd, $n \equiv 3 \pmod 6$, $4 \mid a^2 + 3b$, and $v_2(m) < k$, then*

$$m \text{ is even, } v_2(m) \geq k + 1 - v_2(a^2 + 3b), \text{ and } U_n^{v_2(m)} \parallel m.$$

Proof. Some parts of the proof are similar to those of Theorem 10, so we skip some details.
Case 1. a is odd and b is even. Similar to Case 1 of Theorem 10, we have U_n is odd. For any prime $p \mid U_n$, we obtain by Lemma 2 that

$$v_p(U_n^k) + v_p(U_n) = v_p(U_n^{k+1}) \leq v_p(U_{nm}) = v_p(U_n) + v_p(m), \tag{3.6}$$

which implies $U_n^k \mid m$. If $U_n^{k+1} \mid m$, then by Theorem 9, we have $U_n^{k+2} \mid U_{nm}$ which contradicts $U_n^{k+1} \parallel U_{nm}$. Therefore $U_n^{k+1} \nmid m$, and thus $U_n^k \parallel m$.
Case 2. a is even and b is odd. Then U_n is even if and only if n is even. So if $2 \nmid n$, then for any prime $p \mid U_n$, we have p is odd, (3.6) holds, and so $U_n^k \mid m$. If $2 \mid n$, then we can still apply Lemma 2(i) or Lemma 2(ii) to obtain (3.6) and conclude that $U_n^k \mid m$. If $U_n^{k+1} \mid m$, then by Theorem 9, we have $U_n^{k+2} \mid U_{nm}$ which contradicts $U_n^{k+1} \parallel U_{nm}$. So $U_n^{k+1} \nmid m$ and therefore $U_n^k \parallel m$.

We now assume throughout that a and b are odd and divide the consideration into four cases according to the additional conditions in (iii) to (vi).
Case 3. $n \not\equiv 3 \pmod 6$. If $n \equiv 1, 2, 4, 5 \pmod 6$, then we apply Lemma 3 to obtain $v_2(U_n^k) = 0 \leq v_2(m)$, and use Lemma 2 to show that for any odd prime $p \mid U_n$,

$$v_p(U_n) + v_p(U_n^k) = v_p(U_n^{k+1}) \leq v_p(U_{nm}) = v_p(m) + v_p(U_n). \tag{3.7}$$

If $n \equiv 0 \pmod 6$, then n is even and we can apply Lemma 2(i) or Lemma 2(ii) to obtain (3.7) for any prime $p \mid U_n$. In any case, we have $U_n^k \mid m$. Again, by Theorem 9, we have $U_n^{k+1} \nmid m$, and so $U_n^k \parallel m$. This proves (iii).
Case 4. $n \equiv 3 \pmod 6$ and $2 \parallel a^2 + 3b$. Similar to Case 4 in the proof of Theorem 10 we have $v_2(U_6) = v_2(U_3) + 1$. If m is odd, then $nm \equiv 3 \pmod 6$ and we obtain by Lemma 3 that $v_2(U_{nm}) = v_2(U_3) < (k + 1)v_2(U_3) = v_2(U_n^{k+1})$, which contradicts the assumption $U_n^{k+1} \mid U_{nm}$. So m is even, and thus $nm \equiv 0 \pmod 6$. By Lemma 3 and the fact that $n \equiv 3 \pmod 6$ is odd, we obtain $v_2(m) + v_2(U_6) - 1 = v_2(U_{nm}) \geq v_2(U_n^{k+1}) = v_2(U_n^k) + v_2(U_n) = v_2(U_n^k) + v_2(U_3) = v_2(U_n^k) + v_2(U_6) - 1$, which implies $v_2(m) \geq v_2(U_n^k)$. If p is odd and $p \mid U_n$, then we apply Lemma 2 to obtain (3.7) Therefore $v_p(U_n^k) \leq v_p(m)$ for every prime p dividing U_n. Thus $U_n^k \mid m$. By Theorem 9, $U_n^{k+1} \nmid m$. Hence $U_n^k \parallel m$. This proves (iv).
Case 5. $n \equiv 3 \pmod 6$, $4 \mid a^2 + 3b$, and $v_2(m) \geq k$. Then $U_3 = a^2 + b = (a^2 + 3b) - 2b \equiv 2 \pmod 4$, and so $v_2(U_3) = 1$. By Lemma 3, we obtain $v_2(m) \geq kv_2(U_3) = kv_2(U_n) = v_2(U_n^k)$. By Lemma 2, if p is

an odd prime dividing U_n, then (3.6) holds, and so we conclude that $v_p(U_n^k) \le v_p(m)$ for every prime p dividing U_n. Therefore $U_n^k \mid m$. By Theorem 9, $U_n^{k+1} \nmid m$ and so $U_n^k \parallel m$. This proves (v).
Case 6. $n \equiv 3 \pmod 6$, $4 \mid a^2 + 3b$, and $v_2(m) < k$. For convenience, let $t = v_2(m)$. Similar to Case 4, we have m is even. In addition, $v_2(U_6) = v_2(U_3) + v_2(a^2 + 3b) = 1 + v_2(a^2 + 3b)$. So $k > t \ge 1$ and $v_2(m) = tv_2(U_3) = tv_2(U_n) = v_2(U_n^t)$. By Lemma 2, if p is odd and $p \mid U_n$, then

$$v_p(U_n) + v_p(U_n^t) \le v_p(U_n) + v_p(U_n^k) = v_p(U_n^{k+1}) \le v_p(U_{nm}) = v_p(m) + v_p(U_n).$$

From the above inequalities, we obtain that $v_p(U_n^t) \le v_p(m)$ for every prime p dividing U_n. Therefore $U_n^t \mid m$. If $U_n^{t+1} \mid m$, then we obtain by Lemma 3 that $t = v_2(m) \ge v_2(U_n^{t+1}) = t + 1$, which is false. So $U_n^{t+1} \nmid m$. Therefore $U_n^t \parallel m$. From $U_n^{k+1} \parallel U_{nm}$, we also obtain $k + 1 = v_2(U_n^{k+1}) \le v_2(U_{nm}) = v_2(m) + v_2(U_6) - 1 = v_2(m) + v_2(a^2 + 3b)$, which implies $v_2(m) \ge k + 1 - v_2(a^2 + 3b)$. This completes the proof. □

The next example shows that $v_2(m)$ in Theorem 12(vi) can be any positive integer in $[1, k)$.

Example 13. Let $k \ge 1$ and $1 \le M < k$ be integers. We show that there are m, n, a, b satisfying the conditions in Theorem 12(vi) with $v_2(m) = M$. Choose $n \in \mathbb{N}$ and $n \equiv 3 \pmod 6$.
Case 1. $k - M$ is odd. Choose $a = 1$, $b = \frac{2^{k-M+1}-1}{3}$, and $m = \frac{U_n^k}{2^{k-M}}$. Then a and b are odd integers, $(a, b) = 1$, and $v_2(a^2 + 3b) = k - M + 1 \ge 2$. Since $v_2(U_6) = v_2(U_3) + v_2(a^2 + 3b) \ge v_2(U_3) + 2$, we obtain by Lemmas 3 and 8 that $v_2(U_n) = v_2(U_3) = 1$ and $v_2(U_6) = k - M + 2$. By Lemma 2, for $p > 2$ and $p \mid U_n$ we obtain

$$v_p(U_{nm}) = v_p(m) + v_p(U_n) = v_p(U_n^k) + v_p(U_n) = v_p(U_n^{k+1}).$$

By Lemma 3, we have

$$v_2(m) = v_2(U_n^k) - v_2(2^{k-M}) = k - k + M = M$$

and

$$v_2(U_{nm}) = v_2(m) + v_2(U_6) - 1 = M + k - M + 2 - 1 = v_2(U_n^{k+1}).$$

From these, we obtain $U_n^{k+1} \parallel U_{nm}$ and $U_n^M \parallel m$. Therefore k, m, n, a, b satisfy all the conditions in Theorem 12(vi).
Case 2. $k - M$ is even. Choose $a = 1$, $b = \frac{5 \cdot 2^{k-M+1}-1}{3}$, and $m = \frac{U_n^k}{2^{k-M}}$. The verification is the same as that in Case 1. So we leave the details to the reader.

Substituting $a = b = 1$ in Theorems 10 and 12, (U_n) becomes the Fibonacci sequence $(F_n)_{n\ge0}$ and we obtain our previous results [11, 18] as a corollary.

Corollary 14. [18, Theorem 2] and [11, Theorem 3.2] *Let $n \ge 3$. Then the following statements hold:*

(i) *if $F_n^k \parallel m$ and $n \not\equiv 3 \pmod 6$, then $F_n^{k+1} \parallel F_{nm}$;*
(ii) *if $F_n^k \parallel m$, $n \equiv 3 \pmod 6$ and $\frac{F_n^{k+1}}{2} \nmid m$, then $F_n^{k+1} \parallel F_{nm}$;*
(iii) *if $F_n^k \parallel m$, $n \equiv 3 \pmod 6$ and $\frac{F_n^{k+1}}{2} \mid m$, then $F_n^{k+2} \parallel F_{nm}$;*
(iv) *if $F_n^{k+1} \parallel F_{nm}$ and $n \not\equiv 3 \pmod 6$, then $F_n^k \parallel m$;*
(v) *if $F_n^{k+1} \parallel F_{nm}$, $n \equiv 3 \pmod 6$, and $2^k \mid m$, then $F_n^k \parallel m$;*
(vi) *if $F_n^{k+1} \parallel F_{nm}$, $n \equiv 3 \pmod 6$, and $2^k \nmid m$, then $F_n^{k-1} \parallel m$.*

Substituting $a = 6$ and $b = -1$, in our theorems, (U_n) reduces to the sequence (B_n) of balancing numbers and we obtain the results of Patra, Panda, and Khemaratchatakumthorn.

Corollary 15. [14, Theorem 9] *For all $k \geq 1$ and $m, n \geq 2$, we obtain $B_n^k \parallel m$ if and only if $B_n^{k+1} \parallel B_{nm}$.*

Similarly by, substituting $a = 2$ and $b = 1$ in our theorems, we obtain the exact divisibility results for the Pell sequence $(P_n)_{n\geq 0}$ as follows.

Corollary 16. *For all $k \geq 1$ and $m, n \geq 2$, we obtain $P_n^k \parallel m$ if and only if $P_n^{k+1} \parallel P_{nm}$.*

We also plan to solve this problem for the Lucas sequence of the second kind in the future. The answers will appear in our next article.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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Research article

Exact divisibility by powers of the integers in the Lucas sequences of the first and second kinds

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Abstract: Lucas sequences of the first and second kinds are, respectively, the integer sequences $(U_n)_{n\geq 0}$ and $(V_n)_{n\geq 0}$ depending on parameters $a, b \in \mathbb{Z}$ and defined by the recurrence relations $U_0 = 0$, $U_1 = 1$, and $U_n = aU_{n-1} + bU_{n-2}$ for $n \geq 2$, $V_0 = 2$, $V_1 = a$, and $V_n = aV_{n-1} + bV_{n-2}$ for $n \geq 2$. In this article, we obtain exact divisibility results concerning U_n^k and V_n^k for all positive integers n and k. This and our previous article extend many results in the literature and complete a long investigation on this problem from 1970 to 2021.

Keywords: Lucas sequence; Lucas number; Fibonacci number; exact divisibility; p-adic valuation
Mathematics Subject Classification: 11B39; 11B37; 11A05

1. Introduction

Throughout this article, let a and b be relatively prime integers and let $(U_n)_{n\geq 0}$ and $(V_n)_{n\geq 0}$ be the Lucas sequences of the first and second kinds which are defined by the recurrence relations

$$U_0 = 0,\ U_1 = 1,\ U_n = aU_{n-1} + bU_{n-2} \text{ for } n \geq 2,$$

$$V_0 = 2,\ V_1 = a, \text{ and } V_n = aV_{n-1} + bV_{n-2} \text{ for } n \geq 2.$$

To avoid triviality, we also assume that $b \neq 0$ and α/β is not a root of unity where α and β are the roots of the characteristic polynomial $x^2 - ax - b$. In particular, this implies that $\alpha \neq \beta$, $\alpha \neq -\beta$, the discriminant $D = a^2 + 4b \neq 0$, $U_n \neq 0$, and $V_n \neq 0$ for all $n \geq 1$. If $a = b = 1$, then $(U_n)_{n\geq 0}$ reduces to the sequence of Fibonacci numbers F_n; if $a = 6$ and $b = -1$, then $(U_n)_{n\geq 0}$ becomes the sequence of balancing numbers; if $a = 2$ and $b = 1$, then $(U_n)_{n\geq 0}$ is the sequence of Pell numbers; and many other famous integer sequences are just special cases of the Lucas sequences of the first and second kinds.

The divisibility by powers of the Fibonacci numbers has attracted some attentions because it is used in Matijasevich's solution to Hilbert's 10th problem [5, 6, 7]. More precisely, Matijasevich show that

$$F_n^2 \mid F_{nm} \quad \text{if and only if} \quad F_n \mid m. \tag{1.1}$$

From that point, Hoggatt and Bicknell-Johnson [4], Benjamin and Rouse [1], Seibert and Trojovský [19], Pongsriiam [15], Onphaeng and Pongsriiam [9, 10], Panraksa and Tangboonduangjit [11], and Patra, Panda, and Khemaratchatakumthorn [12] have made some contributions on the extensions of (1.1). For more details about the timeline and the development of this problem, we refer the reader to the introduction of our previous article [8]. In fact, the most general results in this direction has recently been given by us [8] as follows.

Theorem 1. [8, Theorem 10] *Let $k, m, n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $(a, b) = 1$, $n \geq 2$, and $U_n^k \parallel m$. Then*

(i) *if a is odd and b is even, then $U_n^{k+1} \parallel U_{nm}$;*
(ii) *if a is even and b is odd, then $U_n^{k+1} \parallel U_{nm}$;*
(iii) *if a and b are odd and $n \not\equiv 3 \pmod 6$, then $U_n^{k+1} \parallel U_{nm}$;*
(iv) *if a and b are odd, $n \equiv 3 \pmod 6$, and $\frac{U_n^{k+1}}{2} \nmid m$, then $U_n^{k+1} \parallel U_{nm}$;*
(v) *if a and b are odd, $n \equiv 3 \pmod 6$, $\frac{U_n^{k+1}}{2} \mid m$, and $2 \parallel a^2 + 3b$, then $U_n^{k+1} \parallel U_{nm}$;*
(vi) *if a and b are odd, $n \equiv 3 \pmod 6$, $\frac{U_n^{k+1}}{2} \mid m$, and $4 \mid a^2 + 3b$, then $U_n^{k+t+1} \parallel U_{nm}$, where*

$$t = \min(\{v_2(U_6) - 2\} \cup \{y_p - k \mid p \text{ is an odd prime factor of } U_n\}) \text{ and}$$
$$y_p = \left\lfloor \frac{v_p(m)}{v_p(U_n)} \right\rfloor \text{ for each odd prime } p \text{ dividing } U_n.$$

Theorem 2. [8, Theorem 12] *Let $k, m, n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $(a, b) = 1$, $n \geq 2$, and $U_n^{k+1} \parallel U_{nm}$. Then*

(i) *if a is odd and b is even, then $U_n^k \parallel m$;*
(ii) *if a is even and b is odd, then $U_n^k \parallel m$;*
(iii) *if a and b are odd and $n \not\equiv 3 \pmod 6$, then $U_n^k \parallel m$;*
(iv) *if a and b are odd, $n \equiv 3 \pmod 6$, and $2 \parallel a^2 + 3b$, then $U_n^k \parallel m$;*
(v) *if a and b are odd, $n \equiv 3 \pmod 6$, $4 \mid a^2 + 3b$, and $v_2(m) \geq k$, then $U_n^k \parallel m$;*
(vi) *if a and b are odd, $n \equiv 3 \pmod 6$, $4 \mid a^2 + 3b$, and $v_2(m) < k$, then*

$$m \text{ is even, } v_2(m) \geq k + 1 - v_2(a^2 + 3b), \text{ and } U_n^{v_2(m)} \parallel m.$$

For other related and recent results on Fibonacci, Lucas, balancing, and Lucas-balancing numbers, see for example in [3, 13, 14, 16, 17, 20] and references there in.

In this article, we extend Theorems 1 and 2 to the case of V_n and the mix of U_n and V_n. For example, we obtain in Theorem 15 that if a and m are even, b is odd, and $V_n^{k+1} \parallel U_{nm}$, then $2 \mid n$ implies $V_n^{\min(k, v_2(m))} \parallel m$; while $2 \nmid n$ implies $V_n^k \mid m$ and the exponent k can be replaced by $k + 1$ if and only if $\frac{V_n^{k+2}}{2} \mid U_{nm}$.

2. Preliminaries and Lemmas

In this section, we recall some definition and well known results, and give some useful lemmas for the reader's convenience. The order (or the rank) of appearance of $n \in \mathbb{N}$ in the Lucas sequence

$(U_n)_{n\geq 0}$ is defined as the smallest positive integer m such that $n \mid U_m$ and is denoted by $\tau(n)$. The exact divisibility $m^k \parallel n$ means that $m^k \mid n$ and $m^{k+1} \nmid n$. The letter p is always a prime. For $n \in \mathbb{N}$, the p-adic valuation of n, denoted by $v_p(n)$ is the power of p in the prime factorization of n. We sometimes write the expression such as $a \mid b \mid c = d$ to mean that $a \mid b$, $b \mid c$, and $c = d$. For each $x \in \mathbb{R}$, we write $\lfloor x \rfloor$ to denote the largest integer less than or equal to x. So $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$. We let $D = a^2 + 4b$ be the discriminant and let α and β be the roots of the characteristic polynomial $x^2 - ax - b$. Then it is well known that if $D \neq 0$, then the Binet formula

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = \alpha^n + \beta^n \text{ holds for all } n \geq 0.$$

Next, we recall Sanna's result [18] on the p-adic valuation of the Lucas sequence of the first kind.

Lemma 3. [18, Theorem 1.5] *Let p be a prime number such that $p \nmid b$. Then, for each positive integer n,*

$$v_p(U_n) = \begin{cases} v_p(n) + v_p(U_p) - 1 & \text{if } p \mid D \text{ and } p \mid n, \\ 0 & \text{if } p \mid D \text{ and } p \nmid n, \\ v_p(n) + v_p(U_{p\tau(p)}) - 1 & \text{if } p \nmid D,\ \tau(p) \mid n, \text{ and } p \mid n, \\ v_p(U_{\tau(p)}) & \text{if } p \nmid D,\ \tau(p) \mid n, \text{ and } p \nmid n, \\ 0 & \text{if } p \nmid D \text{ and } \tau(p) \nmid n. \end{cases}$$

In particular, if p is an odd prime such that $p \nmid b$, then, for each positive integer n,

$$v_p(U_n) = \begin{cases} v_p(n) + v_p(U_p) - 1 & \text{if } p \mid D \text{ and } p \mid n, \\ 0 & \text{if } p \mid D \text{ and } p \nmid n, \\ v_p(n) + v_p(U_{\tau(p)}) & \text{if } p \nmid D \text{ and } \tau(p) \mid n, \\ 0 & \text{if } p \nmid D \text{ and } \tau(p) \nmid n. \end{cases}$$

From Lemma 3, and the fact that $V_n = U_{2n}/U_n$, we easily obtain the following result.

Lemma 4. *If p is an odd prime and $p \nmid b$. Then, for each positive integer n,*

$$v_p(V_n) = \begin{cases} v_p(n) + v_p(U_{\tau(p)}) & \text{if } p \nmid D,\ \tau(p) \nmid n \text{ and } \tau(p) \mid 2n, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. This follows from the application of Lemma 3, a straightforward calculation, and the fact that $v_p(V_n) = v_p\left(\frac{U_{2n}}{U_n}\right) = v_p(U_{2n}) - v_p(U_n)$. □

Next, we give some old and new lemmas that are needed in the proof of main theorems.

Lemma 5. *Let $n \geq 1$ and $(a, b) = 1$. If $p \mid U_n$ or $p \mid V_n$, then $p \nmid b$. Consequently, $(U_n, b) = (V_n, b) = 1$ for all $n \geq 1$.*

Proof. The case for U_n is already given in [8, Lemma 7]. So suppose by way of contradiction that $p \mid V_n$ and $p \mid b$. Since $V_n = aV_{n-1} + bV_{n-2}$ and $(a, b) = 1$, we obtain $p \mid V_{n-1}$. Repeating this argument, we see that $p \mid V_m$ for $1 \leq m \leq n$. In particular, $p \mid V_1 = a$ contradicting $(a, b) = 1$. So if $p \mid V_n$, then $p \nmid b$, and the proof is complete. □

Lemma 6. [8, Lemma 8] *Let a and b be odd, $(a, b) = 1$, and $v_2(U_6) \geq v_2(U_3) + 2$. Then $v_2(U_3) = 1$.*

For convenience, we also calculate the 2-adic valuation of U_n and V_n as follows.

Lemma 7. *Assume that a is odd, b is even, and $n \geq 1$. Then $v_2(U_n) = v_2(V_n) = 0$.*

Proof. Since $U_1 = 1$ and $U_2 = a$ are odd, and $U_r = aU_{r-1} + bU_{r-2} \equiv U_{r-1} \pmod 2$ for $r \geq 3$, it follows by induction that U_n is odd. Since $V_n = \frac{U_{2n}}{U_n}$, V_n is also odd. This proves this lemma. □

Lemma 8. *Assume that a is even, b is odd, and $n \geq 1$. Then*

$$v_2(U_n) = \begin{cases} v_2(n) + v_2(a) - 1 & \text{if } 2 \mid n, \\ 0 & \text{if } 2 \nmid n, \end{cases}$$

$$v_2(V_n) = \begin{cases} 1 & \text{if } 2 \mid n, \\ v_2(a) & \text{if } 2 \nmid n, \end{cases}$$

Proof. Since $2 \mid D$, we obtain by Lemma 3 that for each $n \in \mathbb{N}$, $v_2(U_n) = v_2(n) + v_2(U_2) - 1$ if $2 \mid n$ and $v_2(U_n) = 0$ if $2 \nmid n$. Since $U_2 = a$, the formula for $v_2(U_n)$ is verified. Then $v_2(V_n)$ can be obtained from a straightforward calculation and the fact that $V_n = \frac{U_{2n}}{U_n}$. This completes the proof. □

Lemma 9. *Assume that a and b are odd, and $n \geq 1$. Then*

$$v_2(U_n) = \begin{cases} v_2(n) + v_2(U_6) - 1 & \text{if } n \equiv 0 \pmod 6, \\ v_2(U_3) & \text{if } n \equiv 3 \pmod 6, \\ 0 & \text{if } n \not\equiv 0 \pmod 3, \end{cases}$$

$$v_2(V_n) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 6, \\ v_2(U_6) - v_2(U_3) & \text{if } n \equiv 3 \pmod 6, \\ 0 & \text{if } n \not\equiv 0 \pmod 3, \end{cases}$$

Proof. Since U_1 and U_2 are odd, and $U_3 = a^2 + b$ is even, we have $\tau(2) = 3$. In addition, $2 \nmid D$. Furthermore, $3 \mid n$ and $2 \mid n$ if and only if $n \equiv 0 \pmod 6$; $3 \mid n$ and $2 \nmid n$ if and only if $n \equiv 3 \pmod 6$. Then applying Lemma 3 and the fact that $V_n = \frac{U_{2n}}{U_n}$, we obtain the desired result. □

3. Main Results

We begin with the simplest theorem of this paper.

Theorem 10. *Assume that $k, m, n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $(a, b) = 1$, and m is odd. Then*

(i) *if $V_n^k \mid m$, then $V_n^{k+1} \mid V_{nm}$;*
(ii) *if $V_n^k \parallel m$, then $V_n^{k+1} \parallel V_{nm}$;*
(iii) *if $V_n^k \mid V_{nm}$, then $V_n^{k-1} \mid m$;*
(iv) *if $V_n^k \parallel V_{nm}$, then $V_n^{k-1} \parallel m$.*

Proof. We use Lemma 5 without reference. For (i), assume that $V_n^k \mid m$. Since m is odd, V_n is also odd, and so $v_2(V_n^{k+1}) = 0$. If $p > 2$ and $p \mid V_n$, then $p \nmid b$ and we obtain by Lemma 4 that

$$\begin{aligned} v_p(V_{nm}) &= v_p(mn) + v_p(U_{\tau(p)}) \\ &= v_p(m) + v_p(n) + v_p(U_{\tau(p)}) \\ &\geq v_p(V_n^k) + v_p(V_n) = v_p(V_n^{k+1}). \end{aligned}$$

Therefore $v_p(V_{nm}) \geq v_p(V_n^{k+1})$ for all primes p dividing V_n. This implies $V_n^{k+1} \mid V_{nm}$.

For (ii), assume that $V_n^k \parallel m$. By (i), it is enough to show that $V_n^{k+2} \nmid V_{nm}$. Since $V_n^{k+1} \nmid m$, there exists a prime p dividing V_n such that $v_p(V_n^{k+1}) > v_p(m)$. Here we remark that the letter p in the proof of (i) and in the proof of (ii) may be different or may be the same. We believe that there is no ambiguity since (i) is already done. Now since $V_n^k \mid m$ and m is odd, V_n is also odd, and so $v_2(V_n^{k+1}) = v_2(m) = 0$. Therefore p is odd. By Lemma 4, we obtain

$$\begin{aligned} v_p(V_{nm}) = v_p(nm) + v_p(U_{\tau(p)}) &= v_p(m) + v_p(n) + v_p(U_{\tau(p)}) \\ &= v_p(m) + v_p(V_n) < v_p(V_n^{k+1}) + v_p(V_n) = v_p(V_n^{k+2}). \end{aligned}$$

This shows that $V_n^{k+2} \nmid V_{nm}$, as required.

For (iii), assume that $V_n^k \mid V_{nm}$. We show that $v_p(V_n^{k-1}) \leq v_p(m)$ for all primes p dividing V_n. If p is odd and $p \mid V_n$, then we apply Lemma 4 to obtain that

$$\begin{aligned} v_p(V_n) + v_p(V_n^{k-1}) = v_p(V_n^k) \leq v_p(V_{nm}) &= v_p(nm) + v_p(U_{\tau(p)}) \\ &= v_p(m) + v_p(n) + v_p(U_{\tau(p)}) \\ &= v_p(m) + v_p(V_n), \end{aligned}$$

and so $v_p(V_n^{k-1}) \leq v_p(m)$. It remains to show that $v_2(V_n^{k-1}) \leq v_2(m)$. If a is odd and b is even, then it follows from Lemma 7 that $v_2(V_n^{k-1}) = 0 \leq v_2(m)$. Recall that $(a, b) = 1$, so a and b cannot be both even. So we have the following two remaining cases: (a is even and b is odd) or (a and b are odd).

Case 1 a is even and b is odd. We will show that k must be 1, and so $v_2(V_n^{k-1}) = 0 \leq v_2(m)$. If $2 \mid n$, then we apply Lemma 8 and the assumption that $V_n^k \mid V_{nm}$ to obtain

$$1 \leq k = v_2(V_n^k) \leq v_2(V_{nm}) = 1.$$

Similarly, if $2 \nmid n$, then $2 \nmid nm$ and we can use Lemma 8 again to obtain

$$kv_2(a) = v_2(V_n^k) \leq v_2(V_{nm}) = v_2(a).$$

In any case, $k = 1$, as asserted.

Case 2 a and b are odd. We use Lemma 9 in this case. If $n \not\equiv 0 \pmod 3$, then $v_2(V_n^{k-1}) = 0 \leq v_2(m)$. If $n \equiv 0 \pmod 6$, then $nm \equiv 0 \pmod 6$, and so $k = v_2(V_n^k) \leq v_2(V_{nm}) = 1$; thus $v_2(V_n^{k-1}) = 0 \leq v_2(m)$. We now suppose $n \equiv 3 \pmod 6$. Since m is odd, $nm \equiv 3 \pmod 6$. Therefore

$$k(v_2(U_6) - v_2(U_3)) = v_2(V_n^k) \leq v_2(V_{nm}) = v_2(U_6) - v_2(U_3).$$

So $k = 1$ and thus $v_2(V_n^{k-1}) = 0 \leq v_2(m)$. Hence $v_p(V_n^{k-1}) \leq v_p(m)$ for all primes p dividing V_n, as desired. This proves (iii).

For (iv), assume that $V_n^k \parallel V_{nm}$. By (iii), we have $V_n^{k-1} \mid m$. If $V_n^k \mid m$, then we obtain by (i) that $V_n^{k+1} \mid V_{nm}$ which contradicts $V_n^k \parallel V_{nm}$. Therefore $V_n^{k-1} \parallel m$. This completes the proof. □

In the next example, we show that a version of Theorem 10 where m is even does not exist.

Example 11. Let $k, m, n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $(a, b) = 1$, and m is even. Let p be an odd prime dividing V_n. By Lemma 4, we have $p \nmid D$, $\tau(p) \nmid n$ and $\tau(p) \mid 2n$. Since m is even and $\tau(p) \mid 2n$, we obtain $\tau(p) \mid mn$. By Lemma 4, we have $p \nmid V_{nm}$, and so $V_n \nmid V_{nm}$. This shows that m in Theorem 10 cannot be even.

Remark 12. The argument in Example 11 works provided that there exists an odd prime p dividing V_n. The case $V_n = 2^k$ for some $k \in \mathbb{N} \cup \{0\}$ may occur but it is very rare. For example, when $a = b = 1$, we know from the result of Bugeaud, Mignotte, and Siksek [2] that V_n is 1 or is a power of 2 if and only if $n = 0, 1, 3$. Therefore we do not consider this rare case in our theorems.

We now have the exact divisibility results for U_n and V_n separately. In the next theorem, we consider them together. In other words, we investigate the relations of the type $V_n^c \mid m$ implies $V_n^d \mid U_{nm}$; and $V_n^c \parallel U_{nm}$ implies $V_n^d \parallel m$. We divide the results into 5 Theorems according to the parities of a and b. From this point on, we apply Lemma 5 without reference.

Theorem 13. *Suppose that $k, m, n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $(a, b) = 1$, a is odd, b is even, and m is even. Then*

(i) *if $V_n^k \mid m$, then $V_n^{k+1} \mid U_{nm}$;*
(ii) *if $V_n^k \parallel m$, then $V_n^{k+1} \parallel U_{nm}$;*
(iii) *if $V_n^{k+1} \mid U_{nm}$, then $V_n^k \mid m$;*
(iv) *if $V_n^{k+1} \parallel U_{nm}$, then $V_n^k \parallel m$.*

Proof. For (i), assume that $V_n^k \mid m$. We show that $v_p(V_n^{k+1}) \le v_p(U_{nm})$ for all primes p dividing V_n. By Lemma 7, we have $v_2(V_n) = 0$. So let p be an odd prime dividing V_n. By Lemma 4, $p \nmid D$, $\tau(p) \nmid n$, and $\tau(p) \mid 2n$. Then $\tau(p) \mid nm$. By Lemmas 3 and 4, we obtain

$$\begin{aligned} v_p(U_{nm}) &= v_p(nm) + v_p(U_{\tau(p)}) = v_p(m) + v_p(n) + v_p(U_{\tau(p)}) \ge v_p(V_n^k) + v_p(n) + v_p(U_{\tau(p)}) \\ &= v_p(V_n^k) + v_p(V_n) = v_p(V_n^{k+1}), \text{ as required.} \end{aligned}$$

For (ii), assume that $V_n^k \parallel m$. By (i), it is enough to show that $V_n^{k+2} \nmid U_{nm}$. Since $V_n^{k+1} \nmid m$, there exists a prime p such that $v_p(V_n^{k+1}) > v_p(m)$. By Lemma 7, $v_2(V_n^{k+1}) = 0$, and so $p \neq 2$. Since $p \mid V_n$, we know that $p \nmid D$ and $\tau(p) \mid nm$. Therefore we obtain by Lemmas 3 and 4 that

$$\begin{aligned} v_p(U_{nm}) &= v_p(nm) + v_p(U_{\tau(p)}) = v_p(m) + v_p(n) + v_p(U_{\tau(p)}) = v_p(m) + v_p(V_n) \\ &< v_p(V_n^{k+1}) + v_p(V_n) = v_p(V_n^{k+2}), \text{ as desired.} \end{aligned}$$

For (iii), assume that $V_n^{k+1} \mid U_{nm}$. By Lemma 7, $v_2(m) \ge 0 = v_2(V_n^k)$. If p is odd and $p \mid V_n$, then we apply Lemmas 3 and 4 again to obtain

$$\begin{aligned} v_p(V_n) + v_p(V_n^k) = v_p(V_n^{k+1}) \le v_p(U_{nm}) &= v_p(nm) + v_p(U_{\tau(p)}) \\ &= v_p(m) + v_p(n) + v_p(U_{\tau(p)}) \\ &= v_p(m) + v_p(V_n). \end{aligned}$$

This shows that $v_p(V_n^k) \le v_p(m)$ for every prime p dividing V_n. So $V_n^k \mid m$.

For (iv), suppose $V_n^{k+1} \parallel U_{nm}$. By (iii), it is enough to show that $V_n^{k+1} \nmid m$. If $V_n^{k+1} \mid m$, we apply (i) to obtain $V_n^{k+2} \mid U_{nm}$ contradicting $V_n^{k+1} \parallel U_{nm}$. Therefore the proof is complete. □

We show in Example 18 that m in Theorems 13 to 17 cannot be odd.

Theorem 14. *Assume that $k, m, n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $(a, b) = 1$, a is even, b is odd and m is even. Let*

$$t = \min(\{v_2(n) + v_2(a) - 2\} \cup \{y_p - k \mid p \text{ is an odd prime factor of } V_n\}) \text{ and}$$
$$y_p = \left\lfloor \frac{v_p(m)}{v_p(V_n)} \right\rfloor \text{ for each odd prime } p \text{ dividing } V_n.$$

Then

(i) *if $V_n^k \mid m$ and $2 \mid n$, then $V_n^{k+1} \mid U_{nm}$;*
if $V_n^k \mid m$ and $2 \nmid n$, then $\frac{V_n^{k+1}}{2} \mid U_{nm}$;
if $V_n^k \mid m$, $2 \nmid n$, and $v_2(m) \geq v_2(V_n^k) + 1$, then $V_n^{k+1} \mid U_{nm}$;
if $V_n^k \mid m$, $2 \mid n$, and $\frac{V_n^{k+1}}{2} \mid m$, then $t \geq 0$, $v_2(m) \geq k$, and $V_n^{k+t+1} \mid U_{nm}$;

(ii) *if $V_n^k \parallel m$, $2 \mid n$ and $\frac{V_n^{k+1}}{2} \nmid m$, then $V_n^{k+1} \parallel U_{nm}$;*

(iii) *if $V_n^k \parallel m$, $2 \mid n$ and $\frac{V_n^{k+1}}{2} \mid m$, then $V_n^{k+t+1} \parallel U_{nm}$;*

(iv) *if $V_n^k \parallel m$, $2 \nmid n$ and $v_2(m) = v_2(V_n^k)$, then $V_n^k \parallel U_{nm}$;*

(v) *if $V_n^k \parallel m$, $2 \nmid n$ and $v_2(m) \geq v_2(V_n^k) + 1$, then $V_n^{k+1} \parallel U_{nm}$.*

Proof. For (i), assume that $V_n^k \mid m$. If p is an odd prime and $p \mid V_n$, then $p \nmid D$, $\tau(p) \mid nm$, and we can apply Lemmas 3 and 4, to obtain

$$\begin{aligned} v_p(U_{nm}) &= v_p(nm) + v_p(U_{\tau(p)}) \\ &= v_p(m) + v_p(n) + v_p(U_{\tau(p)}) \\ &\geq v_p(V_n^k) + v_p(V_n) = v_p(V^{k+1}). \end{aligned}$$

From this point on, we sometimes use Lemmas 3 and 4 without reference. Next, we consider $v_2(V_n^{k+1})$ and $v_2(U_{nm})$. If $2 \mid n$, then we apply Lemma 8 to obtain

$$\begin{aligned} v_2(U_{nm}) = v_2(nm) + v_2(a) - 1 &= v_2(m) + v_2(n) + v_2(a) - 1 \\ &\geq v_2(V_n^k) + v_2(n) + v_2(a) - 1 \\ &\geq v_2(V_n^k) + 1 = v_2(V_n^k) + v_2(V_n) = v_2(V_n^{k+1}). \end{aligned}$$

This implies the first part of (i). Since m is even, $2 \mid nm$. So if $2 \nmid n$, then we can still apply Lemma 8 to obtain

$$\begin{aligned} v_2(U_{nm}) &= v_2(nm) + v_2(a) - 1 \\ &= v_2(m) + v_2(a) - 1 \\ &\geq v_2(V_n^k) + v_2(a) - 1 = v_2(V_n^k) + v_2(V_n) - 1 = v_2\left(\frac{V_n^{k+1}}{2}\right). \end{aligned} \tag{3.1}$$

This implies the second part of (i). For the third part of (i), we assume that $2 \nmid n$ and $v_2(m) \geq v_2(V_n^k) + 1$, and then we repeat the argument used in the second part to obtain

$$v_2(U_{nm}) = v_2(m) + v_2(a) - 1 \geq v_2(V_n^k) + v_2(a) = v_2(V_n^{k+1}).$$

Therefore $v_p(U_{nm}) \geq v_p(V_n^{k+1})$ for all primes p, which implies the desired result. Next, we prove the last part of (i). Assume that $V_n^k \mid m$, $2 \mid n$, and $\frac{V_n^{k+1}}{2} \mid m$. Since a and n are even, $v_2(n) + v_2(a) - 2 \geq 0$. In addition, $v_p(m) \geq v_p(V_n^k) = kv_p(V_n)$, and so $y_p \geq k$. Therefore $t \geq 0$ and $t + 1 \leq v_2(n) + v_2(a) - 1$. By Lemma 8, we have $v_p(V_n) = 1$, and therefore $v_p(m) \geq k$ and

$$v_2(U_{nm}) = v_2(nm) + v_2(a) - 1 = v_2(m) + v_2(n) + v_2(a) - 1 \geq k + t + 1 = v_2(V_n^{k+t+1}).$$

If p is an odd prime and $p \mid V_n$, then

$$\begin{aligned} v_p(U_{nm}) &= v_p(m) + v_p(n) + v_p(U_{\tau(p)}) = v_p(m) + v_p(V_n) \geq y_p v_p(V_n) + v_p(V_n) \\ &= (y_p + 1)v_p(V_n) \geq (k + t + 1)v_p(V_n) = v_p(V_n^{k+t+1}). \end{aligned}$$

1 Hence $v_p(U_{nm}) \geq v_p(V_n^{k+t+1})$ for all primes p dividing V_n. Thus $V_n^{k+t+1} \mid U_{nm}$, as desired.

Next, we prove (ii). Assume that $V_n^k \parallel m$, $2 \mid n$ and $\frac{V_n^{k+1}}{2} \nmid m$. By (i), it is enough to show that $V_n^{k+2} \nmid U_{nm}$. By Lemma 8, we know that $v_2(V_n) = 1$. Then $v_2(m) \geq v_2(V_n^k) = v_2\left(\frac{V_n^{k+1}}{2}\right)$. Since $\frac{V_n^{k+1}}{2} \nmid m$, there exists an odd prime p dividing V_n such that $v_p(V_n^{k+1}) > v_p(m)$. Then $p \nmid D$, $\tau(p) \mid nm$, and

$$\begin{aligned} v_p(V_n^{k+2}) = v_p(V_n^{k+1}) + v_p(V_n) &> v_p(m) + v_p(V_n) \\ &= v_p(m) + v_p(n) + v_p(U_{\tau(p)}) \\ &= v_p(U_{nm}). \end{aligned}$$

2 This implies $V_n^{k+2} \nmid U_{nm}$.

For (iii), assume that $V_n^k \parallel m$, $2 \mid n$, and $\frac{V_n^{k+1}}{2} \mid m$. By (i), we obtain $t \geq 0$, $v_2(m) \geq k$, and $V_n^{k+t+1} \mid U_{nm}$. So it remains to show that $V_n^{k+t+2} \nmid U_{nm}$. We first observe that since $\frac{V_n^{k+1}}{2} \mid m$, we obtain $v_p(V_n^{k+1}) \leq v_p(m)$ for every odd prime p. If $v_2(m) \geq k + 1$, then $v_2(m) \geq v_2(V_n^{k+1})$ which implies $V_n^{k+1} \mid m$ contradicting the assumption $V_n^k \parallel m$. Therefore $v_2(m) = k$. Next, we show that $V_n^{k+t+2} \nmid U_{nm}$. If $t = y_p - k$ for some odd prime p dividing V_n, then we apply Lemmas 3 and 4 to obtain

$$\begin{aligned} v_p(U_{nm}) &= v_p(nm) + v_p(U_{\tau(p)}) \\ &= v_p(m) + v_p(V_n) = \left(\frac{v_p(m)}{v_p(V_n)} + 1\right) v_p(V_n) \\ &< (y_p + 2)v_p(V_n) = (k + t + 2)v_p(V_n) = v_p(V_n^{k+t+2}), \end{aligned}$$

and so $V_n^{k+t+2} \nmid U_{nm}$. If $t = v_2(n) + v_2(a) - 2$, then we obtain by Lemma 8 that

$$v_2(U_{nm}) = v_2(nm) + v_2(a) - 1 = v_2(m) + v_2(n) + v_2(a) - 1 = k + t + 1 < v_2(V_n^{k+t+2}),$$

3 and so $V_n^{k+t+2} \nmid U_{nm}$. This proves (iii).

Next, we prove (iv). Assume that $V_n^k \parallel m$, $2 \nmid n$ and $v_2(m) = v_2(V_n^k)$. By (i), we have $\frac{V_n^{k+1}}{2} \mid U_{nm}$. To show that $V_n^k \mid U_{nm}$, it suffices to prove that $v_2(V_n^k) \leq v_2(U_{nm})$. Recall from (3.1) in the proof of the second part of (i) that

$$v_2(U_{nm}) = v_2(m) + v_2(a) - 1 = v_2(V_n^k) + v_2(a) - 1 \geq v_2(V_n^k),$$

and

$$v_2(U_{nm}) = v_2(m) + v_2(a) - 1 = v_2(V_n^k) + v_2(V_n) - 1 < v_2(V_n^{k+1}).$$

So $V_n^k \mid U_{nm}$ and $V_n^{k+1} \nmid U_{nm}$. Thus $V_n^k \parallel U_{nm}$.

For (v), assume that $V_n^k \parallel m$, $2 \nmid n$, and $v_2(m) \geq v_2(V_n^k)+1$. By (i), it suffices to show that $V_n^{k+2} \nmid U_{nm}$. Since $V_n^{k+1} \nmid m$, there exists a prime p dividing V_n such that $v_p(V_n^{k+1}) > v_p(m)$. If $p = 2$, then we obtain by Lemma 8 that

$$v_2(U_{nm}) = v_2(m) + v_2(a) - 1 < v_2(V_n^{k+1}) + v_2(V_n) - 1 < v_2(V_n^{k+2}),$$

and so $V_n^{k+2} \nmid U_{nm}$. If $p > 2$, then we obtain

$$v_p(U_{nm}) = v_p(nm) + v_p(U_{\tau(p)}) = v_p(m) + v_p(V_n) < v_p(V_n^{k+1}) + v_p(V_n) = v_p(V_n^{k+2}),$$

which implies $V_n^{k+2} \nmid U_{nm}$. This completes the proof. □

From this point on, we apply Lemmas 3, 4, 5, and 8 without reference.

Theorem 15. *Suppose that $k, m, n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $(a, b) = 1$, a is even, b is odd, and m is even. Then*

(i) *for all odd primes p, if $v_p(V_n^{k+1}) \leq v_p(U_{nm})$, then $v_p(V_n^k) \leq v_p(m)$;*
(ii) *if $V_n^{k+1} \mid U_{nm}$ and $2 \mid n$, then $V_n^{\min(k, v_2(m))} \mid m$;*
if $V_n^{k+1} \parallel U_{nm}$ and $2 \mid n$, then $V_n^{\min(k, v_2(m))} \parallel m$;
(iii) *if $V_n^{k+1} \mid U_{nm}$ and $2 \nmid n$, then $V_n^k \mid m$;*
(iv) *if $V_n^{k+1} \parallel U_{nm}$, $2 \nmid n$ and $\frac{V_n^{k+2}}{2} \nmid U_{nm}$, then $V_n^k \parallel m$;*
(v) *if $V_n^{k+1} \parallel U_{nm}$, $2 \nmid n$, and $\frac{V_n^{k+2}}{2} \mid U_{nm}$, then $V_n^{k+1} \parallel m$.*

Proof. For (i), assume that p is an odd prime and $v_p(V_n^{k+1}) \leq v_p(U_{nm})$. If $p \mid V_n$, then

$$\begin{aligned} v_p(V_n) + v_p(V_n^k) = v_p(V_n^{k+1}) \leq v_p(U_{nm}) &= v_p(nm) + v_p(U_{\tau(p)}) \\ &= v_p(m) + v_p(n) + v_p(U_{\tau(p)}) \\ &= v_p(m) + v_p(V_n), \end{aligned}$$

which implies (i). By (i), we only need to consider the 2-adic valuation in the proofs of (ii), (iii), (iv), and (v).

For (ii), assume that $V_n^{k+1} \mid U_{nm}$ and $2 \mid n$. For convenience, let $c = \min(k, v_2(m))$. If $v_2(m) \geq k$, then $v_2(V_n^k) = k \leq v_2(m)$, and so $V_n^k \mid m$. If $v_2(m) < k$, then $v_2(V_n^{v_2(m)}) = v_2(m)$ and $v_p(V_n^{v_2(m)}) \leq v_p(V_n^k) \leq v_p(m)$ for all odd primes p, and therefore $V_n^{v_2(m)} \mid m$. In any case, we obtain $V_n^c \mid m$. This proves the first part of (ii). Suppose further that $V_n^{k+1} \parallel U_{nm}$ but $V_n^{c+1} \mid m$. Then

$$v_2(m) \geq v_2(V_n^{c+1}) = \min(k, v_2(m)) + 1,$$

which implies $c = k$. Then $V_n^{k+1} = V_n^{c+1} \mid m$. By (i) of Theorem 14, we obtain $V_n^{k+2} \mid U_{nm}$ contradicting $V_n^{k+1} \parallel U_{nm}$. This completes the proof of (ii).

For (iii), assume that $V_n^{k+1} \mid U_{nm}$ and $2 \nmid n$. Then

$$v_2(a) + v_2(V_n^k) = v_2(V_n^{k+1}) \leq v_2(U_{nm}) = v_2(nm) + v_2(a) - 1 = v_2(m) + v_2(a) - 1.$$

Therefore $v_2(V_n^k) < v_2(m)$, and so $V_n^k \mid m$.

For (iv), assume that $V_n^{k+1} \parallel U_{nm}$, $2 \nmid n$, and $\frac{V_n^{k+2}}{2} \nmid U_{nm}$. By (iii), $V_n^k \mid m$. If $V_n^{k+1} \mid m$, then we obtain from (i) of Theorem 14 that $\frac{V_n^{k+2}}{2} \mid U_{nm}$, a contradiction. So $V_n^k \parallel m$.

For (v), assume that $V_n^{k+1} \parallel U_{nm}$, $2 \nmid n$, and $\frac{V_n^{k+2}}{2} \mid U_{nm}$. If p is odd, then $v_p(V_n^{k+2}) \le v_p(U_{nm})$, and so we obtain by (i) that $v_p(V_n^{k+1}) \le v_p(m)$. In addition,

$$v_2(V_n^{k+1}) + v_2(a) - 1 = v_2(V_n^{k+2}) - 1 \le v_2(U_{nm}) = v_2(nm) + v_2(a) - 1 = v_2(m) + v_2(a) - 1,$$

and so $v_2(V_n^{k+1}) \le v_2(m)$. Therefore $V_n^{k+1} \mid m$. If $V_n^{k+2} \mid m$, we obtain from (i) of Theorem 14 that $\frac{V_n^{k+3}}{2} \mid U_{nm}$, which implies $V_n^{k+2} \mid U_{nm}$ contradicting $V_n^{k+1} \parallel U_{nm}$. Therefore $V_n^{k+1} \parallel m$ and the proof is complete. □

Theorem 16. *Suppose that $k, m, n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $(a, b) = 1$, a and b are odd, and m is even. Let $c = v_2(U_6) - 1$,*

$$t = \min(\{v_2(n) + c - 1\} \cup \{y_p - k \mid p \text{ is an odd prime factor of } V_n\}),$$
$$s = \min(\{c - 1\} \cup \{y_p - k \mid p \text{ is an odd prime factor of } V_n\}), \text{ and}$$
$$y_p = \left\lfloor \frac{v_p(m)}{v_p(V_n)} \right\rfloor \text{ for each odd prime } p \text{ dividing } V_n.$$

Then

(i) *if $V_n^k \mid m$, then $V_n^{k+1} \mid U_{nm}$;*
(ii) *if $V_n^k \parallel m$ and $n \not\equiv 0 \pmod 3$, then $V_n^{k+1} \parallel U_{nm}$;*
(iii) *if $V_n^k \parallel m$, $n \equiv 0 \pmod 6$ and $\frac{V_n^{k+1}}{2} \nmid m$, then $V_n^{k+1} \parallel U_{nm}$;*
(iv) *if $V_n^k \mid m$, $n \equiv 0 \pmod 6$, and $\frac{V_n^{k+1}}{2} \mid m$, then $t \ge 0$ and $V_n^{k+t+1} \mid U_{nm}$;*
if $V_n^k \parallel m, n \equiv 0 \pmod 6$ and $\frac{V_n^{k+1}}{2} \mid m$, then $V_n^{k+t+1} \parallel U_{nm}$;
(v) *if $V_n^k \parallel m$, $n \equiv 3 \pmod 6$, $2 \parallel a^2 + 3b$ and $\frac{V_n^{k+1}}{2} \nmid m$, then $V_n^{k+1} \parallel U_{nm}$;*
(vi) *if $V_n^k \mid m$, $n \equiv 3 \pmod 6$, $2 \parallel a^2 + 3b$, and $\frac{V_n^{k+1}}{2} \mid m$, then $s \ge 0$ and $V_n^{k+s+1} \mid U_{nm}$;*
if $V_n^k \parallel m$, $n \equiv 3 \pmod 6$, $2 \parallel a^2 + 3b$ and $\frac{V_n^{k+1}}{2} \mid m$, then $V_n^{k+s+1} \parallel U_{nm}$;
(vii) *if $V_n^k \parallel m$, $n \equiv 3 \pmod 6$, $4 \mid a^2 + 3b$ and $\frac{V_n^{k+1}}{2^c} \nmid m$, then $V_n^{k+1} \parallel U_{nm}$;*
(viii) *if $V_n^k \mid m$, $n \equiv 3 \pmod 6$, $4 \mid a^2 + 3b$ and $\frac{V_n^{k+1}}{2^c} \mid m$, then $V_n^{k+2} \mid 2^c U_{nm}$;*
if $V_n^k \parallel m$, $n \equiv 3 \pmod 6$, $4 \mid a^2 + 3b$ and $\frac{V_n^{k+1}}{2^c} \mid m$, then $V_n^{k+2} \parallel 2^c U_{nm}$.

Proof. As usual, to prove that $V_n^d \mid U_{nm}$, we show that $v_p(V_n^d) \le v_p(U_{nm})$ for all primes p dividing V_n. Similarly, if we would like to prove that $V_n^d \nmid U_{nm}$, then we show that $v_p(V_n^d) > v_p(U_{nm})$ for some prime p. If p is odd, then we apply Lemmas 3 and 4; if $p = 2$, then we use Lemma 9; and we will do this without further reference. For (i), assume that $V_n^k \mid m$. If p is odd and $p \mid V_n$, then

$$v_p(U_{nm}) = v_p(nm) + v_p(U_{\tau(p)}) = v_p(m) + v_p(n) + v_p(U_{\tau(p)}) \ge v_p(V_n^k) + v_p(V_n) = v_p(V^{k+1}).$$

So it remains to show that $v_2(U_{nm}) \ge v_2(V_n^{k+1})$. If $n \not\equiv 0 \pmod 3$, then $v_2(V_n^{k+1}) = 0 \le v_2(U_{nm})$. So suppose that $n \equiv 0 \pmod 3$. Then $nm \equiv 0 \pmod 6$ and so

$$v_2(U_{nm}) = v_2(nm) + v_2(U_6) - 1 \ge v_2(V_n^k) + v_2(n) + v_2(U_6) - 1. \tag{3.2}$$

Since $U_3 = a^2 + b$ is even and $U_6 = a(a^2 + 3b)U_3$, we know that $v_2(U_3) \geq 1$ and $v_2(U_6) \geq 1$. So if $n \equiv 0 \pmod 6$, then $v_2(n) \geq 1$ and (3.2) implies that

$$v_2(U_{nm}) \geq v_2(V_n^k) + v_2(U_6) \geq v_2(V_n^k) + v_2(V_n) = v_2(V_n^{k+1}).$$

If $n \equiv 3 \pmod 6$, then (3.2) implies

$$v_2(U_{nm}) \geq v_2(V_n^k) + v_2(U_6) - 1 \geq v_2(V_n^k) + v_2(U_6) - v_2(U_3) = v_2(V_n^{k+1}).$$

1 In any case, $v_2(U_{nm}) \geq v_2(V_n^{k+1})$. This proves (i).

For (ii), assume that $V_n^k \parallel m$ and $n \not\equiv 0 \pmod 3$. By (i), it is enough to show that $V_n^{k+2} \nmid U_{nm}$. Since $V_n^{k+1} \nmid m$, there exists a prime p dividing V_n such that $v_p(V_n^{k+1}) > v_p(m)$. Since $v_2(V_n^{k+1}) = 0$, we see that $p \neq 2$. Then

$$v_p(U_{nm}) = v_p(nm) + v_p(U_{\tau(p)}) = v_p(m) + v_p(n) + v_p(U_{\tau(p)}) < v_p(V_n^{k+1}) + v_p(V_n) = v_p(V_n^{k+2}), \text{ as desired.}$$

For (iii), assume that $V_n^k \parallel m$, $n \equiv 0 \pmod 6$, and $\frac{V_n^{k+1}}{2} \nmid m$. By (i), it is enough to show that $V_n^{k+2} \nmid U_{nm}$. Since $\frac{V_n^{k+1}}{2} \nmid m$ and $v_2(\frac{V_n^{k+1}}{2}) = v_2(V_n^k) \leq v_2(m)$, we see that there exists an odd prime p dividing V_n such that $v_p(V_n^{k+1}) > v_p(m)$. Then

$$v_p(U_{nm}) = v_p(nm) + v_p(U_{\tau(p)}) = v_p(m) + v_p(V_n) < v_p(V_n^{k+1}) + v_p(V_n) = v_p(V_n^{k+2}).$$

2 Therefore $V_n^{k+2} \nmid U_{nm}$, as required.

For (iv), we first assume that $V_n^k \mid m$, $n \equiv 0 \pmod 6$, and $\frac{V_n^{k+1}}{2} \mid m$. Since $v_2(n) \geq 1$ and $v_2(U_6) \geq v_2(U_3) \geq 1$, it is not difficult to see that $t \geq 0$. If p is an odd prime dividing V_n, then

$$\begin{aligned} v_p(U_{nm}) &= v_p(nm) + v_p(U_{\tau(p)}) = v_p(m) + v_p(V_n) \\ &\geq y_p v_p(V_n) + v_p(V_n) = (y_p + 1)v_p(V_n) \\ &\geq (k + t + 1)v_p(V_n) = v_p(V_n^{k+t+1}). \end{aligned}$$

In addition,

$$\begin{aligned} v_2(U_{nm}) &= v_2(nm) + v_2(U_6) - 1 = v_2(m) + v_2(n) + v_2(U_6) - 1 \\ &\geq v_2(V_n^k) + t + 1 = k + t + 1 = v_2(V_n^{k+t+1}). \end{aligned}$$

Therefore $V_n^{k+t+1} \mid U_{nm}$. This proves the first part of (iv). Next, assume further that $V_n^k \parallel m$. It is enough to show that $V_n^{k+t+2} \nmid U_{nm}$. Recall that $y_p = \left\lfloor \frac{v_p(m)}{v_p(V_n)} \right\rfloor$, so $v_p(m) < (y_p + 1)v_p(V_n)$. So if $t = y_p - k$ for some odd prime p dividing V_n, then

$$v_p(U_{nm}) = v_p(nm) + v_p(U_{\tau(p)}) = v_p(m) + v_p(V_n) < (y_p + 2)v_p(V_n) = (k + t + 2)v_p(V_n) = v_p(V_n^{k+t+2}),$$

which implies $V_n^{k+t+2} \nmid U_{nm}$. So suppose $t = v_2(n) + v_2(U_6) - 2$. Since $\frac{V_n^{k+1}}{2} \mid m$, we see that $v_p(m) \geq v_p(V_n^{k+1})$ for all odd primes p. If $v_2(m) \geq k + 1$, then $v_2(m) \geq v_2(V_n^{k+1})$,which implies $V_n^{k+1} \mid m$ contradicting the assumption $V_n^k \parallel m$. Therefore $v_2(m) \leq k$. Then

$$v_2(U_{nm}) = v_2(nm) + v_2(U_6) - 1 = v_2(m) + v_2(n) + v_2(U_6) - 1 \leq k + t + 1 < v_2(V_n^{k+t+2}).$$

Therefore, $V_n^{k+t+2} \nmid U_{nm}$ as required.

For (v), assume that $V_n^k \parallel m$, $n \equiv 3 \pmod 6$, $2 \parallel a^2+3b$, and $\frac{V_n^{k+1}}{2} \nmid m$. By (i), it suffices to show that $V_n^{k+2} \nmid U_{nm}$. Since $U_6 = a(a^2+3b)U_3$ and $2 \parallel a^2+3b$, we obtain $v_2(V_n) = v_2(U_6) - v_2(U_3) = 1$. Since $\frac{V_n^{k+1}}{2} \nmid m$ and $v_2\left(\frac{V_n^{k+1}}{2}\right) = v_2(V_n^k) \leq v_2(m)$, there exists an odd prime p dividing V_n such that $v_p(V_n^{k+1}) > v_p(m)$. Therefore

$$v_p(U_{nm}) = v_p(nm) + v_p(U_{\tau(p)}) = v_p(m) + v_p(V_n) < v_p(V_n^{k+1}) + v_p(V_n) = v_p(V_n^{k+2}), \text{ as desired.}$$

For (vi), assume that $V_n^k \mid m$, $n \equiv 3 \pmod 6$, $2 \parallel a^2+3b$, and $\frac{V_n^{k+1}}{2} \mid m$. Since a^2+3b and U_3 are even, and $U_6 = a(a^2+3b)U_3$, we have $v_2(U_6) - 2 \geq 0$. Since $V_n^k \mid m$, we have $y_p \geq k$ for all odd primes p dividing V_n. Therefore $s \geq 0$. By the same argument as in the proof of (v), we obtain $v_2(V_n) = 1$. In addition, $v_2(m) \geq v_2(V_n^k) = k$ and $v_p(V_n^{k+1}) = v_p(\frac{V_n^{k+1}}{2}) \leq v_p(m)$ for every odd prime p. If $V_n^k \parallel m$ and $v_2(m) \geq k+1 = v_2(V_n^{k+1})$, then $V_n^{k+1} \mid m$ which is a contradiction. Therefore,

$$\text{if } V_n^k \parallel m, \text{ then } v_2(m) = k. \tag{3.3}$$

We will apply (3.3) later. For now, we only need to apply $v_2(m) \geq k$. We obtain

$$v_2(U_{nm}) = v_2(nm) + v_2(U_6) - 1 = v_2(m) + v_2(U_6) - 1 \geq k + v_2(U_6) - 1 \geq k + s + 1 = v_2(V_n^{k+s+1}).$$

If $p > 2$ and $p \mid V_n$, then

$$v_p(U_{nm}) = v_p(nm) + v_p(U_{\tau(p)}) = v_p(m) + v_p(V_n) \geq (y_p+1)v_p(V_n) \geq (k+s+1)v_p(V_n) = v_p(V_n^{k+s+1}).$$

This implies $V_n^{k+s+1} \mid U_{nm}$. Next, assume further that $V_n^k \parallel m$. It remains to show that $V_n^{k+s+2} \nmid U_{nm}$. By the definition of y_p, we know that $(y_p+1)v_p(V_n) > v_p(m)$. So if $s = y_p - k$ for some odd prime p dividing V_n, then

$$v_p(U_{nm}) = v_p(nm) + v_p(U_{\tau(p)}) = v_p(m) + v_p(V_n) < (y_p+2)v_p(V_n) = (k+s+2)v_p(V_n) = v_p(V_n^{k+s+2}),$$

which implies $V_n^{k+s+2} \nmid U_{nm}$. By (3.3), we know that $v_2(m) = k$. So if $s = v_2(U_6) - 2$, then

$$v_2(U_{nm}) = v_2(nm) + v_2(U_6) - 1 = v_2(m) + v_2(U_6) - 1 = k + s + 1 < v_2(V_n^{k+s+2}).$$

So in any case, $V_n^{k+s+2} \nmid U_{nm}$, as required.

For (vii), we let $c = v_2(U_6) - 1$ and assume that $V_n^k \parallel m$, $n \equiv 3 \pmod 6$, $4 \mid a^2+3b$, and $\frac{V_n^{k+1}}{2^c} \nmid m$. By (i), it is enough to show that $V_n^{k+2} \nmid U_{nm}$. Since $4 \mid a^2+3b$ and $U_6 = a(a^2+3b)U_3$, we have $v_2(U_6) \geq v_2(U_3) + 2$. By Lemma 6, we obtain $v_2(U_3) = 1$, and so $v_2(V_n) = v_2(U_6) - v_2(U_3) = v_2(U_6) - 1 = c$. Since $\frac{V_n^{k+1}}{2^c} \nmid m$ and

$$v_2\left(\frac{V_n^{k+1}}{2^c}\right) = (k+1)v_2(V_n) - v_2(V_n) = v_2(V_n^k) \leq v_2(m),$$

there exists an odd prime p dividing V_n such that $v_p(V_n^{k+1}) > v_p(m)$. Then

$$v_p(U_{nm}) = v_p(nm) + v_p(U_{\tau(p)}) = v_p(m) + v_p(V_n) < v_p(V_n^{k+1}) + v_p(V_n) = V_p(V_n^{k+2}).$$

Therefore $V_n^{k+2} \nmid U_{nm}$.

For (viii), assume that $V_n^k \mid m$, $n \equiv 3 \pmod 6$, $4 \mid a^2 + 3b$, and $\frac{V_n^{k+1}}{2^c} \mid m$. Then for each odd prime p dividing V_n, we have

$$v_p(V_n^{k+1}) = v_p\left(\frac{V_n^{k+1}}{2^c}\right) \le v_p(m). \tag{3.4}$$

Since $4 \mid a^2+3b$ and $U_6 = a(a^2+3b)U_3$, we obtain $v_2(U_6) \ge v_2(U_3)+2$. By the same argument as in the proof of (vii), we obtain $v_2(V_n) = v_2(U_6) - 1 = c$. Since $V_n^k \mid m$, we see that $v_2(m) \ge v_2(V_n^k) = kv_2(V_n)$. If $V_n^k \parallel m$ and $v_2(m) \ge (k+1)v_2(V_n)$, then $v_p(m) \ge v_p(V_n^{k+1})$ for all primes p, and so $V_n^{k+1} \mid m$, a contradiction. Therefore

$$v_2(m) \ge kv_2(V_n), \tag{3.5}$$

and

$$\text{if } V_n^k \parallel m, \text{ then } kv_2(V_n) \le v_2(m) < (k+1)v_2(V_n). \tag{3.6}$$

We will apply (3.6) later. For now (3.5) is good enough. We obtain

$$\begin{aligned}
v_2(2^c U_{nm}) &= v_2(U_6) - 1 + v_2(U_{nm}) = v_2(U_6) - 1 + v_2(nm) + v_2(U_6) - 1 \\
&= 2(v_2(U_6) - 1) + v_2(m) \\
&\ge 2(v_2(U_6) - 1) + kv_2(V_n) \\
&= 2(v_2(U_6) - 1) + k(v_2(U_6) - 1) \\
&= (k+2)(v_2(U_6) - 1) = v_2(V_n^{k+2}).
\end{aligned}$$

If $p > 2$ and $p \mid V_n$, then

$$v_p(2^c U_{nm}) = v_p(U_{nm}) = v_p(nm) + v_p(U_{\tau(p)}) = v_p(m) + v_p(V_n) \ge v_p(V_n^{k+1}) + v_p(V_n) = v_p(V_n^{k+2}),$$

where the last inequality is obtained from (3.4). This implies that $V_n^{k+2} \mid 2^c U_{nm}$. So the first part of (viii) is proved. Next, assume further that $V_n^k \parallel m$. To prove the second part, it now suffices to show that $V_n^{k+3} \nmid 2^c U_{nm}$. We have

$$\begin{aligned}
v_2(2^c U_{nm}) &= v_2(U_6) - 1 + v_2(U_{nm}) \\
&= v_2(U_6) - 1 + v_2(nm) + v_2(U_6) - 1 \\
&= 2(v_2(U_6) - 1) + v_2(m) \\
&< 2(v_2(U_6) - 1) + (k+1)(v_2(U_6) - 1) \\
&= (k+3)(v_2(U_6) - 1) = v_2(V_n^{k+3}),
\end{aligned}$$

where the inequality is obtained form (3.6) and the fact that $v_2(V_n) = v_2(U_6) - 1$. This completes the proof. □

Theorem 17. *Suppose that $k, m, n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $(a,b) = 1$, a and b are odd and m is even. Then*

(i) *for every odd prime p dividing V_n, if $v_p(V_n^{k+1}) \le v_p(U_{nm})$, then $v_p(V_n^k) \le v_p(m)$;*

(ii) *if $V_n^{k+1} \mid U_{nm}$ and $n \not\equiv 0 \pmod 3$, then $V_n^k \mid m$;*
if $V_n^{k+1} \parallel U_{nm}$ and $n \not\equiv 0 \pmod 3$, then $V_n^k \parallel m$;

(iii) *if* $V_n^{k+1} \mid U_{nm}$, $n \equiv 0 \pmod 6$, *and* $v_2(m) \geq k$, *then* $V_n^k \mid m$;
if $V_n^{k+1} \parallel U_{nm}$, $n \equiv 0 \pmod 6$, *and* $v_2(m) \geq k$, *then* $V_n^k \parallel m$;
if $V_n^{k+1} \mid U_{nm}$, $n \equiv 0 \pmod 6$, *and* $v_2(m) < k$, *then* $V_n^{v_2(m)} \parallel m$;

(iv) *if* $V_n^{k+1} \mid U_{nm}$, $n \equiv 3 \pmod 6$, $2 \parallel a^2 + 3b$, *and* $v_2(m) \geq k$, *then* $V_n^k \mid m$;
if $V_n^{k+1} \parallel U_{nm}$, $n \equiv 3 \pmod 6$, $2 \parallel a^2 + 3b$, *and* $v_2(m) \geq k$, *then* $V_n^k \parallel m$;
if $V_n^{k+1} \mid U_{nm}$, $n \equiv 3 \pmod 6$, $2 \parallel a^2 + 3b$, *and* $v_2(m) < k$, *then* $V_2^{v_2(m)} \parallel m$;

(v) *if* $V_n^{k+1} \mid U_{nm}$, $n \equiv 3 \pmod 6$, *and* $4 \mid a^2 + 3b$, *then* $V_n^k \mid m$;
if $V_n^{k+1} \parallel U_{nm}$, $n \equiv 3 \pmod 6$, *and* $4 \mid a^2 + 3b$, *then* $V_n^k \parallel m$.

Proof. We apply Lemmas 3, 4, and 9 throughout the proof without reference. For (i), assume that p is an odd prime dividing V_n and $v_p(V_n^{k+1}) \leq v_p(U_{nm})$. Then

$$v_p(V_n) + v_p(V_n^k) = v_p(V_n^{k+1}) \leq v_p(U_{nm}) \leq v_p(nm) + v_p(U_{\tau(p)}) = v_p(m) + v_p(V_n),$$

which implies (i). Therefore we only need to consider the 2-adic valuation in the proof of (ii) to (v).

For (ii), assume that $V_n^{k+1} \mid U_{nm}$ and $n \not\equiv 0 \pmod 3$. Since $v_2(V_n^k) = 0 \leq v_2(m)$, we obtain by (i) that $V_n^k \mid m$. Suppose futher that $V_n^{k+1} \parallel U_{nm}$. If $V_n^{k+1} \mid m$, then (i) of Theorem 16 implies $V_n^{k+2} \mid U_{nm}$, which contradicts $V_n^{k+1} \parallel U_{nm}$, and so $V_n^k \parallel m$.

For (iii), assume that $V_n^{k+1} \mid U_{nm}$ and $n \equiv 0 \pmod 6$.

Case 1 $v_2(m) \geq k$. Then $v_2(V_n^k) = k \leq v_2(m)$. So we obtain by (i) that $V_n^k \mid m$. If $V_n^{k+1} \parallel U_{nm}$, then we obtain by (i) of Theorem 16 that $V_n^{k+1} \nmid m$, and so $V_n^k \parallel m$. This proves (iii) in the case $v_2(m) \geq k$.

Case 2 $v_2(m) < k$. For convenience, let $d = v_2(m)$. Since $v_2(V_n^d) = d = v_2(m)$ and $v_p(V_n^d) \leq v_p(V_n^k) \leq v_p(m)$ for every odd prime p dividing V_n, we obtain $V_n^d \mid m$. If $V_n^{d+1} \mid m$, then $d+1 = v_2(V_n^{d+1}) \leq v_2(m) = d$, a contradiction. So $V_n^d \parallel m$.

For (iv), assume that $V_n^{k+1} \mid U_{nm}$, $n \equiv 3 \pmod 6$, and $2 \parallel a^2 + 3b$. Since $U_6 = a(a^2 + 3b)U_3$ and $2 \parallel a^2 + 3b$, we obtain $v_2(V_n) = v_2(U_6) - v_2(U_3) = 1$.

Case 1 $v_2(m) \geq k$. Then $v_2(V_n^k) = k \leq v_2(m)$, and so we obtain by (i) that $V_n^k \mid m$. If $V_n^{k+1} \parallel U_{nm}$, then we obtain by (i) of Theorem 16 that $V_n^k \parallel m$. This proves (iv) in the case $v_2(m) \geq k$.

Case 2 $v_2(m) < k$. For convenience, let $d = v_2(m)$. Then $v_2(V_n^d) = d = v_2(m)$ and $v_p(V_n^d) \leq v_p(V_n^k) \leq v_p(m)$. Therefore $V_n^d \mid m$. If $V_n^{d+1} \mid m$, then $d + 1 = v_2(V_n^{d+1}) \leq v_2(m) = d$, a contradiction. Therefore $V_n^d \parallel m$.

For (v), assume that $V_n^{k+1} \mid U_{nm}$, $n \equiv 3 \pmod 6$, and $4 \mid a^2 + 3b$. Since $U_6 = a(a^2 + 3b)U_3$ and $4 \mid a^2 + 3b$, we obtain $v_2(U_6) \geq v_2(U_3) + 2$. By Lemma 6, we have $v_2(U_3) = 1$. Then $v_2(V_n) = v_2(U_6) - v_2(U_3) = v_2(U_6) - 1$ and

$$v_2(V_n^k) + v_2(V_n) = v_2(V_n^{k+1}) \leq v_2(U_{nm}) = v_2(nm) + v_2(U_6) - 1 = v_2(m) + v_2(V_n).$$

So $v_2(V_n^k) \leq v_2(m)$. By (i), we obtain $V_n^k \mid m$. If $V_n^{k+1} \parallel U_{nm}$, then we obtain by (i) of Theorem 16 that $V_n^{k+1} \nmid m$, and so $V_n^k \parallel m$. This completes the proof. □

The next example shows that m in Theorems 13 to 17 is necessarily even.

Example 18. Let $k, m, n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $(a, b) = 1$ and m is odd. Let p be an odd prime dividing V_n. By Lemma 4, we have $p \nmid D$, $\tau(p) \nmid n$ and $\tau(p) \mid 2n$. Therefore $\tau(p)$ is even and $v_2(\tau(p)) = v_2(n) + 1$. So $\tau(p) \nmid nm$. By Lemma 3, $v_p(U_{nm}) = 0$. Therefore $V_n \nmid U_{nm}$. This shows that m in Theorems 13 to 17 cannot be odd.

Example 19. Let $k, m, n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $(a, b) = 1$. Let $p > 2$ and $p \mid U_n$. By Lemma 3, we have (i) $v_p(U_n) = v_p(n) + v_p(U_p) - 1$ if $p \mid D$ and $p \mid n$, and (ii) $v_p(U_n) = v_p(n) + v_p(U_{\tau(p)})$ if $p \nmid D$ and $\tau(p) \mid n$. For (i), we have $p \mid D$ and so $v_p(V_{nm}) = 0$ and $U_n \nmid V_{nm}$. For (ii), we have $\tau(p) \mid nm$ and so $v_p(V_{nm}) = 0$ and $U_n \nmid V_{nm}$. This shows that there is no interesting divisibility relation such as $U_n^k \mid V_{nm}$.

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Conflict of Interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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