PROBLEMS CONCERNING THE SUM OF DIVISORS AND GENERALIZATIONS OF PERFECT NUMBERS

By

Miss Saralee AURSUKAREE

A Thesis Submitted in Partial Fulfillment of the Requirements

for Doctor of Philosophy (MATHEMATICS)

Department of MATHEMATICS

Graduate School, Silpakorn University

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ปัญหาเกี่ยวกับผลบวกของตัวหารและนัยทั่วไปของจำนวนสมบูรณ์

โดย
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In this thesis, we conduct a study on problems concerning the sum of divisors and generalizations of perfect numbers. We determine all odd exactly 3-deficient-perfect numbers with at most two distinct prime factors.

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Saralee AURSUKAREE

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Chapter 1

Introduction

Throughout this thesis, let n, k be positive integers and let $\sigma(n)$ be the sum of all positive divisors of n. We call n a *perfect number* if it is equal to the sum of its proper divisors, that is $\sigma(n) = 2n$. For example, 6 and 28 are perfect numbers since $\sigma(6) = 1+2+3+6 = 12 = 2\times6$ and $\sigma(28) = 1+2+4+7+14+28 = 56 = 2\times28$. Euler proved in 1747 that an even perfect number is of the form $2^{p-1}(2^p-1)$, where both p and 2^p-1 are primes. The primes of the form 2^p-1 are called a *Mersenne prime* where p is prime. List of Mersenne primes is shown as A000668 in the On-Line Encyclopedia of Integer Sequences (OEIS). We do not know whether the set of Mersenne primes is finite or infinite, so we do not know whether there are infinitely many even perfect numbers. Currently only 51 even perfect numbers are known and the largest known even perfect number is $2^{82589932}(2^{82589933}-1)$[10]. On the other hand, it is not known whether an odd perfect number exists. Ochem and Rao proved in 2012 [18] that any odd perfect number is greater than 10^{1500} and it has at least 101 not necessarily distinct prime factors and that its largest prime power divisor is greater than 10^{62}.

Various authors have defined concepts that are closely related to perfect numbers. A positive integer n is called *deficient* if $\sigma(n) < 2n$ and n is called *abundant* if $\sigma(n) > 2n$. In 1965, Sierpiński [29] defined n to be *pseudoperfect* if n can be written as a sum of some (or all) of its proper divisors. In 2012, Pollack and Shevelev [21] studied a subclass of pseudoperfect numbers and introduced the concept of k-near-perfect numbers. A positive integer n is called *k-near-perfect* if n is the sum of all of its proper divisors with at most k exceptions (called *redundant divisors*). It is called *exactly k-near-perfect number* if it is a sum of all of its proper divisors with exactly k exceptions. A positive integer n is called *near-perfect* with *redundant divisor* d if d is a proper divisor of n and $\sigma(n) = 2n + d$ and n is called *quasiperfect* if $\sigma(n) = 2n + 1$. Pollack and Shevelev [21] presented an upper bound on the count of near-perfect numbers and proved that there are infinitely many exactly k-near-perfect numbers for all large k. In 2013, Ren and Chen [27] determined all near-perfect numbers with two distinct prime divisors. Tang, Ren, and Li [35] proved that there are no odd near-perfect numbers with three distinct prime divisors. In 2015, Li and Liao [15] gave two equivalent conditions of all even near-perfect numbers of the form $2^\alpha p_1 p_2$ and $2^\alpha p_1^2 p_2$ where p_1 and p_2 are odd primes with $p_1 < p_2$. The following year, Tang, Ma, and Feng [34] showed that the only odd near-perfect number with four distinct prime divisors is $173369889 = 3^4 7^2 11^2 19^2$.

A positive integer n is said to be *exactly k-deficient-perfect* if $\sigma(n) = 2n - d_1 - d_2 - \cdots - d_k$ for some distinct proper divisors $d_1, d_2, \ldots, d_k$ of n. In this case, $d_1, d_2, \ldots, d_k$ are also called deficient divisors of n. For $k = 1$, it is called *deficient-perfect*. Moreover, n is called *almost perfect* if $\sigma(n) = 2n - 1$. In 2013, Tang, Ren and Li [35] determined all deficient-perfect numbers with at most two distinct prime divisors. In 2014, Tang and Feng [33] showed that there are no odd deficient-perfect numbers with three distinct primes divisors. Recently, Chen [5] determined all odd exactly 2-deficient-perfect numbers with two distinct prime divisors. Sun and He [32] also showed that the only odd deficient-perfect number with four distinct prime divisors is $9018009 = 3^2 7^2 11^2 13^2$. In this work, we show that the only odd exactly 3-deficient-perfect number with at most two distinct prime divisors is $1521 = 3^2 \cdot 13^2$.

This thesis is composed of 3 chapters. In Chapter 1, we introduce problems concerning the sum of divisors, generalization of perfect numbers, and literature review. After this introduction, definitions, preliminaries, and lemmas concerning the sum of divisors, generalizations of perfect numbers, and p-adic valuation are described in Chapter 2. In Chapter 3, we present the main results. For related problems of the divisor functions or divisibility problems can be found in [1, 2, 3, 8, 9, 12, 13, 15, 16, 17, 19, 20, 22, 23, 24, 25, 26, 28, 31, 36].

Chapter 2
Preliminaries and Lemmas

In this chapter, we give definitions, preliminaries, and lemmas concerning the sum of divisors, generalizations of perfect numbers, and p-adic valuation.

Definition 2.1. For each positive integer n, we define $\sigma(n)$ to be the sum of all positive divisors of n.

Definition 2.2. An arithmetical function f is said to be *multiplicative* if f is not identically zero and if $f(mn) = f(m)f(n)$ for every $m, n \in \mathbb{N}$ with $(m, n) = 1$.

Theorem 2.3. *The function σ is multiplicative and satisfies*

$$\sigma(n) = \prod_{j=1}^{k} \frac{p_j^{a_j+1} - 1}{p_j - 1}$$

where $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is the canonical factorization of n.

Definition 2.4. A positive integer n is said to be *perfect* if $\sigma(n) = 2n$.

Definition 2.5. A positive integer n is said to be *k-near-perfect* if n is expressible as the sum of all its proper divisors with at most k exceptions (called *redundant divisors*). Moreover, we say that n is *exactly k-near-perfect* if n can be written as a sum of all of its divisors with exactly k exceptions that is $\sigma(n) = 2n + d_1 + d_2 \cdots + d_k$ and it is called *near-perfect* if $\sigma(n) = 2n + d$.

Definition 2.6. A positive integer n is said to be *exactly k-deficient-perfect* if $\sigma(n) = 2n - d_1 - d_2 \cdots - d_k$ for some distinct positive proper divisors $d_1, d_2, \ldots, d_k$ of n and $d_1, d_2, \ldots, d_k$ are called the *deficient divisors* of n. Furthermore, n is *k-deficient perfect* if n is perfect or n is exactly l-deficient perfect for some $l = 1, 2, \ldots, k$. In addition, a number that is 1- deficient-perfect is called *a deficient-perfect number.*

The following two lemmas concerning deficient-perfect numbers are stated and we will extend these lemmas in Chapter 3.

Lemma 2.7. *[33] Let $n = \prod_{i=1}^{t} p_i^{\alpha_i}$ be the canonical factorization of n. If n is an odd deficient-perfect number, then the exponents α_i are even for all i.*

Lemma 2.8. *[35] Let n be a prime power. If n is a deficient-perfect number, then $n = 2^\alpha$ with deficient divisor $d = 1$.*

Definition 2.9. For each positive integer n and each prime p, $v_p(n)$ denotes the p-adic valuation of n, which is defined to be the exponent of p in the canonical factorization of n.

Theorem 2.10. *For positive integers m and n, the following statements hold.*

(i) $v_p(mn) = v_p(m) + v_p(n)$.

(ii) $v_p(m + n) \geq \min\{v_p(m), v_p(n)\}$.

Chapter 3
Main Results

In this chapter, we give some lemmas and main results. As shown in Lemma 2.7, Tang and Feng showed that if n is odd and n is deficient-perfect, then n is a square. We can extend their result for exactly k-deficient-perfect numbers as follows.

Lemma 3.1. *Let n and k be positive integers. Suppose that n is odd exactly k-deficient-perfect. Then n is a square if and only if k is odd. In particular, if n is odd exactly 3-deficient-perfect, then n is a square.*

Proof. Since 1 has no proper divisor, we assume that $n > 1$ and write $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where $p_1, p_2, \ldots, p_r$ are distinct primes and $\alpha_1, \alpha_2, \ldots, \alpha_r$ are positive integers. Then there are divisors $d_1, d_2, \ldots, d_k$ of n such that

$$2n - d_1 - d_2 - \cdots - d_k = \sigma(n) = \prod_{i=1}^{r} \sigma(p_i^{\alpha_i}) = \prod_{i=1}^{r} (1 + p_i + \cdots + p_i^{\alpha_i}). \qquad (3.1)$$

Since n is odd, we obtain that d_i and p_j are odd for every $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, r$. Reducing (3.1) mod 2, we get $k \equiv \prod_{i=1}^{r} (\alpha_i + 1) \pmod 2$. We have the equivalence: k is odd $\Leftrightarrow$ α_i is even for all i $\Leftrightarrow$ n is a square. □

Moreover, we extend the result of Tang, Ren, and Li in Lemma 2.8 from deficient-perfect number to exactly k-deficient-perfect numbers.

Lemma 3.2. *Let $n \geq 2$, $k \geq 1$ be integers. If n is a prime power and n is an exactly k-deficient-perfect number, then $k = 1$ and n is a power of 2. Consequently, if n is an exactly k-deficient-perfect number and $k \geq 2$, then n has at least two distinct prime divisors. In particular, every exactly 3-deficient-perfect number has at least two distinct prime divisors.*

Proof. Suppose that $n = p^\alpha$ is an exactly k-deficient-perfect number with k deficient divisors $d_i = p^{\beta_i}$, where p is a prime and α, β_i are integers with $\alpha > \beta_1 > \beta_2 > \ldots > \beta_k \geq 0$. Then

$$\begin{aligned}
\sigma(n) &= 2n - d_1 - d_2 - \cdots - d_k \\
\sigma(p^\alpha) &= 2p^\alpha - p^{\beta_1} - p^{\beta_2} - \cdots - p^{\beta_k} \\
p^{\alpha+1} - 1 &= 2p^\alpha(p-1) - (p^{\beta_1} + p^{\beta_2} + \cdots + p^{\beta_k})(p-1) \\
(p^{\beta_1} + p^{\beta_2} + \cdots + p^{\beta_k})(p-1) - 1 &= p^{\alpha+1} - 2p^\alpha \\
(p^{\beta_1} + p^{\beta_2} + \cdots + p^{\beta_k})(p-1) - 1 &= p^\alpha(p-2). \qquad (3.2)
\end{aligned}$$

If $p \geq 3$, we have that

$$\begin{aligned}
p^\alpha \leq p^\alpha(p-2) &= (p^{\beta_1} + p^{\beta_2} + \cdots + p^{\beta_k})(p-1) - 1 \\
&\leq (p^{\alpha-1} + p^{\alpha-2} + \cdots + p^{\alpha-k})(p-1) - 1 \\
&= (p^\alpha + p^{\alpha-1} + \cdots + p^{\alpha-k+1}) - (p^{\alpha-1} + p^{\alpha-2} + \cdots + p^{\alpha-k}) - 1 \\
&= p^\alpha - p^{\alpha-k} - 1,
\end{aligned}$$

which is a contradiction. Therefore $p = 2$ and n is a power of 2. By (3.2), we obtain $d_1 + \cdots + d_k = p^{\beta_1} + p^{\beta_2} + \cdots + p^{\beta_k} = 1$, which implies $k = 1$ and $\beta_1 = 0$. □

Now, the main result of this research is presented.

Theorem 3.3. *The only odd exactly 3-deficient-perfect number with two distinct prime divisors is* $1521 = 3^2 \cdot 13^2$ *and deficient divisors are* $d_1 = 507$, $d_2 = 117$, *and* $d_3 = 39$.

Proof. Assume that $n = p_1^{2\alpha} p_2^{2\beta}$ is an exactly 3-deficient-perfect number with distinct deficient divisors d_1, d_2, and d_3, where p_1 and p_2 are two distinct primes such that $2 < p_1 < p_2$, $\alpha, \beta \geq 1$, and $d_1 > d_2 > d_3$. Then $\sigma(p_1^{2\alpha} p_2^{2\beta}) = 2p_1^{2\alpha} p_2^{2\beta} - d_1 - d_2 - d_3$, where $d_1 = p_1^{a_1} p_2^{b_1}$, $d_2 = p_1^{a_2} p_2^{b_2}$, $d_3 = p_1^{a_3} p_2^{b_3}$, $D_1 = \dfrac{n}{d_1}$, $D_2 = \dfrac{n}{d_2}$, and $D_3 = \dfrac{n}{d_3}$.

Then

$$\begin{aligned}
2 &= \frac{\sigma(p_1^{2\alpha}p_2^{2\beta})}{p_1^{2\alpha}p_2^{2\beta}} + \frac{d_1}{p_1^{2\alpha}p_2^{2\beta}} + \frac{d_2}{p_1^{2\alpha}p_2^{2\beta}} + \frac{d_3}{p_1^{2\alpha}p_2^{2\beta}} \\
&= \frac{({p_1}^{2\alpha+1}-1)({p_2}^{2\beta+1}-1)}{(p_1-1)(p_2-1)p_1^{2\alpha}p_2^{2\beta}} + \frac{1}{D_1} + \frac{1}{D_2} + \frac{1}{D_3} \\
&< \frac{{p_1}^{2\alpha+1}{p_2}^{2\beta+1}}{(p_1-1)(p_2-1)p_1^{2\alpha}p_2^{2\beta}} + \frac{1}{D_1} + \frac{1}{D_2} + \frac{1}{D_3} \\
&= \frac{p_1p_2}{(p_1-1)(p_2-1)} + \frac{1}{D_1} + \frac{1}{D_2} + \frac{1}{D_3}.
\end{aligned}$$

If $p_1 \geq 5$, then

$$2 < \frac{p_1p_2}{(p_1-1)(p_2-1)} + \frac{1}{D_1} + \frac{1}{D_2} + \frac{1}{D_3} \leq \frac{5}{4}\cdot\frac{7}{6} + \frac{1}{5} + \frac{1}{7} + \frac{1}{25} = 1.8411\ldots,$$

which is a contradiction. So $p_1 = 3$ and

$$2 = \frac{\sigma(3^{2\alpha}p_2^{2\beta})}{3^{2\alpha}p_2^{2\beta}} + \frac{1}{D_1} + \frac{1}{D_2} + \frac{1}{D_3} < \frac{3}{2}\cdot\frac{p_2}{p_2-1} + \frac{1}{D_1} + \frac{1}{D_2} + \frac{1}{D_3}. \tag{3.3}$$

If $p_2 \geq 83$, then from (3.3), we get

$$2 = \frac{\sigma(3^{2\alpha}p_2^{2\beta})}{3^{2\alpha}p_2^{2\beta}} + \frac{1}{D_1} + \frac{1}{D_2} + \frac{1}{D_3} < \frac{3}{2}\cdot\frac{83}{82} + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} = 1.9997\ldots,$$

which is not possible.

So $5 \leq p_2 \leq 79$, that is $p_2 \in \{5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79\}$.

For $p_2 \geq 11$, if $D_1 > 3$, then from (3.3), we get

$$2 = \frac{\sigma(3^{2\alpha}p_2^{2\beta})}{3^{2\alpha}p_2^{2\beta}} + \frac{1}{D_1} + \frac{1}{D_2} + \frac{1}{D_3} < \frac{3}{2}\cdot\frac{11}{10} + \frac{1}{9} + \frac{1}{11} + \frac{1}{27} = 1.8890\ldots,$$

which is impossible. So $D_1 = 3$.

For $p_2 \geq 23$, if $D_2 > 9$, then from (3.3), we get

$$2 = \frac{\sigma(3^{2\alpha}p_2^{2\beta})}{3^{2\alpha}p_2^{2\beta}} + \frac{1}{D_1} + \frac{1}{D_2} + \frac{1}{D_3} < \frac{3}{2}\cdot\frac{23}{22} + \frac{1}{3} + \frac{1}{23} + \frac{1}{27} = 1.9820\ldots,$$

which is also impossible. So $D_2 = 9$.

Consider p_2, we have the following eleven subcases.

Case 1. $p_2 \in \{47, 53, 59, 61, 67, 71, 73, 79\}$.

We have $D_1 = 3, D_2 = 9$, and $D_3 \in \{27, p_2, 81, \ldots\}$.

If $D_3 \geq p_2$, then from (3.3), we get

$$2 = \frac{\sigma(3^{2\alpha}p_2^{2\beta})}{3^{2\alpha}p_2^{2\beta}} + \frac{1}{D_1} + \frac{1}{D_2} + \frac{1}{D_3} < \frac{3}{2} \cdot \frac{47}{46} + \frac{1}{3} + \frac{1}{9} + \frac{1}{47} = 1.9983\ldots,$$

we have a contradiction. So $D_3 = 27$ implies $2\alpha \geq 3$. Then

$$\begin{aligned}
\sigma(n) &= \sigma(3^{2\alpha}{p_2}^{2\beta}) \\
&= \frac{(3^{2\alpha+1}-1)}{2}\frac{({p_2}^{2\beta+1}-1)}{p_2-1} \\
&= \frac{3^{2\alpha+1}{p_2}^{2\beta+1} - 3^{2\alpha+1} - {p_2}^{2\beta+1} + 1}{2(p_2-1)} \\
&= \frac{81 \cdot 3^{2\alpha-3}{p_2}^{2\beta+1} - 3^{2\alpha+1} - {p_2}^{2\beta+1} + 1}{2(p_2-1)} \qquad (3.4)
\end{aligned}$$

and

$$\begin{aligned}
\sigma(n) &= \sigma(3^{2\alpha}{p_2}^{2\beta}) \\
&= 2 \cdot 3^{2\alpha}{p_2}^{2\beta} - 3^{2\alpha-1}p_2^{2\beta} - 3^{2\alpha-2}p_2^{2\beta} - 3^{2\alpha-3}{p_2}^{2\beta} \\
&= 3^{2\alpha-3}{p_2}^{2\beta}(2 \cdot 3^3 - 3^2 - 3 - 1) \\
&= 41 \cdot 3^{2\alpha-3}{p_2}^{2\beta}. \qquad (3.5)
\end{aligned}$$

From (3.5) and (3.4), we get

$$\begin{aligned}
81 \cdot 3^{2\alpha-3}{p_2}^{2\beta+1} - 3^{2\alpha+1} - {p_2}^{2\beta+1} + 1 &= 82(p_2-1)3^{2\alpha-3}{p_2}^{2\beta} \\
81 \cdot 3^{2\alpha-3}{p_2}^{2\beta+1} - 3^{2\alpha+1} - {p_2}^{2\beta+1} + 1 &= 82 \cdot 3^{2\alpha-3}{p_2}^{2\beta+1} - 82 \cdot 3^{2\alpha-3}{p_2}^{2\beta} \\
3^{2\alpha-3} &= \frac{{p_2}^{2\beta+1} - 1}{82{p_2}^{2\beta} - {p_2}^{2\beta+1} - 81}. \qquad (3.6)
\end{aligned}$$

From (3.6), we obtain

$$\begin{aligned}
p_2 = 47;\ 3^{2\alpha-3} &= \frac{47 \cdot 47^{2\beta} - 1}{35 \cdot 47^{2\beta} - 81} = 1 + \frac{12 \cdot 47^{2\beta} + 80}{35 \cdot 47^{2\beta} - 81} \\
&= 1 + \frac{12 + \frac{80}{47^{2\beta}}}{35 - \frac{81}{47^{2\beta}}} \in (1,2) \\
p_2 = 53;\ 3^{2\alpha-3} &= \frac{53 \cdot 53^{2\beta} - 1}{29 \cdot 53^{2\beta} - 81} = 1 + \frac{24 \cdot 53^{2\beta} + 80}{29 \cdot 53^{2\beta} - 81} \\
&= 1 + \frac{24 + \frac{80}{53^{2\beta}}}{29 - \frac{81}{53^{2\beta}}} \in (1,2)
\end{aligned}$$

$$\begin{aligned}
p_2 = 59;\ 3^{2\alpha-3} &= \frac{59 \cdot 59^{2\beta} - 1}{23 \cdot 59^{2\beta} - 81} = 2 + \frac{13 \cdot 59^{2\beta} + 161}{23 \cdot 59^{2\beta} - 81} \\
&= 2 + \frac{13 + \frac{161}{59^{2\beta}}}{23 - \frac{81}{59^{2\beta}}} \in (2,3) \\
p_2 = 61;\ 3^{2\alpha-3} &= \frac{61 \cdot 61^{2\beta} - 1}{21 \cdot 61^{2\beta} - 81} = 2 + \frac{19 \cdot 61^{2\beta} + 161}{21 \cdot 61^{2\beta} - 81} \\
&= 2 + \frac{19 + \frac{161}{61^{2\beta}}}{21 - \frac{81}{61^{2\beta}}} \in (2,3) \\
p_2 = 67;\ 3^{2\alpha-3} &= \frac{67 \cdot 67^{2\beta} - 1}{15 \cdot 67^{2\beta} - 81} = 4 + \frac{7 \cdot 67^{2\beta} + 323}{15 \cdot 67^{2\beta} - 81} \\
&= 4 + \frac{7 + \frac{323}{67^{2\beta}}}{15 - \frac{81}{67^{2\beta}}} \in (4,5) \\
p_2 = 71;\ 3^{2\alpha-3} &= \frac{71 \cdot 71^{2\beta} - 1}{11 \cdot 71^{2\beta} - 81} = 6 + \frac{5 \cdot 71^{2\beta} + 485}{11 \cdot 71^{2\beta} - 81} \\
&= 6 + \frac{5 + \frac{485}{71^{2\beta}}}{11 - \frac{81}{71^{2\beta}}} \in (6,7) \\
p_2 = 73;\ 3^{2\alpha-3} &= \frac{73 \cdot 73^{2\beta} - 1}{9 \cdot 73^{2\beta} - 81} = 8 + \frac{73^{2\beta} + 647}{9 \cdot 73^{2\beta} - 81} \\
&= 8 + \frac{1 + \frac{647}{73^{2\beta}}}{9 - \frac{81}{73^{2\beta}}} \in (8,9) \\
p_2 = 79;\ 3^{2\alpha-3} &= \frac{79 \cdot 79^{2\beta} - 1}{3 \cdot 79^{2\beta} - 81} = 26 + \frac{79^{2\beta} + 2105}{3 \cdot 79^{2\beta} - 81} \\
&= 26 + \frac{1 + \frac{2105}{79^{2\beta}}}{3 - \frac{81}{79^{2\beta}}} \in (26,27).
\end{aligned}$$

Thus this case cannot hold.

Case 2. $p_2 \in \{37, 41, 43\}$.

We have $D_1 = 3, D_2 = 9$, and $D_3 \in \{27, p_2, 81, \ldots\}$.

If $D_3 \geq 81$, then

$$2 = \frac{\sigma(3^{2\alpha} p_2^{2\beta})}{3^{2\alpha} p_2^{2\beta}} + \frac{1}{D_1} + \frac{1}{D_2} + \frac{1}{D_3} < \frac{3}{2} \cdot \frac{37}{36} + \frac{1}{3} + \frac{1}{9} + \frac{1}{81} = 1.9984\ldots,$$

which is not true.

So $D_1 = 3$, $D_2 = 9$, and $D_3 \in \{27, p_2\}$.

Case 2.1. $D_1 = 3$, $D_2 = 9$, and $D_3 = 27$. Then $2\alpha \geq 3$. From (3.6), we get

$$\begin{aligned}
p_2 = 37; \quad 3^{2\alpha-3} &= \frac{37 \cdot 37^{2\beta} - 1}{45 \cdot 37^{2\beta} - 81} \in (0,1) \\
p_2 = 41; \quad 3^{2\alpha-3} &= \frac{41 \cdot 41^{2\beta} - 1}{41 \cdot 41^{2\beta} - 81} = 1 + \frac{80}{41 \cdot 41^{2\beta} - 81} \in (1,2) \\
p_2 = 43; \quad 3^{2\alpha-3} &= \frac{43 \cdot 43^{2\beta} - 1}{39 \cdot 43^{2\beta} - 81} = 1 + \frac{4 \cdot 43^{2\beta} + 80}{39 \cdot 43^{2\beta} - 81} \in (1,2).
\end{aligned}$$

Thus this case cannot hold.

Case 2.2. $D_1 = 3$, $D_2 = 9$, and $D_3 = p_2$.

For $p_2 = 37$, we get

$$\begin{aligned}
\sigma(n) &= \sigma(3^{2\alpha}37^{2\beta}) \\
&= \frac{(3^{2\alpha+1} - 1)}{2}\frac{(37^{2\beta+1} - 1)}{36} \\
&= \frac{3^{2\alpha+1}37^{2\beta+1} - 3^{2\alpha+1} - 37^{2\beta+1} + 1}{72} \\
&= \frac{36963 \cdot 3^{2\alpha-2}37^{2\beta-1} - 3^{2\alpha+1} - 37^{2\beta+1} + 1}{72}
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
\sigma(n) &= \sigma(3^{2\alpha}37^{2\beta}) \\
&= 2 \cdot 3^{2\alpha}37^{2\beta} - 3^{2\alpha-1}37^{2\beta} - 3^{2\alpha-2}37^{2\beta} - 3^{2\alpha}37^{2\beta-1} \\
&= 3^{2\alpha-2}37^{2\beta-1}(2 \cdot 3^2 \cdot 37 - 3 \cdot 37 - 37 - 3^2) \\
&= 509 \cdot 3^{2\alpha-2}37^{2\beta-1}.
\end{aligned} \tag{3.8}$$

From (3.7) and (3.8), we get

$$\begin{aligned}
36963 \cdot 3^{2\alpha-2}37^{2\beta-1} - 3^{2\alpha+1} - 37^{2\beta+1} + 1 &= 36648 \cdot 3^{2\alpha-2}37^{2\beta-1} \\
3^{2\alpha-2} &= \frac{1369 \cdot 37^{2\beta-1} - 1}{315 \cdot 37^{2\beta-1} - 27} \\
&= 4 + \frac{109 \cdot 37^{2\beta-1} + 107}{315 \cdot 37^{2\beta-1} - 27} \in (4,5),
\end{aligned}$$

which is a contradiction.

For $p_2 = 41$, we get

$$\begin{aligned}
\sigma(n) &= \sigma(3^{2\alpha}41^{2\beta}) \\
&= \frac{(3^{2\alpha+1}-1)}{2}\frac{(41^{2\beta+1}-1)}{40} \\
&= \frac{3^{2\alpha+1}37^{2\beta+1} - 3^{2\alpha+1} - 37^{2\beta+1} + 1}{80} \\
&= \frac{45387 \cdot 3^{2\alpha-2}41^{2\beta-1} - 3^{2\alpha+1} - 41^{2\beta+1} + 1}{80} \qquad (3.9)
\end{aligned}$$

and

$$\begin{aligned}
\sigma(n) &= \sigma(3^{2\alpha}41^{2\beta}) \\
&= 2 \cdot 3^{2\alpha}41^{2\beta} - 3^{2\alpha-1}41^{2\beta} - 3^{2\alpha-2}41^{2\beta} - 3^{2\alpha}41^{2\beta-1} \\
&= 3^{2\alpha-2}41^{2\beta-1}(2 \cdot 3^2 \cdot 41 - 3 \cdot 41 - 41 - 3^2) \\
&= 565 \cdot 3^{2\alpha-2}41^{2\beta-1}. \qquad (3.10)
\end{aligned}$$

From (3.9) and (3.10), we get

$$\begin{aligned}
45387 \cdot 3^{2\alpha-2}41^{2\beta-1} - 3^{2\alpha+1} - 41^{2\beta+1} + 1 &= 45200 \cdot 3^{2\alpha-2}41^{2\beta-1} \\
3^{2\alpha-2} &= \frac{1681 \cdot 41^{2\beta-1} - 1}{187 \cdot 41^{2\beta-1} - 27} \\
&= 8 + \frac{185 \cdot 41^{2\beta-1} + 215}{187 \cdot 41^{2\beta-1} - 27} \in (8, 10).
\end{aligned}$$

We must have

$$\begin{aligned}
3^{2\alpha-2} = 8 + \frac{185 \cdot 41^{2\beta-1} + 215}{187 \cdot 41^{2\beta-1} - 27} &= 9 \\
185 \cdot 41^{2\beta-1} + 215 &= 187 \cdot 41^{2\beta-1} - 27 \\
41^{2\beta-1} &= \frac{242}{2},
\end{aligned}$$

which is impossible.

For $p_2 = 43$, we get

$$\begin{aligned}
\sigma(n) &= \sigma(3^{2\alpha}43^{2\beta}) \\
&= \frac{(3^{2\alpha+1}-1)}{2}\frac{(43^{2\beta+1}-1)}{42} \\
&= \frac{3^{2\alpha+1}43^{2\beta+1} - 3^{2\alpha+1} - 43^{2\beta+1} + 1}{84} \\
&= \frac{49923 \cdot 3^{2\alpha-2}43^{2\beta-1} - 3^{2\alpha+1} - 43^{2\beta+1} + 1}{84} \qquad (3.11)
\end{aligned}$$

and

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}43^{2\beta})\\ &= 2\cdot 3^{2\alpha}43^{2\beta} - 3^{2\alpha-1}43^{2\beta} - 3^{2\alpha-2}43^{2\beta} - 3^{2\alpha}43^{2\beta-1}\\ &= 3^{2\alpha-2}43^{2\beta-1}(2\cdot 3^2\cdot 43 - 3\cdot 43 - 43 - 3^2)\\ &= 593\cdot 3^{2\alpha-2}43^{2\beta-1}. \end{aligned} \tag{3.12}$$

From (3.11) and (3.12), we get

$$\begin{aligned}49923\cdot 3^{2\alpha-2}43^{2\beta-1} - 3^{2\alpha+1} - 43^{2\beta+1} + 1 &= 49812\cdot 3^{2\alpha-2}43^{2\beta-1}\\ 3^{2\alpha-2} &= \frac{1849\cdot 43^{2\beta-1} - 1}{111\cdot 43^{2\beta-1} - 27}\\ &= 16 + \frac{73\cdot 43^{2\beta-1} + 431}{111\cdot 43^{2\beta-1} - 27} \in (16, 17),\end{aligned}$$

which is a contradiction.

Case 3. $p_2 = 31$.

We have $D_1 = 3$, $D_2 = 9$, and $D_3 \in \{27, 31, 81, 93, 243, \ldots\}$.

If $D_3 \geq 243$, then

$$2 = \frac{\sigma(3^{2\alpha}31^{2\beta})}{3^{2\alpha}31^{2\beta}} + \frac{1}{D_1} + \frac{1}{D_2} + \frac{1}{D_3} < \frac{3}{2}\cdot\frac{31}{30} + \frac{1}{3} + \frac{1}{9} + \frac{1}{243} = 1.9985\ldots,$$

which is not possible.

So $D_1 = 3$, $D_2 = 9$, and $D_3 \in \{27, 31, 81, 93\}$

Case 3.1. $D_1 = 3$, $D_2 = 9$, and $D_3 = 27$.

From (3.6), we have $3^{2\alpha-3} = \dfrac{31\cdot 31^{2\beta} - 1}{51\cdot 31^{2\beta} - 81} \in (0, 1)$, which is false.

Case 3.2. $D_1 = 3$, $D_2 = 9$ and $D_3 = 31$. We have

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}31^{2\beta})\\ &= \frac{(3^{2\alpha+1}-1)}{2}\frac{(31^{2\beta+1}-1)}{30}\\ &= \frac{3^{2\alpha+1}31^{2\beta+1} - 3^{2\alpha+1} - 31^{2\beta+1} + 1}{60}\\ &= \frac{25947\cdot 3^{2\alpha-2}31^{2\beta-1} - 3^{2\alpha+1} - 31^{2\beta+1} + 1}{60} \end{aligned} \tag{3.13}$$

and

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}31^{2\beta})\\ &= 2\cdot 3^{2\alpha}31^{2\beta} - 3^{2\alpha-1}31^{2\beta} - 3^{2\alpha-2}31^{2\beta} - 3^{2\alpha}31^{2\beta-1}\\ &= 3^{2\alpha-2}31^{2\beta-1}(2\cdot 3^2\cdot 31 - 3\cdot 31 - 31 - 3^2)\\ &= 425\cdot 3^{2\alpha-2}31^{2\beta-1}. \qquad (3.14)\end{aligned}$$

From (3.13) and (3.14), we get

$$\begin{aligned}25947\cdot 3^{2\alpha-2}31^{2\beta-1} - 3^{2\alpha+1} - 31^{2\beta+1} + 1 &= 25500\cdot 3^{2\alpha-2}31^{2\beta-1}\\ 3^{2\alpha-2} &= \frac{961\cdot 31^{2\beta-1}-1}{447\cdot 31^{2\beta-1}-27}\\ &= 2 + \frac{67\cdot 31^{2\beta-1}+53}{447\cdot 31^{2\beta-1}-27} \in (2,3),\end{aligned}$$

which is a contradiction.

Case 3.3. $D_1 = 3$, $D_2 = 9$, and $D_3 = 81$. Then $2\alpha \geq 4$ and so

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}31^{2\beta})\\ &= \frac{(3^{2\alpha+1}-1)}{2}\frac{(31^{2\beta+1}-1)}{30}\\ &= \frac{3^{2\alpha+1}31^{2\beta+1} - 3^{2\alpha+1} - 31^{2\beta+1} + 1}{60}\\ &= \frac{7533\cdot 3^{2\alpha-4}31^{2\beta} - 3^{2\alpha+1} - 31^{2\beta+1} + 1}{60} \qquad (3.15)\end{aligned}$$

and

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}31^{2\beta})\\ &= 2\cdot 3^{2\alpha}31^{2\beta} - 3^{2\alpha-1}31^{2\beta} - 3^{2\alpha-2}31^{2\beta} - 3^{2\alpha-4}31^{2\beta}\\ &= 3^{2\alpha-4}31^{2\beta}(2\cdot 3^4 - 3^3 - 3^2 - 1)\\ &= 125\cdot 3^{2\alpha-4}29^{2\beta}. \qquad (3.16)\end{aligned}$$

From (3.15) and (3.16), we get

$$\begin{aligned}7533\cdot 3^{2\alpha-4}31^{2\beta} - 3^{2\alpha+1} - 31^{2\beta+1} + 1 &= 7500\cdot 3^{2\alpha-4}29^{2\beta}\\ 3^{2\alpha-4} &= \frac{31\cdot 31^{2\beta}-1}{33\cdot 31^{2\beta}-243} \in (0,1),\end{aligned}$$

which is false.

Case 3.4. $D_1 = 3$, $D_2 = 9$ and $D_3 = 93$. We have

$$
\begin{aligned}
\sigma(n) &= \sigma(3^{2\alpha}31^{2\beta}) \\
&= \frac{(3^{2\alpha+1}-1)}{2}\frac{(31^{2\beta+1}-1)}{30} \\
&= \frac{3^{2\alpha+1}31^{2\beta+1} - 3^{2\alpha+1} - 31^{2\beta+1} + 1}{60} \\
&= \frac{25947 \cdot 3^{2\alpha-2}31^{2\beta-1} - 3^{2\alpha+1} - 31^{2\beta+1} + 1}{60} \qquad (3.17)
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma(n) &= \sigma(3^{2\alpha}31^{2\beta}) \\
&= 2 \cdot 3^{2\alpha}31^{2\beta} - 3^{2\alpha-1}31^{2\beta} - 3^{2\alpha-2}31^{2\beta} - 3^{2\alpha-1}31^{2\beta-1} \\
&= 3^{2\alpha-2}31^{2\beta-1}(2 \cdot 3^2 \cdot 31 - 3 \cdot 31 - 31 - 3) \\
&= 431 \cdot 3^{2\alpha-2}31^{2\beta-1}. \qquad (3.18)
\end{aligned}
$$

From (3.17) and (3.18), we get

$$
\begin{aligned}
25947 \cdot 3^{2\alpha-2}31^{2\beta-1} - 3^{2\alpha+1} - 31^{2\beta+1} + 1 &= 25860 \cdot 3^{2\alpha-2}31^{2\beta-1} \\
3^{2\alpha-2} &= \frac{961 \cdot 31^{2\beta-1} - 1}{87 \cdot 31^{2\beta-1} - 27} \\
&= 11 + \frac{4 \cdot 31^{2\beta-1} + 296}{87 \cdot 31^{2\beta-1} - 27} \in (11, 12),
\end{aligned}
$$

which is not possible.

Case 4. $p_2 = 29$. We have $D_1 = 3$, $D_2 = 9$, and $D_3 \in \{27, 29, 81, 87, 243, 261, 729, \ldots\}$.

If $D_3 \geq 729$, then

$$
2 = \frac{\sigma(3^{2\alpha}29^{2\beta})}{3^{2\alpha}29^{2\beta}} + \frac{1}{D_1} + \frac{1}{D_2} + \frac{1}{D_3} < \frac{3}{2} \cdot \frac{29}{28} + \frac{1}{3} + \frac{1}{9} + \frac{1}{729} = 1.9993\ldots,
$$

which is not true. So $D_1 = 3$, $D_2 = 9$, and $D_3 \in \{27, 29, 81, 87, 243, 261\}$.

Case 4.1. $D_1 = 3$, $D_2 = 9$, and $D_3 = 27$. Then $2\alpha \geq 3$. From (3.6), we get $3^{2\alpha-3} = \dfrac{29 \cdot 29^{2\beta} - 1}{53 \cdot 29^{2\beta} - 81} \in (0, 1)$, which is a contradiction.

Case 4.2. $D_1 = 3$, $D_2 = 9$, and $D_3 = 29$. We have

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}29^{2\beta}) \\ &= \frac{(3^{2\alpha+1}-1)}{2}\frac{(29^{2\beta+1}-1)}{28} \\ &= \frac{3^{2\alpha+1}29^{2\beta+1} - 3^{2\alpha+1} - 29^{2\beta+1} + 1}{56} \\ &= \frac{22707 \cdot 3^{2\alpha-2}29^{2\beta-1} - 3^{2\alpha+1} - 29^{2\beta+1} + 1}{56} \end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}29^{2\beta}) \\ &= 2 \cdot 3^{2\alpha}29^{2\beta} - 3^{2\alpha-1}29^{2\beta} - 3^{2\alpha-2}29^{2\beta} - 3^{2\alpha}29^{2\beta-1} \\ &= 3^{2\alpha-2}29^{2\beta-1}(2 \cdot 3^2 \cdot 29 - 3 \cdot 29 - 29 - 3^2) \\ &= 397 \cdot 3^{2\alpha-2}29^{2\beta-1}. \end{aligned} \tag{3.20}$$

From (3.19) and (3.20), we get

$$\begin{aligned} 22707 \cdot 3^{2\alpha-2}29^{2\beta-1} - 3^{2\alpha+1} - 29^{2\beta+1} + 1 &= 22232 \cdot 3^{2\alpha-2}29^{2\beta-1} \\ 3^{2\alpha-2} &= \frac{841 \cdot 29^{2\beta-1} - 1}{475 \cdot 29^{2\beta-1} - 27} \\ &= 1 + \frac{366 \cdot 29^{2\beta-1} + 26}{475 \cdot 29^{2\beta-1} - 27} \in (1,2), \end{aligned}$$

which is false.

Case 4.3. $D_1 = 3$, $D_2 = 9$, and $D_3 = 81$. We have $2\alpha \geq 4$. Then

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}29^{2\beta}) \\ &= \frac{(3^{2\alpha+1}-1)}{2}\frac{(29^{2\beta+1}-1)}{28} \\ &= \frac{3^{2\alpha+1}29^{2\beta+1} - 3^{2\alpha+1} - 29^{2\beta+1} + 1}{56} \\ &= \frac{7047 \cdot 3^{2\alpha-4}29^{2\beta} - 3^{2\alpha+1} - 29^{2\beta+1} + 1}{56} \end{aligned} \tag{3.21}$$

and

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}29^{2\beta})\\ &= 2\cdot 3^{2\alpha}29^{2\beta} - 3^{2\alpha-1}29^{2\beta} - 3^{2\alpha-2}29^{2\beta} - 3^{2\alpha-4}29^{2\beta}\\ &= 3^{2\alpha-4}29^{2\beta}(2\cdot 3^4 - 3^3 - 3^2 - 1)\\ &= 125\cdot 3^{2\alpha-4}29^{2\beta}. \end{aligned} \tag{3.22}$$

From (3.21) and (3.22), we get

$$\begin{aligned} 7047\cdot 3^{2\alpha-4}29^{2\beta} - 3^{2\alpha+1} - 29^{2\beta+1} + 1 &= 7000\cdot 3^{2\alpha-4}29^{2\beta}\\ 3^{2\alpha-4} &= \frac{29\cdot 29^{2\beta}-1}{47\cdot 29^{2\beta}-243} \in (0,1), \end{aligned}$$

which is a contradiction.

Case 4.4. $D_1 = 3$, $D_2 = 9$, and $D_3 = 87$. We have

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}29^{2\beta})\\ &= \frac{(3^{2\alpha+1}-1)}{2}\frac{(29^{2\beta+1}-1)}{28}\\ &= \frac{3^{2\alpha+1}29^{2\beta+1} - 3^{2\alpha+1} - 29^{2\beta+1} + 1}{56}\\ &= \frac{22707\cdot 3^{2\alpha-2}29^{2\beta-1} - 3^{2\alpha+1} - 29^{2\beta+1} + 1}{56} \end{aligned} \tag{3.23}$$

and

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}29^{2\beta})\\ &= 2\cdot 3^{2\alpha}29^{2\beta} - 3^{2\alpha-1}29^{2\beta} - 3^{2\alpha-2}29^{2\beta} - 3^{2\alpha-1}29^{2\beta-1}\\ &= 3^{2\alpha-2}29^{2\beta-1}(2\cdot 3^2\cdot 29 - 3\cdot 29 - 29 - 3)\\ &= 403\cdot 3^{2\alpha-2}29^{2\beta-1}. \end{aligned} \tag{3.24}$$

From (3.23) and (3.24), we get

$$\begin{aligned} 22707\cdot 3^{2\alpha-2}29^{2\beta-1} - 3^{2\alpha+1} - 29^{2\beta+1} + 1 &= 22568\cdot 3^{2\alpha-2}29^{2\beta-1}\\ 3^{2\alpha-2} &= \frac{841\cdot 29^{2\beta-1}-1}{139\cdot 29^{2\beta-1}-27}\\ &= 6 + \frac{7\cdot 29^{2\beta-1}+161}{139\cdot 29^{2\beta-1}-27} \in (6,7), \end{aligned}$$

we have a contradiction.

Case 4.5. $D_1 = 3$, $D_2 = 9$, and $D_3 = 243$. Then $2\alpha \geq 5$. We have

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}29^{2\beta}) \\ &= \frac{(3^{2\alpha+1}-1)}{2}\frac{(29^{2\beta+1}-1)}{28} \\ &= \frac{3^{2\alpha+1}29^{2\beta+1} - 3^{2\alpha+1} - 29^{2\beta+1} + 1}{56} \\ &= \frac{21141 \cdot 3^{2\alpha-5}29^{2\beta} - 3^{2\alpha+1} - 29^{2\beta+1} + 1}{56} \end{aligned} \tag{3.25}$$

and

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}29^{2\beta}) \\ &= 2 \cdot 3^{2\alpha}29^{2\beta} - 3^{2\alpha-1}29^{2\beta} - 3^{2\alpha-2}29^{2\beta} - 3^{2\alpha-5}29^{2\beta} \\ &= 3^{2\alpha-5}29^{2\beta}(2 \cdot 3^5 - 3^4 - 3^3 - 1) \\ &= 377 \cdot 56 \cdot 3^{2\alpha-5}29^{2\beta}. \end{aligned} \tag{3.26}$$

From (3.25) and (3.26), we get

$$\begin{aligned} 21141 \cdot 3^{2\alpha-5}29^{2\beta} - 3^{2\alpha+1} - 29^{2\beta+1} + 1 &= 21112 \cdot 56 \cdot 3^{2\alpha-5}29^{2\beta} \\ 3^{2\alpha-5} &= \frac{29^{2\beta+1}-1}{29^{2\beta+1}-729} \\ &= 1 + \frac{728}{29^{2\beta+1}-729} \in (1,2), \end{aligned}$$

which is false.

Case 4.6. $D_1 = 3$, $D_2 = 9$, and $D_3 = 261$. We have

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}29^{2\beta}) \\ &= \frac{(3^{2\alpha+1}-1)}{2}\frac{(29^{2\beta+1}-1)}{28} \\ &= \frac{3^{2\alpha+1}29^{2\beta+1} - 3^{2\alpha+1} - 29^{2\beta+1} + 1}{56} \\ &= \frac{22707 \cdot 3^{2\alpha-2}29^{2\beta-1} - 3^{2\alpha+1} - 29^{2\beta+1} + 1}{56} \end{aligned} \tag{3.27}$$

and

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}29^{2\beta})\\ &= 2\cdot 3^{2\alpha}29^{2\beta} - 3^{2\alpha-1}29^{2\beta} - 3^{2\alpha-2}29^{2\beta} - 3^{2\alpha-2}29^{2\beta-1}\\ &= 3^{2\alpha-2}29^{2\beta-1}(2\cdot 3^2\cdot 29 - 3\cdot 29 - 29 - 1)\\ &= 405\cdot 56\cdot 3^{2\alpha-2}29^{2\beta-1}. \end{aligned} \tag{3.28}$$

From (3.27) and (3.28), we get

$$\begin{aligned} 22707\cdot 3^{2\alpha-2}29^{2\beta-1} - 3^{2\alpha+1} - 29^{2\beta+1} + 1 &= 22680\cdot 56\cdot 3^{2\alpha-2}29^{2\beta-1}\\ 3^{2\alpha-2} &= \frac{841\cdot 29^{2\beta-1}-1}{27\cdot 29^{2\beta-1}-27}\\ &= 31 + \frac{4\cdot 29^{2\beta-1}+836}{27\cdot 29^{2\beta-1}-27} \in (31,33), \end{aligned}$$

which is impossible.

Case 5. $p_2 = 23$. We have $D_1 = 3$ and $D_2 = 9$. Recall that $d_3 = 3^{a_3}23^{b_3}$. Then

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}23^{2\beta})\\ &= \frac{(3^{2\alpha+1}-1)}{2}\frac{(23^{2\beta+1}-1)}{22}\\ &= \frac{3^{2\alpha+1}23^{2\beta+1} - 3^{2\alpha+1} - 23^{2\beta+1} + 1}{44}\\ &= \frac{621\cdot 3^{2\alpha-2}23^{2\beta} - 3^{2\alpha+1} - 23^{2\beta+1} + 1}{44} \end{aligned} \tag{3.29}$$

and

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}23^{2\beta})\\ &= 2\cdot 3^{2\alpha}23^{2\beta} - 3^{2\alpha-1}23^{2\beta} - 3^{2\alpha-2}23^{2\beta} - 3^{a_3}23^{b_3}\\ &= 3^{2\alpha-2}23^{2\beta}(2\cdot 3^2 - 3 - 1) - 44\cdot 3^{a_3}23^{b_3}\\ &= 14\cdot 3^{2\alpha-2}23^{2\beta} - 44\cdot 3^{a_3}23^{b_3}. \end{aligned} \tag{3.30}$$

From (3.29) and (3.30), we get

$$\begin{aligned}
621\cdot 3^{2\alpha-2}23^{2\beta}-3^{2\alpha+1}-23^{2\beta+1}+1 &= 616\cdot 3^{2\alpha-2}23^{2\beta}-44\cdot 3^{a_3}23^{b_3}\\
5\cdot 3^{2\alpha-2}23^{2\beta}-3^{2\alpha+1}-23^{2\beta+1} &= -1-44\cdot 3^{a_3}23^{b_3}\\
\left(5\cdot 3^{2\alpha-2}-23\right)\left(23^{2\beta}-\frac{27}{5}\right) &= 23\cdot\frac{27}{5}-1-44\cdot 3^{a_3}\cdot 23^{b_3}\\
\left(5\cdot 3^{2\alpha-2}-23\right)\left(5\cdot 23^{2\beta}-27\right) &= 616-220\cdot 3^{a_3}\cdot 23^{b_3}. \qquad (3.31)
\end{aligned}$$

If $\alpha \geq 2$, then the left-hand side of (3.31) is more than 616, we get a contradiction.

So $\alpha = 1$, and so $-18\left(5\cdot 23^{2\beta}-27\right) = 616-220\cdot 3^{a_3}\cdot 23^{b_3}$. We see that $3|18$ and $3\nmid 616$, so $a_3 = 0$. That is $-18\left(5\cdot 23^{2\beta}-27\right) = 616-220\cdot 23^{b_3}$. We see that $-18\left(5\cdot 23^{2\beta}-27\right)\equiv 5(0-4)\equiv 3 \pmod{23}$ but

$$616-220\cdot 23^{b_3}\equiv\begin{cases}18-0\equiv 18 \pmod{23}, & \text{if } b_3\geq 1;\\ 396\equiv 5 \pmod{23}, & \text{if } b_3 = 0.\end{cases}$$

Thus this case cannot hold.

Case 6. $p_2 = 19$. We have $D_1 = 3$ and $\{D_2, D_3\}\subset\{9, 19, 27, 57, \ldots\}$.

If $D_2\geq 19$ and $D_3\geq 57$, then from (3.3) implies that

$$2=\frac{\sigma(3^{2\alpha}19^{2\beta})}{3^{2\alpha}19^{2\beta}}+\frac{1}{D_1}+\frac{1}{D_2}+\frac{1}{D_3}<\frac{3}{2}\cdot\frac{19}{18}+\frac{1}{3}+\frac{1}{19}+\frac{1}{57}=1.9868\ldots,$$

which is not true.

So $(D_2 = 9)$ or $(D_2 = 19$ and $D_3 = 27)$.

Case 6.1. $D_1 = 3$ and $D_2 = 9$. Recall that $d_3 = 3^{a_3}\cdot 19^{b_3}$. Then

$$\begin{aligned}
\sigma(n) &= \sigma(3^{2\alpha}19^{2\beta})\\
&= \frac{\left(3^{2\alpha+1}-1\right)}{2}\frac{\left(19^{2\beta+1}-1\right)}{18}\\
&= \frac{3^{2\alpha+1}19^{2\beta+1}-3^{2\alpha+1}-19^{2\beta+1}+1}{36}\\
&= \frac{57\cdot 3^{2\alpha}19^{2\beta}-3^{2\alpha+1}-19^{2\beta+1}+1}{36} \qquad (3.32)
\end{aligned}$$

and

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}19^{2\beta})\\ &= 2\cdot 3^{2\alpha}19^{2\beta} - 3^{2\alpha-1}19^{2\beta} - 3^{2\alpha-2}19^{2\beta} - 3^{a_3}19^{b_3}\\ &= 3^{2\alpha-2}19^{2\beta}(2\cdot 3^2 - 3 - 1) - 3^{a_3}19^{b_3}\\ &= 14\cdot 3^{2\alpha-2}19^{2\beta} - 3^{a_3}19^{b_3}. \end{aligned} \tag{3.33}$$

From (3.32) and (3.33), we get

$$\begin{aligned} 57\cdot 3^{2\alpha}19^{2\beta} - 3^{2\alpha+1} - 19^{2\beta+1} + 1 &= 56\cdot 3^{2\alpha}19^{2\beta} - 36\cdot 3^{a_3}19^{b_3}\\ 3^{2\alpha}19^{2\beta} - 3^{2\alpha+1} - 19^{2\beta+1} &= -1 - 36\cdot 3^{a_3}19^{b_3}\\ \left(3^{2\alpha} - 19\right)\left(19^{2\beta} - 3\right) &= 56 - 36\cdot 3^{a_3}\cdot 23^{b_3}. \end{aligned} \tag{3.34}$$

If $\alpha \geq 2$, then the left-hand side of (3.34) is more than 56, which is impossible. So $\alpha = 1$, we get

$$\begin{aligned} -10\left(19^{2\beta} - 3\right) &= 56 - 36\cdot 3^{a_3}\cdot 19^{b_3}\\ -5\cdot 19^{2\beta} + 15 &= 28 - 18\cdot 3^{a_3}\cdot 19^{b_3}\\ -5\cdot 19^{2\beta} &= 13 - 18\cdot 3^{a_3}\cdot 19^{b_3}. \end{aligned}$$

Observe that $19 | 19^{2\beta}$ but $19 \nmid 13$, thus $b_3 = 0$, hence $-5\cdot 19^{2\beta} = 13 - 18\cdot 3^{a_3}$. As $a_3 \leq 2\alpha$, so $a_3 = 0$, 1 or 2. But this equation has no solution for $a_3 = 0$, 1 or 2 and $\beta \geq 1$.

Case 6.2. $D_1 = 3$, $D_2 = 19$, and $D_3 = 27$. We have

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}19^{2\beta})\\ &= \frac{(3^{2\alpha+1}-1)}{2}\frac{(19^{2\beta+1}-1)}{18}\\ &= \frac{3^{2\alpha+1}19^{2\beta+1} - 3^{2\alpha+1} - 19^{2\beta+1} + 1}{36}\\ &= \frac{29241\cdot 3^{2\alpha-3}19^{2\beta-1} - 3^{2\alpha+1} - 19^{2\beta+1} + 1}{36} \end{aligned} \tag{3.35}$$

and

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}19^{2\beta})\\ &= 2\cdot 3^{2\alpha}19^{2\beta} - 3^{2\alpha-1}19^{2\beta} - 3^{2\alpha}19^{2\beta-1} - 3^{2\alpha-3}19^{2\beta}\\ &= 3^{2\alpha-3}19^{2\beta-1}(2\cdot 3^3\cdot 19 - 3^2\cdot 19 - 3^3 - 19)\\ &= 809\cdot 3^{2\alpha-3}19^{2\beta-1}. \qquad (3.36)\end{aligned}$$

From (3.35) and (3.36), we get

$$\begin{aligned}29241\cdot 3^{2\alpha-3}19^{2\beta-1} - 3^{2\alpha+1} - 19^{2\beta+1} + 1 &= 29124\cdot 3^{2\alpha-3}19^{2\beta-1}\\ 3^{2\alpha-3} &= \frac{361\cdot 19^{2\beta-1}-1}{117\cdot 19^{2\beta-1}-81}\\ &= 3 + \frac{10\cdot 19^{2\beta-1}+242}{117\cdot 19^{2\beta-1}-81} \in (3,4),\end{aligned}$$

which is a contradiction.

Case 7. $p_2 = 17$. We have $D_1 = 3$ and $\{D_2, D_3\} \subset \{9, 17, 27, 51, 81, \ldots\}$.

If $D_2 \geq 27$, then from (3.3) we obtain

$$2 = \frac{\sigma(3^{2\alpha}17^{2\beta})}{3^{2\alpha}17^{2\beta}} + \frac{1}{D_1} + \frac{1}{D_2} + \frac{1}{D_3} < \frac{3}{2}\cdot\frac{17}{16} + \frac{1}{3} + \frac{1}{27} + \frac{1}{51} = 1.9837\ldots,$$

we get a contradiction.

If $D_2 \geq 17$ and $D_3 \geq 81$, then from (3.3) we obtain

$$2 = \frac{\sigma(3^{2\alpha}17^{2\beta})}{3^{2\alpha}17^{2\beta}} + \frac{1}{D_1} + \frac{1}{D_2} + \frac{1}{D_3} < \frac{3}{2}\cdot\frac{17}{16} + \frac{1}{3} + \frac{1}{17} + \frac{1}{81} = 1.9982\ldots,$$

which is not true.

So $(D_2 = 9)$ or $(D_2 = 17$ and $D_3 \in \{27,\ 51\})$.

Case 7.1. $D_1 = 3$ and $D_2 = 9$. Recall that $d_3 = 3^{a_3}17^{b_3}$ We have

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}17^{2\beta})\\ &= \frac{(3^{2\alpha+1}-1)}{2}\frac{(17^{2\beta+1}-1)}{16}\\ &= \frac{3^{2\alpha+1}17^{2\beta+1} - 3^{2\alpha+1} - 17^{2\beta+1} + 1}{32}\\ &= \frac{459\cdot 3^{2\alpha-2}17^{2\beta} - 3^{2\alpha+1} - 17^{2\beta+1} + 1}{32} \qquad (3.37)\end{aligned}$$

and

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}17^{2\beta})\\ &= 2\cdot 3^{2\alpha}17^{2\beta} - 3^{2\alpha-1}17^{2\beta} - 3^{2\alpha-2}17^{2\beta} - 3^{a_3}17^{b_3}\\ &= 3^{2\alpha-2}17^{2\beta}(2\cdot 3^2 - 3 - 1) - 3^{a_3}17^{b_3}\\ &= 14\cdot 3^{2\alpha-2}17^{2\beta} - 3^{a_3}17^{b_3}. \qquad (3.38)\end{aligned}$$

From (3.37) and (3.38), we get

$$\begin{aligned}459\cdot 3^{2\alpha-2}17^{2\beta} - 3^{2\alpha+1} - 17^{2\beta+1} + 1 &= 448\cdot 3^{2\alpha-2}17^{2\beta} - 32\cdot 3^{a_3}17^{b_3}\\ 11\cdot 3^{2\alpha-2}17^{2\beta} - 3^{2\alpha+1} - 17^{2\beta+1} &= -1 - 32\cdot 3^{a_3}17^{b_3}\\ \left(11\cdot 3^{2\alpha-2} - 17\right)\left(17^{2\beta} - \frac{27}{11}\right) &= 17\cdot\frac{27}{11} - 1 - 32\cdot 3^{a_3}\cdot 17^{b_3}\\ \left(11\cdot 3^{2\alpha-2} - 17\right)\left(11\cdot 17^{2\beta} - 27\right) &= 448 - 11\cdot 32\cdot 3^{a_3}\cdot 17^{b_3}. \qquad (3.39)\end{aligned}$$

If $\alpha \geq 2$, then the left-hand side of (3.39) is more than 448. Thus $\alpha = 1$, and so (3.39) becomes

$$\begin{aligned}-6\left(11\cdot 17^{2\beta} - 27\right) &= 448 - 11\cdot 32\cdot 3^{a_3}\cdot 17^{b_3}\\ -33\cdot 17^{2\beta} + 81 &= 224 - 11\cdot 16\cdot 3^{a_3}\cdot 17^{b_3}\\ -33\cdot 17^{2\beta} &= 143 - 11\cdot 16\cdot 3^{a_3}\cdot 17^{b_3}\\ -3\cdot 17^{2\beta} &= 13 - 16\cdot 3^{a_3}\cdot 17^{b_3}.\end{aligned}$$

Observe that $17|17^{2\beta}$ but $17\nmid 13$, so $b_3 = 0$. That is $-3\cdot 17^{2\beta} = 13 - 16\cdot 3^{a_3}$. As $3|-3$ and $3\nmid 13$, so $a_3 = 0$. Now, we have that $-3\cdot 17^{2\beta} = 13 - 16$, so $17^{2\beta} = 1$. That is $\beta = 0$, we get a contradiction.

Case 7.2. $D_1 = 3$, $D_2 = 17$, and $D_3 = 27$. Then $2\alpha \geq 3$ and we have

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}17^{2\beta})\\ &= \frac{(3^{2\alpha+1}-1)}{2}\frac{(17^{2\beta+1}-1)}{16}\\ &= \frac{3^{2\alpha+1}17^{2\beta+1} - 3^{2\alpha+1} - 17^{2\beta+1} + 1}{32}\\ &= \frac{23409\cdot 3^{2\alpha-3}17^{2\beta-1} - 3^{2\alpha+1} - 17^{2\beta+1} + 1}{32} \qquad (3.40)\end{aligned}$$

and

$$\begin{aligned}
\sigma(n) &= \sigma(3^{2\alpha}17^{2\beta}) \\
&= 2 \cdot 3^{2\alpha}17^{2\beta} - 3^{2\alpha-1}17^{2\beta} - 3^{2\alpha}17^{2\beta-1} - 3^{2\alpha-3}17^{2\beta} \\
&= 3^{2\alpha-3}17^{2\beta-1}(2 \cdot 3^3 \cdot 17 - 3^2 \cdot 17 - 3^3 - 17) \\
&= 721 \cdot 3^{2\alpha-3}17^{2\beta-1}. \qquad (3.41)
\end{aligned}$$

From (3.40) and (3.41), we get

$$\begin{aligned}
23409 \cdot 3^{2\alpha-3}17^{2\beta-1} - 3^{2\alpha+1} - 17^{2\beta+1} + 1 &= 23072 \cdot 3^{2\alpha-3}17^{2\beta-1} \\
337 \cdot 3^{2\alpha-3}17^{2\beta-1} - 81 \cdot 3^{2\alpha-3} &= 17^{2\beta+1} - 1 \\
3^{2\alpha-3} &= \frac{289 \cdot 17^{2\beta-1} - 1}{337 \cdot 17^{2\beta-1} - 81} \in (0,1),
\end{aligned}$$

which is not true.

Case 7.3. $D_1 = 3$, $D_2 = 17$, and $D_3 = 51$. We have

$$\begin{aligned}
\sigma(n) &= \sigma(3^{2\alpha}17^{2\beta}) \\
&= \frac{(3^{2\alpha+1} - 1)}{2}\frac{(17^{2\beta+1} - 1)}{16} \\
&= \frac{3^{2\alpha+1}17^{2\beta+1} - 3^{2\alpha+1} - 17^{2\beta+1} + 1}{32} \\
&= \frac{2601 \cdot 3^{2\alpha-1}17^{2\beta-1} - 3^{2\alpha+1} - 17^{2\beta+1} + 1}{32} \qquad (3.42)
\end{aligned}$$

and

$$\begin{aligned}
\sigma(n) &= \sigma(3^{2\alpha}17^{2\beta}) \\
&= 2 \cdot 3^{2\alpha}17^{2\beta} - 3^{2\alpha-1}17^{2\beta} - 3^{2\alpha}17^{2\beta-1} - 3^{2\alpha-1}17^{2\beta-1} \\
&= 3^{2\alpha-1}17^{2\beta-1}(2 \cdot 3 \cdot 17 - 17 - 3 - 1) \\
&= 81 \cdot 3^{2\alpha-1}17^{2\beta-1}. \qquad (3.43)
\end{aligned}$$

From (3.42) and (3.43), we get

$$\begin{aligned}
2601 \cdot 3^{2\alpha-1}17^{2\beta-1} - 3^{2\alpha+1} - 17^{2\beta+1} + 1 &= 2592 \cdot 3^{2\alpha-1}17^{2\beta-1} \\
9 \cdot 3^{2\alpha-1}17^{2\beta-1} - 3^{2\alpha+1} &= 17^{2\beta+1} - 1 \\
3^{2\alpha-1} &= \frac{289 \cdot 17^{2\beta-1} - 1}{9 \cdot 17^{2\beta-1} - 9} \\
&= 32 + \frac{17^{2\beta-1} + 287}{9 \cdot 17^{2\beta-1} - 9} \in (32, 35),
\end{aligned}$$

which is false.

Case 8. $p_2 = 13$. We have $D_1 = 3$ and $\{D_2, D_3\} \subset \{9, 13, 27, 39, 81, 117, 169, 243, \ldots\}$.

If $D_2 \geq 39$, then from (3.3) we obtain

$$2 = \frac{\sigma(3^{2\alpha}13^{2\beta})}{3^{2\alpha}13^{2\beta}} + \frac{1}{D_1} + \frac{1}{D_2} + \frac{1}{D_3} < \frac{3}{2} \cdot \frac{13}{12} + \frac{1}{3} + \frac{1}{39} + \frac{1}{81} = 1.9963\ldots,$$

which is impossible.

If $D_2 \geq 27$ and $D_3 \geq 243$, then from (3.3) we obtain

$$2 = \frac{\sigma(3^{2\alpha}13^{2\beta})}{3^{2\alpha}13^{2\beta}} + \frac{1}{D_1} + \frac{1}{D_2} + \frac{1}{D_3} < \frac{3}{2} \cdot \frac{13}{12} + \frac{1}{3} + \frac{1}{27} + \frac{1}{243} = 1.9994\ldots,$$

we get a contradiction. So ($D_2 \in \{9, 13\}$) or ($D_2 = 27$ and $D_3 \in \{39, 81, 117, 169\}$).

Case 8.1. $D_1 = 3$ and $D_2 = 9$. Recall that $d_3 = 3^{a_3}13^{b_3}$. We have

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}13^{2\beta}) \\ &= \frac{(3^{2\alpha+1}-1)}{2}\frac{(13^{2\beta+1}-1)}{12} \\ &= \frac{3^{2\alpha+1}13^{2\beta+1} - 3^{2\alpha+1} - 13^{2\beta+1} + 1}{24} \\ &= \frac{117 \cdot 3^{2\alpha-1}13^{2\beta} - 3^{2\alpha+1} - 13^{2\beta+1} + 1}{24}\end{aligned} \tag{3.44}$$

and

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}13^{2\beta}) \\ &= 2 \cdot 3^{2\alpha}13^{2\beta} - 3^{2\alpha-1}13^{2\beta} - 3^{2\alpha-2}13^{2\beta} - 3^{a_3}13^{b_3} \\ &= 3^{2\alpha-2}13^{2\beta}(2 \cdot 3^2 - 3 - 1) - 3^{a_3}13^{b_3} \\ &= 14 \cdot 3^{2\alpha-2}13^{2\beta} - 3^{a_3}13^{b_3}.\end{aligned} \tag{3.45}$$

From (3.44) and (3.45), we get

$$\begin{aligned}117 \cdot 3^{2\alpha-1}13^{2\beta} - 3^{2\alpha+1} - 13^{2\beta+1} + 1 &= 112 \cdot 3^{2\alpha-1}13^{2\beta} - 24 \cdot 3^{a_3}13^{b_3} \\ 5 \cdot 3^{2\alpha-1}13^{2\beta} - 3^{2\alpha+1} - 13^{2\beta+1} &= -1 - 24 \cdot 3^{a_3}13^{b_3} \\ \left(5 \cdot 3^{2\alpha-1} - 13\right)\left(13^{2\beta} - \frac{9}{5}\right) &= \frac{112}{5} - 24 \cdot 3^{a_3} \cdot 13^{b_3}.\end{aligned} \tag{3.46}$$

As $\alpha \geq 1$ and $\beta \geq 1$, then the left-hand side of (3.46) is more than zero but the right-hand side of (3.46) can only be negative.

Case 8.2. $D_1 = 3$ and $D_2 = 13$. Recall that $d_3 = 3^{a_3}13^{b_3}$. We have

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}13^{2\beta})\\ &= \frac{(3^{2\alpha+1}-1)}{2}\frac{(13^{2\beta+1}-1)}{12}\\ &= \frac{3^{2\alpha+1}13^{2\beta+1} - 3^{2\alpha+1} - 13^{2\beta+1} + 1}{24}\\ &= \frac{507 \cdot 3^{2\alpha}13^{2\beta-1} - 3^{2\alpha+1} - 13^{2\beta+1} + 1}{24} \qquad (3.47)\end{aligned}$$

and

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}13^{2\beta})\\ &= 2 \cdot 3^{2\alpha}13^{2\beta} - 3^{2\alpha-1}13^{2\beta} - 3^{2\alpha}13^{2\beta-1} - 3^{a_3}13^{b_3}\\ &= 3^{2\alpha-1}13^{2\beta-1}(2 \cdot 3 \cdot 13 - 13 - 3) - 3^{a_3}13^{b_3}\\ &= 62 \cdot 3^{2\alpha-1}13^{2\beta-1} - 3^{a_3}13^{b_3}. \qquad (3.48)\end{aligned}$$

From (3.47) and (3.48), we get

$$\begin{aligned}507 \cdot 3^{2\alpha}13^{2\beta-1} - 3^{2\alpha+1} - 13^{2\beta+1} + 1 &= 496 \cdot 3^{2\alpha}13^{2\beta-1} - 24 \cdot 3^{a_3}13^{b_3}\\ 11 \cdot 3^{2\alpha}13^{2\beta-1} - 3^{2\alpha+1} - 13^{2\beta+1} &= -1 - 24 \cdot 3^{a_3}13^{b_3}\\ \left(11 \cdot 3^{2\alpha} - 169\right)\left(13^{2\beta-1} - \frac{3}{11}\right) &= 169 \cdot \frac{3}{11} - 1 - 24 \cdot 3^{a_3} \cdot 13^{b_3}\\ \left(11 \cdot 3^{2\alpha} - 169\right)\left(11 \cdot 13^{2\beta-1} - 3\right) &= 496 - 264 \cdot 3^{a_3} \cdot 13^{b_3}. \qquad (3.49)\end{aligned}$$

If $\alpha \geq 2$, then the left-hand side of (3.49) is more than 496, which is a contradiction. So $\alpha = 1$, and so (3.49) becomes

$$\begin{aligned}-70\left(11 \cdot 13^{2\beta-1} - 3\right) &= 496 - 264 \cdot 3^{a_3} \cdot 13^{b_3}\\ -35 \cdot 11 \cdot 13^{2\beta-1} + 105 &= 248 - 132 \cdot 3^{a_3} \cdot 13^{b_3}\\ -35 \cdot 11 \cdot 13^{2\beta-1} &= 143 - 132 \cdot 3^{a_3} \cdot 13^{b_3}\\ -35 \cdot 13^{2\beta-1} &= 13 - 12 \cdot 3^{a_3} \cdot 13^{b_3}.\end{aligned}$$

As $a_3 \leq 2\alpha$, we have $a_3 = 0$, 1 or 2.

If $a_3 = 0$, then $-35 \cdot 13^{2\beta-1} = 13 - 12 \cdot 13^{b_3}$. We have that $-35 \cdot 13^{2\beta-1} \equiv 0 \pmod 7$ but $13 - 12 \cdot 13^{b_3} \equiv 1$ or $4 \pmod 7$, which is a contradiction.

If $a_3 = 2$, then $-35 \cdot 13^{2\beta-1} = 13 - 12 \cdot 9 \cdot 13^{b_3}$. We have that $-35 \cdot 13^{2\beta-1} \equiv 0 \pmod 7$ but $13 - 12 \cdot 9 \cdot 13^{b_3} \equiv 2$ or $3 \pmod 7$, which is false.

Hence $a_3 = 1$. That is

$$-35 \cdot 13^{2\beta-1} = 13 - 12 \cdot 3 \cdot 13^{b_3}. \tag{3.50}$$

Since $13 | 13^{2\beta-1}$ and $13 | 13$, so $b_3 \geq 1$.

If $b_3 > 1$, if $\beta = 1$, then from (3.50) we get $-35 \cdot 13 = 13 - 36 \cdot 13^{b_3}$, that is $b_3 = 1$, which is false. So $\beta > 1$, we get $13^2 | 13^{2\beta-1}$ and $13^2 | 13^{b_3}$ but $13^2 \nmid 13$, a contradiction.

If $b_3 = 1$, then from (3.50) we get $-35 \cdot 13^{2\beta-1} = 13 - 12 \cdot 3 \cdot 13$, and so $13^{2\beta-1} = 13$, which implies that $\beta = 1$. We obtain $\alpha = 1$, $\beta = 1$, $a_3 = 1$ and $b_3 = 1$. Therefore, $n = 1521 = 3^2 13^2$ is an exactly 3-deficient-perfect number with three deficient divisors $d_1 = 507 = 3 \cdot 13^2$, $d_2 = 117 = 3^2 \cdot 13$, and $d_3 = 39 = 3 \cdot 13$.

Case 8.3. $D_1 = 3$, $D_2 = 27$, and $D_3 = 39$. We have

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha} 13^{2\beta}) \\ &= \frac{(3^{2\alpha+1} - 1)}{2} \frac{(13^{2\beta+1} - 1)}{12} \\ &= \frac{3^{2\alpha+1} 13^{2\beta+1} - 3^{2\alpha+1} - 13^{2\beta+1} + 1}{24} \\ &= \frac{13689 \cdot 3^{2\alpha-3} 13^{2\beta-1} - 3^{2\alpha+1} - 13^{2\beta+1} + 1}{24}\end{aligned} \tag{3.51}$$

and

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha} 13^{2\beta}) \\ &= 2 \cdot 3^{2\alpha} 13^{2\beta} - 3^{2\alpha-1} 13^{2\beta} - 3^{2\alpha-3} 13^{2\beta} - 3^{2\alpha-1} 13^{2\beta-1} \\ &= 3^{2\alpha-3} 13^{2\beta-1} (2 \cdot 3^3 \cdot 13 - 3^2 \cdot 13 - 13 - 3^2) \\ &= 563 \cdot 3^{2\alpha-3} 13^{2\beta-1}.\end{aligned} \tag{3.52}$$

From (3.51) and (3.52), we get

$$13689 \cdot 3^{2\alpha-3} 13^{2\beta-1} - 3^{2\alpha+1} - 13^{2\beta+1} + 1 = 13512 \cdot 3^{2\alpha-3} 13^{2\beta-1}$$

$$3^{2\alpha-3} = \frac{169 \cdot 13^{2\beta-1} - 1}{177 \cdot 13^{2\beta-1} - 81} \in (0, 1),$$

which is false.

Case 8.4. $D_1 = 3$, $D_2 = 27$, and $D_3 = 81$. We have

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}13^{2\beta}) \\ &= \frac{(3^{2\alpha+1}-1)}{2}\frac{(13^{2\beta+1}-1)}{12} \\ &= \frac{3^{2\alpha+1}13^{2\beta+1}-3^{2\alpha+1}-13^{2\beta+1}+1}{24} \\ &= \frac{3159\cdot 3^{2\alpha-4}13^{2\beta}-3^{2\alpha+1}-13^{2\beta+1}+1}{24} \qquad (3.53)\end{aligned}$$

and

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}13^{2\beta}) \\ &= 2\cdot 3^{2\alpha}13^{2\beta}-3^{2\alpha-1}13^{2\beta}-3^{2\alpha-3}13^{2\beta}-3^{2\alpha-4}13^{2\beta} \\ &= 3^{2\alpha-4}13^{2\beta}(2\cdot 3^4-3^3-3-1) \\ &= 131\cdot 3^{2\alpha-4}13^{2\beta}. \qquad (3.54)\end{aligned}$$

From (3.53) and (3.54), we get

$$\begin{aligned}3159\cdot 3^{2\alpha-4}13^{2\beta}-3^{2\alpha+1}-13^{2\beta+1}+1 &= 3144\cdot 3^{2\alpha-4}13^{2\beta} \\ 3^{2\alpha-4} &= \frac{13\cdot 13^{2\beta}-1}{15\cdot 13^{2\beta}-243}\in(0,1),\end{aligned}$$

which is a contradiction.

Case 8.5. $D_1 = 3$, $D_2 = 27$, and $D_3 = 117$. We have

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}13^{2\beta}) \\ &= \frac{(3^{2\alpha+1}-1)}{2}\frac{(13^{2\beta+1}-1)}{12} \\ &= \frac{3^{2\alpha+1}13^{2\beta+1}-3^{2\alpha+1}-13^{2\beta+1}+1}{24} \\ &= \frac{13689\cdot 3^{2\alpha-3}13^{2\beta-1}-3^{2\alpha+1}-13^{2\beta+1}+1}{24} \qquad (3.55)\end{aligned}$$

and

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}13^{2\beta}) \\ &= 2\cdot 3^{2\alpha}13^{2\beta}-3^{2\alpha-1}13^{2\beta}-3^{2\alpha-3}13^{2\beta}-3^{2\alpha-2}13^{2\beta-1} \\ &= 3^{2\alpha-3}13^{2\beta-1}(2\cdot 3^3\cdot 13-3^2\cdot 13-13-3) \\ &= 569\cdot 3^{2\alpha-3}13^{2\beta-1}. \qquad (3.56)\end{aligned}$$

From (3.55) and (3.56), we get

$$13689 \cdot 3^{2\alpha-3}13^{2\beta-1} - 3^{2\alpha+1} - 13^{2\beta+1} + 1 = 13656 \cdot 3^{2\alpha-3}13^{2\beta-1}$$

$$3^{2\alpha-3} = \frac{169 \cdot 13^{2\beta-1} - 1}{33 \cdot 13^{2\beta-1} - 81}$$

$$= 5 + \frac{4 \cdot 13^{2\beta-1} + 404}{33 \cdot 13^{2\beta-1} - 81} \in (5, 7),$$

which is not true.

Case 8.6. $D_1 = 3$, $D_2 = 27$, and $D_3 = 169$. We have

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}13^{2\beta})\\ &= \frac{(3^{2\alpha+1}-1)}{2}\frac{(13^{2\beta+1}-1)}{12}\\ &= \frac{3^{2\alpha+1}13^{2\beta+1} - 3^{2\alpha+1} - 13^{2\beta+1} + 1}{24}\\ &= \frac{177957 \cdot 3^{2\alpha-3}13^{2\beta-2} - 3^{2\alpha+1} - 13^{2\beta+1} + 1}{24}\end{aligned} \tag{3.57}$$

and

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}13^{2\beta})\\ &= 2 \cdot 3^{2\alpha}13^{2\beta} - 3^{2\alpha-1}13^{2\beta} - 3^{2\alpha-3}13^{2\beta} - 3^{2\alpha}13^{2\beta-2}\\ &= 3^{2\alpha-3}13^{2\beta-2}(2 \cdot 3^3 \cdot 13^2 - 3^2 \cdot 13^2 - 13^2 - 3^3)\\ &= 7409 \cdot 3^{2\alpha-3}13^{2\beta-2}.\end{aligned} \tag{3.58}$$

From (3.57) and (3.58), we get

$$177957 \cdot 3^{2\alpha-3}13^{2\beta-2} - 3^{2\alpha+1} - 13^{2\beta+1} + 1 = 177816 \cdot 3^{2\alpha-3}13^{2\beta-2}$$

$$3^{2\alpha-3} = \frac{2197 \cdot 13^{2\beta-2} - 1}{141 \cdot 13^{2\beta-2} - 81}$$

$$= 15 + \frac{82 \cdot 13^{2\beta-2} + 1,214}{141 \cdot 13^{2\beta-2} - 81} \in (15, 37).$$

We must have

$$3^{2\alpha-3} = 15 + \frac{82 \cdot 13^{2\beta-2} + 1,214}{141 \cdot 13^{2\beta-2} - 81} = 27$$

$$82 \cdot 13^{2\beta-2} + 1214 = 1692 \cdot 13^{2\beta-2} - 972$$

$$13^{2\beta-2} = \frac{2186}{1610},$$

which is a contradiction.

Case 9. $p_2 = 11$. We have $D_1 = 3$ and $\{D_2, D_3\} \subset \{9, 11, 27, 33, 81, 99, 121, 243, \ldots\}$.

If $D_2 \geq 81$ and $D_3 \geq 243$, then from (3.3), we have

$$2 = \frac{\sigma(3^{2\alpha}11^{2\beta})}{3^{2\alpha}11^{2\beta}} + \frac{1}{D_1} + \frac{1}{D_2} + \frac{1}{D_3} < \frac{3}{2} \cdot \frac{11}{10} + \frac{1}{3} + \frac{1}{81} + \frac{1}{243} = 1.9997\ldots,$$

we get a contradiction. So $D_2 = 9$, 11, 27, 33 or $D_2 = 81$ and $D_3 \in \{99, 121\}$ or $D_2 = 99$ and $D_3 = 121$.

Now we consider the following seven cases.

Case 9.1. $D_1 = 3$ and $D_2 = 9$. Recall that $d_3 = 3^{a_3}11^{b_3}$. We have

$$\begin{aligned}
\sigma(n) &= \sigma(3^{2\alpha}11^{2\beta}) \\
&= \frac{(3^{2\alpha+1}-1)}{2}\frac{(11^{2\beta+1}-1)}{10} \\
&= \frac{3^{2\alpha+1}11^{2\beta+1} - 3^{2\alpha+1} - 11^{2\beta+1} + 1}{20} \\
&= \frac{297 \cdot 3^{2\alpha-2}11^{2\beta} - 3^{2\alpha+1} - 11^{2\beta+1} + 1}{20}
\end{aligned} \tag{3.59}$$

and

$$\begin{aligned}
\sigma(n) &= \sigma(3^{2\alpha}11^{2\beta}) \\
&= 2 \cdot 3^{2\alpha}11^{2\beta} - 3^{2\alpha-1}11^{2\beta} - 3^{2\alpha-2}11^{2\beta} - 3^{a_3}11^{b_3} \\
&= 3^{2\alpha-2}11^{2\beta}(2 \cdot 3^2 - 3 - 1) - 3^{a_3}11^{b_3} \\
&= 14 \cdot 3^{2\alpha-2}11^{2\beta} - 3^{a_3}11^{b_3}.
\end{aligned} \tag{3.60}$$

From (3.59) and (3.60), we get

$$\begin{aligned}
297 \cdot 3^{2\alpha-2}11^{2\beta} - 3^{2\alpha+1} - 11^{2\beta+1} + 1 &= 280 \cdot 3^{2\alpha-2}11^{2\beta} - 20 \cdot 3^{a_3}11^{b_3} \\
17 \cdot 3^{2\alpha-2}11^{2\beta} - 3^{2\alpha+1} - 11^{2\beta+1} &= -1 - 20 \cdot 3^{a_3}11^{b_3} \\
\left(17 \cdot 3^{2\alpha-2} - 11\right)\left(11^{2\beta} - \frac{27}{17}\right) &= \frac{280}{17} - 20 \cdot 3^{a_3} \cdot 11^{b_3}.
\end{aligned} \tag{3.61}$$

As $\alpha \geq 1$ and $\beta \geq 1$, so the left-hand side of (3.61) is more than $\frac{280}{17}$, which is a contradiction.

Case 9.2. $D_1 = 3$ and $D_2 = 11$. Recall that $d_3 = 3^{a_3}11^{b_3}$. We have

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}11^{2\beta})\\ &= \frac{(3^{2\alpha+1}-1)}{2}\frac{(11^{2\beta+1}-1)}{10}\\ &= \frac{3^{2\alpha+1}11^{2\beta+1} - 3^{2\alpha+1} - 11^{2\beta+1} + 1}{20}\\ &= \frac{1089\cdot 3^{2\alpha-1}11^{2\beta-1} - 3^{2\alpha+1} - 11^{2\beta+1} + 1}{20} \qquad (3.62)\end{aligned}$$

and

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}11^{2\beta})\\ &= 2\cdot 3^{2\alpha}11^{2\beta} - 3^{2\alpha-1}11^{2\beta} - 3^{2\alpha}11^{2\beta-1} - 3^{a_3}11^{b_3}\\ &= 3^{2\alpha-1}11^{2\beta-1}(2\cdot 3\cdot 11 - 11 - 3) - 3^{a_3}11^{b_3}\\ &= 52\cdot 3^{2\alpha-1}11^{2\beta-1} - 3^{a_3}11^{b_3}. \qquad (3.63)\end{aligned}$$

From (3.62) and (3.63), we get

$$\begin{aligned}1089\cdot 3^{2\alpha-1}11^{2\beta-1} - 3^{2\alpha+1} - 11^{2\beta+1} + 1 &= 1040\cdot 3^{2\alpha-1}11^{2\beta-1} - 20\cdot 3^{a_3}11^{b_3}\\ 49\cdot 3^{2\alpha-1}11^{2\beta-1} - 3^{2\alpha+1} - 11^{2\beta+1} &= -1 - 20\cdot 3^{a_3}11^{b_3}\\ (49\cdot 3^{2\alpha-1} - 121)\left(11^{2\beta-1} - \frac{9}{49}\right) &= \frac{1040}{49} - 20\cdot 3^{a_3}\cdot 11^{b_3}. \qquad (3.64)\end{aligned}$$

As $\alpha \geq 1$ and $\beta \geq 1$, so the left-hand side of (3.64) is more than $\frac{1040}{49}$, which is false.

Case 9.3. $D_1 = 3$ and $D_2 = 27$. Recall that $d_3 = 3^{a_3}11^{b_3}$. Then $2\alpha \geq 3$. We get

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}11^{2\beta})\\ &= \frac{(3^{2\alpha+1}-1)}{2}\frac{(11^{2\beta+1}-1)}{10}\\ &= \frac{3^{2\alpha+1}11^{2\beta+1} - 3^{2\alpha+1} - 11^{2\beta+1} + 1}{20}\\ &= \frac{891\cdot 3^{2\alpha-3}11^{2\beta} - 3^{2\alpha+1} - 11^{2\beta+1} + 1}{20} \qquad (3.65)\end{aligned}$$

and

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}11^{2\beta})\\ &= 2\cdot 3^{2\alpha}11^{2\beta} - 3^{2\alpha-1}11^{2\beta} - 3^{2\alpha-3}11^{2\beta} - 3^{a_3}11^{b_3}\\ &= 3^{2\alpha-3}11^{2\beta}(2\cdot 3^3 - 3^2 - 1) - 3^{a_3}11^{b_3}\\ &= 44\cdot 3^{2\alpha-3}11^{2\beta} - 3^{a_3}11^{b_3}. \qquad (3.66)\end{aligned}$$

From (3.65) and (3.66), we get

$$\begin{aligned}891\cdot 3^{2\alpha-3}11^{2\beta} - 3^{2\alpha+1} - 11^{2\beta+1} + 1 &= 880\cdot 3^{2\alpha-3}11^{2\beta} - 20\cdot 3^{a_3}11^{b_3}\\ 11\cdot 3^{2\alpha-3}11^{2\beta} - 3^{2\alpha+1} - 11^{2\beta+1} &= -1 - 20\cdot 3^{a_3}11^{b_3}\\ 3^{2\alpha-3}11^{2\beta+1} - 3^{2\alpha+1} - 11^{2\beta+1} &= -1 - 20\cdot 3^{a_3}11^{b_3}\\ \left(3^{2\alpha-3} - 1\right)\left(11^{2\beta+1} - 81\right) &= 80 - 20\cdot 3^{a_3}\cdot 11^{b_3}. \qquad (3.67)\end{aligned}$$

As $D_2 = 27$, so $2\alpha \geq 3$ implies $2\alpha > 3$. So the left-hand side of (3.67) is more than 80, we get a contradiction.

Case 9.4. $D_1 = 3$ and $D_2 = 33$. Recall that $d_3 = 3^{a_3}11^{b_3}$. We have

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}11^{2\beta})\\ &= \frac{(3^{2\alpha+1}-1)}{2}\frac{(11^{2\beta+1}-1)}{10}\\ &= \frac{3^{2\alpha+1}11^{2\beta+1} - 3^{2\alpha+1} - 11^{2\beta+1} + 1}{20}\\ &= \frac{121\cdot 3^{2\alpha+1}11^{2\beta-1} - 3^{2\alpha+1} - 11^{2\beta+1} + 1}{20} \qquad (3.68)\end{aligned}$$

and

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}11^{2\beta})\\ &= 2\cdot 3^{2\alpha}11^{2\beta} - 3^{2\alpha-1}11^{2\beta} - 3^{2\alpha-1}11^{2\beta-1} - 3^{a_3}11^{b_3}\\ &= 3^{2\alpha-1}11^{2\beta-1}(2\cdot 3\cdot 11 - 11 - 1) - 3^{a_3}11^{b_3}\\ &= 6\cdot 3^{2\alpha+1}11^{2\beta-1} - 6\cdot 3^{a_3}11^{b_3}. \qquad (3.69)\end{aligned}$$

From (3.68) and (3.69), we get

$$121 \cdot 3^{2\alpha+1}11^{2\beta-1} - 3^{2\alpha+1} - 11^{2\beta+1} + 1 = 120 \cdot 3^{2\alpha+1}11^{2\beta-1} - 20 \cdot 3^{a_3}11^{b_3}$$
$$3^{2\alpha+1}11^{2\beta-1} - 3^{2\alpha+1} - 11^{2\beta+1} = -1 - 20 \cdot 3^{a_3}11^{b_3}$$
$$\left(3^{2\alpha+1} - 121\right)\left(11^{2\beta-1} - 1\right) = 120 - 20 \cdot 3^{a_3} \cdot 11^{b_3}. \quad (3.70)$$

If $\alpha \geq 2$, then the left-hand side of (3.70) is more than 120, which is a contradiction. So $\alpha = 1$, we get

$$-94(11^{2\beta+1} - 1) = 120 - 20 \cdot 3^{a_3}11^{b_3}$$
$$-47 \cdot 11^{2\beta+1} + 47 = 60 - 10 \cdot 3^{a_3}11^{b_3}$$
$$-47 \cdot 11^{2\beta+1} = 13 - 10 \cdot 3^{a_3}11^{b_3}.$$

Observe that $11|11^{2\beta+1}$ but $11 \nmid 13$, so $b_3 = 0$. That is $-47 \cdot 11^{2\beta+1} = 13 - 10 \cdot 3^{a_3}$. Since $a_3 \leq 2\alpha$, we have $a_3 = 0$, 1 or 2. We know that $-47 \cdot 11^{2\beta+1} = 13 - 10 \cdot 3^{a_3}$ has no solution.

Case 9.5. $D_1 = 3$, $D_2 = 81$, and $D_3 = 99$. We have

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}11^{2\beta}) \\ &= \frac{(3^{2\alpha+1} - 1)}{2}\frac{(11^{2\beta+1} - 1)}{10} \\ &= \frac{3^{2\alpha+1}11^{2\beta+1} - 3^{2\alpha+1} - 11^{2\beta+1} + 1}{20} \\ &= \frac{29403 \cdot 3^{2\alpha-4}11^{2\beta-1} - 3^{2\alpha+1} - 11^{2\beta+1} + 1}{20} \end{aligned} \quad (3.71)$$

and

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}11^{2\beta}) \\ &= 2 \cdot 3^{2\alpha}11^{2\beta} - 3^{2\alpha-1}11^{2\beta} - 3^{2\alpha-4}11^{2\beta} - 3^{2\alpha-2}11^{2\beta-1} \\ &= 3^{2\alpha-4}11^{2\beta-1}(2 \cdot 3^4 \cdot 11 - 3^3 \cdot 11 - 11 - 3^2) \\ &= 1465 \cdot 3^{2\alpha-4}11^{2\beta-1}. \end{aligned} \quad (3.72)$$

From (3.71) and (3.72), we get

$$29403 \cdot 3^{2\alpha-4}11^{2\beta-1} - 3^{2\alpha+1} - 11^{2\beta+1} + 1 = 29300 \cdot 3^{2\alpha-4}11^{2\beta-1}$$
$$3^{2\alpha-4} = \frac{121 \cdot 11^{2\beta-1} - 1}{103 \cdot 11^{2\beta-1} - 243}$$
$$= 1 + \frac{18 \cdot 11^{2\beta-1} + 242}{103 \cdot 11^{2\beta-1} - 243} \in (1, 2),$$

which is not true.

Case 9.6. $D_1 = 3$, $D_2 = 81$, and $D_3 = 121$. We have

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}11^{2\beta}) \\ &= \frac{(3^{2\alpha+1}-1)}{2}\frac{(11^{2\beta+1}-1)}{10} \\ &= \frac{3^{2\alpha+1}11^{2\beta+1} - 3^{2\alpha+1} - 11^{2\beta+1} + 1}{20} \\ &= \frac{323433 \cdot 3^{2\alpha-4}11^{2\beta-2} - 3^{2\alpha+1} - 11^{2\beta+1} + 1}{20} \end{aligned} \tag{3.73}$$

and

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}11^{2\beta}) \\ &= 2 \cdot 3^{2\alpha}11^{2\beta} - 3^{2\alpha-1}11^{2\beta} - 3^{2\alpha-4}11^{2\beta} - 3^{2\alpha}11^{2\beta-2} \\ &= 3^{2\alpha-4}11^{2\beta-2}(2 \cdot 3^4 \cdot 11^2 - 3^3 \cdot 11^2 - 11^2 - 3^4) \\ &= 16133 \cdot 3^{2\alpha-4}11^{2\beta-2}. \end{aligned} \tag{3.74}$$

From (3.73) and (3.74), we get

$$323433 \cdot 3^{2\alpha-4}11^{2\beta-2} - 3^{2\alpha+1} - 11^{2\beta+1} + 1 = 322660 \cdot 3^{2\alpha-4}11^{2\beta-2}$$
$$3^{2\alpha-4} = \frac{1331 \cdot 11^{2\beta-2} - 1}{773 \cdot 11^{2\beta-2} - 243}$$
$$= 1 + \frac{558 \cdot 11^{2\beta-2} + 242}{773 \cdot 11^{2\beta-2} - 243} \in (1, 3),$$

which is a contradiction.

Case 9.7. $D_1 = 3$, $D_2 = 99$, and $D_3 = 121$. We have

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}11^{2\beta})\\ &= \frac{(3^{2\alpha+1}-1)}{2}\frac{(11^{2\beta+1}-1)}{10}\\ &= \frac{3^{2\alpha+1}11^{2\beta+1}-3^{2\alpha+1}-11^{2\beta+1}+1}{20}\\ &= \frac{35937\cdot 3^{2\alpha-2}11^{2\beta-2}-3^{2\alpha+1}-11^{2\beta+1}+1}{20}\end{aligned} \tag{3.75}$$

and

$$\begin{aligned}\sigma(n) &= \sigma(3^{2\alpha}11^{2\beta})\\ &= 2\cdot 3^{2\alpha}11^{2\beta}-3^{2\alpha-1}11^{2\beta}-3^{2\alpha-2}11^{2\beta-1}-3^{2\alpha}11^{2\beta-2}\\ &= 3^{2\alpha-2}11^{2\beta-2}(2\cdot 3^2\cdot 11^2-3\cdot 11^2-11-3^2)\\ &= 1795\cdot 3^{2\alpha-2}11^{2\beta-2}.\end{aligned} \tag{3.76}$$

From (3.75) and (3.76), we get

$$\begin{aligned}35937\cdot 3^{2\alpha-2}11^{2\beta-2}-3^{2\alpha+1}-11^{2\beta+1}+1 &= 35900\cdot 3^{2\alpha-2}11^{2\beta-2}\\ 3^{2\alpha-2} &= \frac{1331\cdot 11^{2\beta-2}-1}{37\cdot 11^{2\beta-2}-27}\\ &= 35+\frac{36\cdot 11^{2\beta-2}+944}{37\cdot 11^{2\beta-2}-27}\in(35,37)\cup\{133\},\end{aligned}$$

which is not true.

Case 10. $p_2 = 7$. Then $\{D_1, D_2, D_3\} \subset \{3, 7, 9, 21, 27, \ldots\}$.

If $D_1 \geq 7$ and $D_2 \geq 21$, then from (3.3) we get

$$2 = \frac{\sigma(3^{2\alpha}7^{2\beta})}{3^{2\alpha}7^{2\beta}}+\frac{1}{D_1}+\frac{1}{D_2}+\frac{1}{D_3} < \frac{3}{2}\cdot\frac{7}{6}+\frac{1}{7}+\frac{1}{21}+\frac{1}{27} = 1.9775\ldots,$$

we get a contradiction.

So ($D_1 = 3$) or ($D_1 = 7$ and $D_2 = 9$).

Case 10.1. $D_1 = 3$. We have

$$\begin{aligned}\sigma(3^{2\alpha}7^{2\beta}) = \frac{(3^{2\alpha+1}-1)}{2}\frac{(7^{2\beta+1}-1)}{6} &= 2\cdot 3^{2\alpha}7^{2\beta}-3^{2\alpha-1}7^{2\beta}-d_2-d_3\\ 21\cdot 3^{2\alpha}7^{2\beta}-3^{2\alpha+1}-7^{2\beta+1}+1 &= 24\cdot 3^{2\alpha}7^{2\beta}-12(3^{2\alpha-1}7^{2\beta}+d_2+d_3)\\ 12(3^{2\alpha-1}7^{2\beta}+d_2+d_3) &= 3\cdot 3^{2\alpha}7^{2\beta}+3^{2\alpha+1}+7^{2\beta+1}-1.\end{aligned}$$

We obtain that

$$12(3^{2\alpha-1}7^{2\beta}+d_2+d_3)<3^{2\alpha+1}7^{2\beta}+3^{2\alpha+1}+7^{2\beta+1}.$$

We divide both side by $3^{2\alpha}7^{2\beta}$,

$$\frac{12}{3^{2\alpha}7^{2\beta}}(3^{2\alpha-1}7^{2\beta}+d_2+d_3)<3+\frac{3}{7^{2\beta}}+\frac{7}{3^{2\alpha}}$$
$$4<12\left(\frac{1}{3}+\frac{d_2}{3^{2\alpha}7^{2\beta}}+\frac{d_3}{3^{2\beta}7^{2\beta}}\right)<3+\frac{3}{7^2}+\frac{7}{3^2}=3.8390\ldots,$$

we have a contradiction.

Case 10.2. $D_1=7$ and $D_2=9$. Recall thet $d_3=3^{a_3}7^{b_3}$, we have

$$\sigma(3^{2\alpha}7^{2\beta})=\frac{(3^{2\alpha+1}-1)}{2}\frac{(7^{2\beta+1}-1)}{6}=2\cdot3^{2\alpha}7^{2\beta}-3^{2\alpha}7^{2\beta-1}-3^{2\alpha-2}7^{2\beta}-3^{a_3}7^{b_3}$$
$$3^{2\alpha+1}7^{2\beta+1}-3^{2\alpha+1}-7^{2\beta+1}+1=12\cdot3^{2\alpha-2}7^{2\beta-1}(2\cdot3^2\cdot7-3^2-7)-12\cdot3^{a_3}7^{b_3}$$
$$441\cdot3^{2\alpha-1}7^{2\beta-1}-3^{2\alpha+1}-7^{2\beta+1}+1=440\cdot3^{2\alpha-1}7^{2\beta-1}-12\cdot3^{a_3}7^{b_3}$$
$$3^{2\alpha-1}7^{2\beta-1}-3^{2\alpha+1}-7^{2\beta+1}=-1-12\cdot3^{a_3}7^{b_3}$$
$$(3^{2\alpha-1}-49)(7^{2\beta-1}-9)=440-12\cdot3^{a_3}7^{b_3}. \qquad (3.77)$$

If $\alpha=1$, then $a_3=0,1,2$ and from (3.77) we get

$$-46(7^{2\beta-1}-9)=440-12\cdot3^{a_3}7^{b_3}$$
$$-23\cdot7^{2\beta-1}+207=220-6\cdot3^{a_3}7^{b_3}$$
$$-23\cdot7^{2\beta-1}=13-6\cdot3^{a_3}7^{b_3}.$$

Observe that $7|7^{2\beta-1}$ but $7\nmid13$, so $b_3=0$ that is $-23\cdot7^{2\beta-1}=13-6\cdot3^{a_3}$. We know that $-23\cdot7^{2\beta-1}=13-6\cdot3^{a_3}$ has no solution for $a_3=0,1$ or 2.

If $\alpha=2$, then from (3.77) we get

$$-22(7^{2\beta-1}-9)=440-12\cdot3^{a_3}7^{b_3}$$
$$-11\cdot7^{2\beta-1}+99=220-6\cdot3^{a_3}7^{b_3}$$
$$-11\cdot7^{2\beta-1}=121-6\cdot3^{a_3}7^{b_3}.$$

As $11|-11$ and $11|121$ but $11\nmid6\cdot3^{a_3}7^{b_3}$, which is impossible. Hene $\alpha>2$.

If $\beta = 1$, then

$$(3^{2\alpha-1} - 49)(-2) = 440 - 12 \cdot 3^{a_3}7^{b_3}$$
$$3^{2\alpha-1} - 49 = -220 + 6 \cdot 3^{a_3}7^{b_3}$$
$$3^{2\alpha-1} + 171 = 6 \cdot 3^{a_3}7^{b_3}.$$

We have $v_3(3^{2\alpha-1} + 171) = 2$, so $a_3 = 1$, thus

$$3^{2\alpha-1} + 171 = 6 \cdot 3 \cdot 7^{b_3}$$
$$3^{2\alpha-3} + 19 = 2 \cdot 7^{b_3}.$$

We have $3^{2\alpha-3} + 19 \equiv 1 \pmod 3$ but $2 \cdot 7^{b_3} \equiv 2 \pmod 3$, which is impossible.

Now we assume $\alpha > 2$ and $\beta > 1$. Then the left-hand side of (3.77) is more than 440, which is a contradiction.

Case 11. $p_2 = 5$. Recall that $d_1 = 3^{a_1}5^{b_1}$, $d_2 = 3^{a_2}5^{b_2}$, and $d_3 = 3^{a_3}5^{b_3}$.

Suppose $b_1 = b_2 = b_3 = 0$. That is $d_1 = 3^{a_1}$, $d_2 = 3^{a_2}$, and $d_3 = 3^{a_3}$ where $a_1 > a_2 > a_3 \geq 0$. Then

$$\sigma(3^{2\alpha}5^{2\beta}) = \frac{(3^{2\alpha+1} - 1)}{2}\frac{(5^{2\beta+1} - 1)}{4} = 2 \cdot 3^{2\alpha}5^{2\beta} - 3^{a_1} - 3^{a_2} - 3^{a_3}$$
$$(3^{2\alpha+1} - 1)(5^{2\beta+1} - 1) = 16 \cdot 3^{2\alpha}5^{2\beta} - 8 \cdot 3^{a_3}(3^{a_1-a_3} + 3^{a_2-a_3} + 1). \tag{3.78}$$

Consider LHS and RHS of (3.78).

We have $v_3(LHS) = v_3((3^{2\alpha+1} - 1)) + v_3((5^{2\beta+1} - 1)) = 0 + 0 = 0$ and $v_3(RHS) = a_3$. So $a_3 = 0$. Then

$$(3^{2\alpha+1} - 1)(5^{2\beta+1} - 1) = 16 \cdot 3^{2\alpha}5^{2\beta} - 8 \cdot (3^{a_1} + 3^{a_2}) - 8.$$

We see that $(3^{2\alpha+1} - 1)(5^{2\beta+1} - 1) \equiv (-1)(2 - 1) \equiv 2 \pmod 3$ but $16 \cdot 3^{2\alpha}5^{2\beta} - 8 \cdot (3^{a_1} + 3^{a_2}) - 8 \equiv 1 \pmod 3$, a contradiction. Therefore $b_1 + b_2 + b_3 \geq 1$.

Suppose $a_1 = a_2 = a_3 = 0$. That is $d_1 = 5^{b_1}$, $d_2 = 5^{b_2}$, and $d_3 = 5^{b_3}$ where $b_1 > b_2 > b_3 \geq 0$. Then

$$\sigma(3^{2\alpha}5^{2\beta}) = \frac{(3^{2\alpha+1} - 1)}{2}\frac{(5^{2\beta+1} - 1)}{4} = 2 \cdot 3^{2\alpha}5^{2\beta} - 5^{b_1} - 5^{b_2} - 5^{b_3}$$
$$(3^{2\alpha+1} - 1)(5^{2\beta+1} - 1) = 16 \cdot 3^{2\alpha}5^{2\beta} - 8 \cdot 5^{b_3}(5^{b_1-b_3} + 5^{b_2-b_3} + 1). \tag{3.79}$$

Consider LHS and RHS of (3.79).

We have $v_5(LHS) = v_5((3^{2\alpha+1} - 1)) + v_5((5^{2\beta+1} - 1)) = 0 + 0 = 0$ and $v_5(RHS) = b_3$. So $b_3 = 0$. Then

$$(3^{2\alpha+1} - 1)(5^{2\beta+1} - 1) = 16 \cdot 3^{2\alpha}5^{2\beta} - 8 \cdot (5^{b_1} + 5^{b_2}) - 8.$$

We see that $(3^{2\alpha+1}-1)(5^{2\beta+1}-1) \equiv 3$ or $4 \pmod 5$ but $16 \cdot 3^{2\alpha}5^{2\beta} - 8 \cdot (5^{b_1} + 5^{b_2}) - 8 \equiv 2 \pmod 5$, which is false. Therefore $a_1 + a_2 + a_3 \geq 1$.

Now we have

$$(3^{2\alpha+1} - 1)(5^{2\beta+1} - 1) = 16 \cdot 3^{2\alpha}5^{2\beta} - 8(3^{a_1}5^{b_1} + 3^{a_2}5^{b_2} + 3^{a_3}5^{b_3}), \qquad (3.80)$$

where $a_1 + a_2 + a_3 \geq 1$ and $b_1 + b_2 + b_3 \geq 1$.

Consider LHS and RHS of (3.80).

We have $v_3(RHS) = \min\{a_1, a_2, a_3\}$ or $\min\{a_1, a_2, a_3\}+1$. As $v_3(LHS) = 0$, so $v_3(RHS) = \min\{a_1, a_2, a_3\} = 0$.

We have $v_5(RHS) = \min\{b_1, b_2, b_3\}$ or $\min\{b_1, b_2, b_3\} + 1$. As $v_5(LHS) = 0$, so $v_5(RHS) = \min\{b_1, b_2, b_3\} = 0$.

We have $LHS \equiv 2 \pmod 3$ and

$$RHS \equiv 0 + \begin{cases} (0 + 0 + 5^{b_3}) \pmod 3, & \text{if } a_1 \neq 0,\ a_2 \neq 0,\ a_3 = 0; \\ (0 + 5^{b_2} + 0) \pmod 3, & \text{if } a_1 \neq 0,\ a_2 = 0,\ a_3 \neq 0; \\ (5^{b_1} + 0 + 0) \pmod 3, & \text{if } a_1 = 0,\ a_2 \neq 0,\ a_3 \neq 0; \\ (5^{b_1} + 5^{b_2} + 0) \pmod 3, & \text{if } a_1 = 0,\ a_2 = 0,\ a_3 \neq 0; \\ (5^{b_1} + 0 + 5^{b_3}) \pmod 3, & \text{if } a_1 = 0,\ a_2 \neq 0,\ a_3 = 0; \\ (0 + 5^{b_2} + 5^{b_3}) \pmod 3, & \text{if } a_1 \neq 0,\ a_2 = 0,\ a_3 = 0. \end{cases}$$

We can conclude that

if only one $a_i = 0$, then b_i is odd for $i = 1, 2, 3$ and we write $b_i = 2b_i + 1$, where $b_i \geq 0$

if exactly two $a_i = a_j = 0$, then b_i and b_j are even for $1 \leq i < j \leq 3$ and we write

$b_i = 2b_i$ where $b_i \geq 0$. Thus $(3^{2\alpha+1}-1)(5^{2\beta+1}-1)$ is equal to

$$16 \cdot 3^{2\alpha}5^{2\beta} - \begin{cases} 8(3^{a_1}5^{b_1} + 3^{a_2}5^{b_2} + 5^{2b_3+1}), & \text{if } a_1 \neq 0,\ a_2 \neq 0,\ a_3 = 0; \\ 8(3^{a_1}5^{b_1} + 5^{2b_2+1} + 3^{a_3}5^{b_3}), & \text{if } a_1 \neq 0,\ a_2 = 0,\ a_3 \neq 0; \\ 8(5^{2b_1+1} + 3^{a_2}5^{b_2} + 3^{a_3}5^{b_3}), & \text{if } a_1 = 0,\ a_2 \neq 0,\ a_3 \neq 0; \\ 8(5^{2b_1} + 5^{2b_2} + 3^{a_3}5^{b_3}), & \text{if } a_1 = 0,\ a_2 = 0,\ a_3 \neq 0; \\ 8(5^{2b_1} + +3^{a_2}5^{b_2} + 5^{2b_3}), & \text{if } a_1 = 0,\ a_2 \neq 0,\ a_3 = 0; \\ 8(3^{a_1}5^{b_1} + 5^{2b_2} + 5^{2b_3}), & \text{if } a_1 \neq 0,\ a_2 = 0,\ a_3 = 0. \end{cases} \tag{3.81}$$

We have

$$(3^{2\alpha+1}-1)(5^{2\beta+1}-1) \equiv \begin{cases} 3 \pmod 5, & \text{if } \alpha \equiv 0 \pmod 2; \\ 4 \pmod 5, & \text{if } \alpha \equiv 1 \pmod 2. \end{cases} \tag{3.82}$$

Noting that

$$\begin{aligned} \sigma(3^{2\alpha}5^{2\beta}) = \frac{(3^{2\alpha+1}-1)}{2}\frac{(5^{2\beta+1}-1)}{4} &= 2 \cdot 3^{2\alpha}5^{2\beta} - d_1 - d_2 - d_3 \\ (3^{2\alpha+1}-1)(5^{2\beta+1}-1) &= 16 \cdot 3^{2\alpha}5^{2\beta} - 8(d_1+d_2+d_3) \\ 16 \cdot 3^{2\alpha}5^{2\beta} - 8(d_1+d_2+d_3) &< 3^{2\alpha+1}5^{2\beta+1} \\ 3^{2\alpha}5^{2\beta} &< 8(d_1+d_2+d_3), \\ 1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(d_1+d_2+d_3). \end{aligned} \tag{3.83}$$

Noting that

$$\begin{aligned} \sigma(3^{2\alpha}5^{2\beta}) = \frac{(3^{2\alpha+1}-1)}{2}\frac{(5^{2\beta+1}-1)}{4} &= 2 \cdot 3^{2\alpha}5^{2\beta} - d_1 - d_2 - d_3 \\ 15 \cdot 3^{2\alpha}5^{2\beta} - 3^{2\alpha+1} - 5^{2\beta+1} + 1 &= 16 \cdot 3^{2\alpha}5^{2\beta} - 8(d_1+d_2+d_3) \\ 8(d_1+d_2+d_3) &= 3^{2\alpha}5^{2\beta} + 3^{2\alpha+1} + 5^{2\beta+1} - 1. \end{aligned}$$

We obtain that

$$8(d_1+d_2+d_3) < 3^{2\alpha}5^{2\beta} + 3^{2\alpha+1} + 5^{2\beta+1} \tag{3.84}$$

and

$$\frac{8}{3^{2\alpha}5^{2\beta}}(d_1+d_2+d_3) < 1 + \frac{3}{5^{2\beta}} + \frac{5}{3^{2\alpha}} \leq 1 + \frac{3}{5^2} + \frac{5}{3^2} = 1.675\ldots. \tag{3.85}$$

Consider (3.81), we have the following six cases.

Case 11.1. $(3^{2\alpha+1}-1)(5^{2\beta+1}-1) = 16\cdot 3^{2\alpha}5^{2\beta} - 8(3^{a_1}5^{b_1} + 3^{a_2}5^{b_2} + 5^{2b_3+1})$ where $a_1 \neq 0$, $a_2 \neq 0$.

If $b_1 \neq 0$ and $b_2 \neq 0$, that is

$$16\cdot 3^{2\alpha}5^{2\beta} - 8(3^{a_1}5^{b_1} + 3^{a_2}5^{b_2} + 5^{2b_3+1}) \equiv 0 - 3(0+0+0) \equiv 0 \pmod 5,$$

which contradicts with (3.82).

If $b_1 = 0$ and $b_2 = 0$, then from (3.83) we get that

$$\begin{aligned}
1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(3^{a_1} + 3^{a_2} + 5^{2b_3+1}) \\
&\leq \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha} + 3^{2\alpha-1} + 5^{2\beta-1}) \\
&= 8\left(\frac{1}{5^{2\beta}} + \frac{1}{3\cdot 5^{2\beta}} + \frac{1}{3^{2\alpha}\cdot 5}\right) \\
&\leq 8\left(\frac{1}{5^{2}} + \frac{1}{3\cdot 5^{2}} + \frac{1}{3^{2}\cdot 5}\right) \\
&= 0.6044\ldots,
\end{aligned}$$

we get a contradiction.

If $b_1 = 0$ and $b_2 \neq 0$, then from (3.83) we get that

$$\begin{aligned}
1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(3^{a_1} + 3^{a_2}5^{b_2} + 5^{2b_3+1}) \\
&< \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha} + 3^{2\alpha} + 5^{2\beta-1}) \\
&= 8\left(\frac{1}{5^{2\beta}} + \frac{1}{5^{2\beta}} + \frac{1}{3^{2\alpha}\cdot 5}\right) \\
&\leq 8\left(\frac{1}{5^{2}} + \frac{1}{5^{2}} + \frac{1}{3^{2}\cdot 5}\right) \\
&= 0.8177\ldots,
\end{aligned}$$

which is false.

So $b_1 \neq 0$ and $b_2 = 0$, that is $d_1 = 3^{a_1}5^{b_1}$, $d_2 = 3^{a_2}$, and $d_3 = 5^{2b_3+1}$.

Suppose that $a_1 \leq 2\alpha - 3$. From (3.83), it follows that

$$\begin{aligned}
1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(3^{a_1}5^{b_1} + 3^{a_2} + 5^{2b_3+1}) \\
&\leq \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha-3}5^{2\beta} + 3^{2\alpha} + 5^{2\beta-1}) \\
&= 8\left(\frac{1}{3^3} + \frac{1}{5^{2\beta}} + \frac{1}{3^{2\alpha}\cdot 5}\right) \\
&\leq 8\left(\frac{1}{3^3} + \frac{1}{5^2} + \frac{1}{3^2\cdot 5}\right) \\
&= 0.7940\ldots,
\end{aligned}$$

which is not true. So $a_1 = 2\alpha - 2$, $2\alpha - 1$ or 2α.

Suppose that $b_1 \leq 2\beta - 2$. From (3.83), it follows that

$$\begin{aligned}
1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(3^{a_1}5^{b_1} + 3^{a_2} + 5^{2b_3+1}) \\
&\leq \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha}5^{2\beta-2} + 3^{2\alpha} + 5^{2\beta-1}) \\
&= 8\left(\frac{1}{5^2} + \frac{1}{5^{2\beta}} + \frac{1}{3^{2\alpha}\cdot 5}\right) \\
&\leq 8\left(\frac{1}{5^2} + \frac{1}{5^2} + \frac{1}{3^2\cdot 5}\right) \\
&= 0.8177\ldots,
\end{aligned}$$

we have a contradiction. So $b_1 = 2\beta - 1$ or 2β.

Now we consider the following six cases.

Case 11.1.1. $a_1 = 2\alpha - 2$ and $b_1 = 2\beta - 1$. From (3.83), we get that

$$\begin{aligned}
1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha-2}5^{2\beta-1} + 3^{a_2} + 5^{2b_3+1}) \\
&\leq \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha-2}5^{2\beta-1} + 3^{2\alpha} + 5^{2\beta-1}) \\
&= 8\left(\frac{1}{3^2\cdot 5} + \frac{1}{5^{2\beta}} + \frac{1}{3^{2\alpha}\cdot 5}\right) \\
&\leq 8\left(\frac{1}{3^2\cdot 5} + \frac{1}{5^2} + \frac{1}{3^2\cdot 5}\right) \\
&= 0.6755\ldots,
\end{aligned}$$

which is false.

Case 11.1.2. $a_1 = 2\alpha - 2$ and $b_1 = 2\beta$. Since $a_1 \neq 0$, so $\alpha \geq 2$.

If $\beta \geq 2$, then from (3.83) we get that

$$\begin{aligned}
1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha-2}5^{2\beta} + 3^{a_2} + 5^{2b_3+1}) \\
&\leq \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha-2}5^{2\beta} + 3^{2\alpha} + 5^{2\beta-1}) \\
&= 8\left(\frac{1}{3^2} + \frac{1}{5^{2\beta}} + \frac{1}{3^{2\alpha}\cdot 5}\right) \\
&\leq 8\left(\frac{1}{3^2} + \frac{1}{5^4} + \frac{1}{3^4\cdot 5}\right) \\
&= 0.9214\ldots,
\end{aligned}$$

which is not true. So $\beta = 1$ and we get

$$\begin{aligned}
(3^{2\alpha+1}-1)(5^3-1) &= 16\cdot 3^{2\alpha}\cdot 5^2 - 8(3^{2\alpha-2}\cdot 5^2 + 3^{a_2} + 5^{2b_3+1}) \\
(3^{2\alpha+1}-1)\cdot 124 &= 400\cdot 3^{2\alpha} - 8(3^{2\alpha-2}\cdot 25 + 3^{a_2} + 5) \\
(3^{2\alpha+1}-1)\cdot 31 &= 100\cdot 3^{2\alpha} - 2(3^{2\alpha-2}\cdot 25 + 3^{a_2} + 5) \\
837\cdot 3^{2\alpha-2} - 31 &= 900\cdot 3^{2\alpha-2} - 50\cdot 3^{2\alpha-2} - 2\cdot 3^{a_2} - 10 \\
-13\cdot 3^{2\alpha-2} + 2\cdot 3^{a_2} &= 21.
\end{aligned}$$

If $a_2 \leq 2\alpha - 1$, then $-13\cdot 3^{2\alpha-2} + 2\cdot 3^{a_2} \leq -13\cdot 3^{2\alpha-2} + 2\cdot 3^{2\alpha-1} = -13\cdot 3^{2\alpha-2} + 6\cdot 3^{2\alpha-2} < 0$, we have a contradiction.
If $a_2 = 2\alpha$, then $-13\cdot 3^{2\alpha-2} + 2\cdot 3^{a_2} = -13\cdot 3^{2\alpha-2} + 2\cdot 3^{2\alpha} = 5\cdot 3^{2\alpha-2} \equiv 0 \pmod 5$ but $21 \not\equiv 0 \pmod 5$, we get a contradiction.

Case 11.1.3. $a_1 = 2\alpha - 1$ and $b_1 = 2\beta - 1$. If $\alpha \geq 2$ or $\beta \geq 2$ then from (3.83) we get that

$$\begin{aligned}
1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha-1}5^{2\beta-1} + 3^{a_2} + 5^{2b_3+1}) \\
&\leq \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha-1}5^{2\beta-1} + 3^{2\alpha} + 5^{2\beta-1}) \\
&= 8\left(\frac{1}{3\cdot 5} + \frac{1}{5^{2\beta}} + \frac{1}{3^{2\alpha}\cdot 5}\right) \\
&\leq 8\left(\frac{1}{3\cdot 5} + \frac{1}{5^2} + \frac{1}{3^4\cdot 5}\right) \\
&= 0.8730\ldots,
\end{aligned}$$

which is not possible. Hence $\alpha = 1$ and $\beta = 1$, so $a_1 = 1$ and $b_1 = 1$. Then

$$\begin{aligned}(3^{2(1)+1} - 1)(5^{2(1)+1} - 1) &= 16 \cdot 3^2 \cdot 5^2 - 8(3 \cdot 5 + 3^{a_2} + 5^{2b_3+1}) \\ 26 \times 124 &= 3600 - 8(3 \cdot 5 + 3^2 + 5) \\ 3224 &= 3368,\end{aligned}$$

which is a contradiction.

Case 11.1.4. $a_1 = 2\alpha - 1$ and $b_1 = 2\beta$. We have

$$\begin{aligned}\frac{8}{3^{2\alpha}5^{2\beta}}(d_1 + d_2 + d_3) &= \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha-1}5^{2\beta} + 3^{a_2} + 5^{2b_3+1}) \\ &\geq \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha-1}5^{2\beta} + 3^2 + 5) \\ &= 8\left(\frac{1}{3} + \frac{14}{3^{2\alpha}5^{2\beta}}\right) \\ &> 2,\end{aligned}$$

which is contradicts with (3.85).

Case 11.1.5. $a_1 = 2\alpha$ and $b_1 = 2\beta - 1$. We have

$$8(3^{2\alpha}5^{2\beta-1} + 3^{a_2} + 5^{2b_3+1}) > 8(3^{2\alpha}5^{2\beta-1} + 3^2 + 5) = 8 \cdot 3^{2\alpha}5^{2\beta-1} + 112.$$

From (3.84), we get

$$\begin{aligned}8 \cdot 3^{2\alpha}5^{2\beta-1} + 112 &< 3^{2\alpha}5^{2\beta} + 3^{2\alpha+1} + 5^{2\beta+1} \\ 3 \cdot 3^{2\alpha}5^{2\beta-1} - 3^{2\alpha+1} - 5^{2\beta+1} &< -112 \\ 3 \cdot 3^{2\alpha}5^{2\beta-1} - 3^{2\alpha+1} - 5^{2\beta+1} + 25 &< -112 + 25 \\ (3^{2\alpha+1} - 25)(5^{2\beta-1} - 1) &< -87.\end{aligned}$$

As $\alpha,\ \beta \geq 1$, so $(3^{2\alpha+1} - 25)(5^{2\beta-1} - 1) > 0$, which is not possible.

Case 11.1.6. $a_1 = 2\alpha$ and $b_1 = 2\beta$. Then $d_1 = n$, which is impossible.

Case 11.2. $(3^{2\alpha+1}-1)(5^{2\beta+1}-1) = 16 \cdot 3^{2\alpha}5^{2\beta} - 8(3^{a_1}5^{b_1} + 5^{2b_2+1} + 3^{a_3}5^{b_3})$ where $a_1 \neq 0$, $a_3 \neq 0$.

If $b_1 \neq 0$ and $b_3 \neq 0$, that is

$$16 \cdot 3^{2\alpha}5^{2\beta} - 8(3^{a_1}5^{b_1} + 5^{2b_2+1} + 3^{a_3}5^{b_3}) \equiv 0 - 3(0 + 0 + 0) \equiv 0 \pmod 5,$$

which contradicts with (3.82).

If $b_1 = 0$ and $b_3 = 0$, then from (3.83) we get that

$$\begin{aligned}
1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(3^{a_1} + 5^{2b_2+1} + 3^{a_3}) \\
&\le \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha} + 5^{2\beta-1} + 3^{2\alpha-1}) \\
&= 8\left(\frac{1}{5^{2\beta}} + \frac{1}{3^{2\alpha}\cdot 5} + \frac{1}{3\cdot 5^{2\beta}}\right) \\
&\le 8\left(\frac{1}{5^{2}} + \frac{1}{3^{2}\cdot 5} + \frac{1}{3\cdot 5^{2}}\right) \\
&= 0.6044\ldots,
\end{aligned}$$

we get a contradiction.

If $b_1 = 0$ and $b_3 \neq 0$, then from (3.83) we get that

$$\begin{aligned}
1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(3^{a_1} + 5^{2b_2+1} + 3^{a_3}\cdot 5^{b_3}) \\
&< \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha} + 5^{2\beta-1} + 5^{2\beta-1}) \\
&= 8\left(\frac{1}{5^{2\beta}} + \frac{1}{3^{2\alpha}\cdot 5} + \frac{1}{3^{2\alpha}\cdot 5}\right) \\
&\le 8\left(\frac{1}{5^{2}} + \frac{1}{3^{2}\cdot 5} + \frac{1}{3^{2}\cdot 5}\right) \\
&= 0.6755\ldots,
\end{aligned}$$

which is false.

So $b_1 \neq 0$ and $b_3 = 0$, that is $d_1 = 3^{a_1}5^{b_1}$, $d_2 = 5^{2b_2+1}$, and $d_3 = 3^{a_3}$. Suppose that $a_1 \le 2\alpha - 3$. From (3.83), it follows that

$$\begin{aligned}
1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(3^{a_1}5^{b_1} + 5^{2b_2+1} + 3^{a_3}) \\
&\le \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha-3}5^{2\beta} + 5^{2\beta-1} + 3^{2\alpha}) \\
&= 8\left(\frac{1}{3^{3}} + \frac{1}{3^{2\alpha}\cdot 5} + \frac{1}{5^{2\beta}}\right) \\
&\le 8\left(\frac{1}{3^{3}} + \frac{1}{3^{2}\cdot 5} + \frac{1}{5^{2}}\right) \\
&= 0.7940\ldots,
\end{aligned}$$

which is impossible. So $a_1 = 2\alpha - 2$, $2\alpha - 1$ or 2α.

Suppose that $b_1 \leq 2\beta - 2$. From (3.83), it follows that

$$\begin{aligned}
1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(3^{a_1}5^{b_1} + 5^{2b_2+1} + 3^{a_3}) \\
&\leq \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha}5^{2\beta-2} + 5^{2\beta-1} + 3^{2\alpha}) \\
&= 8\left(\frac{1}{5^2} + \frac{1}{3^{2\alpha}\cdot 5} + \frac{1}{5^{2\beta}}\right) \\
&\leq 8\left(\frac{1}{5^2} + \frac{1}{3^2\cdot 5} + \frac{1}{5^2}\right) \\
&= 0.8177\ldots,
\end{aligned}$$

we get a contradiction. So $b_1 = 2\beta - 1$ or 2β.

Now we consider the following six subsubcases.

Case 11.2.1. $a_1 = 2\alpha - 2$ and $b_1 = 2\beta - 1$.

From (3.83), it follows that

$$\begin{aligned}
1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha-2}5^{2\beta-1} + 5^{2b_2+1} + 3^{a_3}) \\
&\leq \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha-2}5^{2\beta-1} + 5^{2\beta-1} + 3^{2\alpha}) \\
&= 8\left(\frac{1}{3^2\cdot 5} + \frac{1}{3^{2\alpha}\cdot 5} + \frac{1}{5^{2\beta}}\right) \\
&\leq 8\left(\frac{1}{3^2\cdot 5} + \frac{1}{3^2\cdot 5} + \frac{1}{5^2}\right) \\
&= 0.6755\ldots,
\end{aligned}$$

which is not true.

Case 11.2.2. $a_1 = 2\alpha - 2$ and $b_1 = 2\beta$. Since $a_1 \neq 0$, so $\alpha \geq 2$.

If $\beta \geq 2$, then from (3.83) we get that

$$\begin{aligned}
1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha-2}5^{2\beta} + 5^{2b_2+1} + 3^{a_3}) \\
&\leq \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha-2}5^{2\beta} + 5^{2\beta-1} + 3^{2\alpha}) \\
&= 8\left(\frac{1}{3^2} + \frac{1}{3^{2\alpha}\cdot 5} + \frac{1}{5^{2\beta}}\right) \\
&\leq 8\left(\frac{1}{3^2} + \frac{1}{3^4\cdot 5} + \frac{1}{5^4}\right) \\
&= 0.9214\ldots,
\end{aligned}$$

we get a contradiction. So $\beta = 1$ and we get

$$\begin{aligned}
(3^{2\alpha+1}-1)(5^3-1) &= 16\cdot 3^{2\alpha}\cdot 5^2 - 8(3^{2\alpha-2}\cdot 5^2 + 5^{2b_2+1} + 3^{a_3})\\
(3^{2\alpha+1}-1)\cdot 124 &= 400\cdot 3^{2\alpha} - 8(3^{2\alpha-2}\cdot 25 + 5 + 3)\\
(3^{2\alpha+1}-1)\cdot 31 &= 100\cdot 3^{2\alpha} - 2(3^{2\alpha-2}\cdot 25 + 8)\\
837\cdot 3^{2\alpha-2} - 31 &= 900\cdot 3^{2\alpha-2} - 50\cdot 3^{2\alpha-2} - 16\\
-13\cdot 3^{2\alpha-2} &= 15,
\end{aligned}$$

which is false.

Subsubcase 11.2.3. $a_1 = 2\alpha - 1$ and $b_1 = 2\beta - 1$.

If $\alpha \geq 2$ or $\beta \geq 2$ then from (3.83) we get that

$$\begin{aligned}
1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha-1}5^{2\beta-1} + 5^{2b_2+1} + 3^{a_3})\\
&\leq \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha-1}5^{2\beta-1} + 5^{2\beta-1} + 3^{2\alpha})\\
&= 8\left(\frac{1}{3\cdot 5} + \frac{1}{3^{2\alpha}\cdot 5} + \frac{1}{5^{2\beta}}\right)\\
&\leq 8\left(\frac{1}{3\cdot 5} + \frac{1}{3^4\cdot 5} + \frac{1}{5^2}\right)\\
&= 0.8730\ldots,
\end{aligned}$$

we get a contradiction. Hence $\alpha = 1$ and $\beta = 1$, so $a_1 = 1$ and $b_1 = 1$. Then

$$\begin{aligned}
(3^{2(1)+1}-1)(5^{2(1)+1}-1) &= 16\cdot 3^2\cdot 5^2 - 8(3\cdot 5 + 5^{2b_3+1} + 3^{a_2})\\
26\times 124 &= 3600 - 8(3\cdot 5 + 5 + 3)\\
3224 &= 3416,
\end{aligned}$$

which is not true.

Case 11.2.4. $a_1 = 2\alpha - 1$ and $b_1 = 2\beta$. We have

$$\begin{aligned}
\frac{8}{3^{2\alpha}5^{2\beta}}(d_1+d_2+d_3) &= \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha-1}5^{2\beta} + 5^{2b_2+1} + 3^{a_3})\\
&\geq \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha-1}5^{2\beta} + 5 + 3)\\
&= 8\left(\frac{1}{3} + \frac{8}{3^{2\alpha}5^{2\beta}}\right)\\
&> 2,
\end{aligned}$$

which contradicts with (3.85).

Case 11.2.5. $a_1 = 2\alpha$ and $b_1 = 2\beta - 1$. We have

$$8(3^{2\alpha}5^{2\beta-1} + 5^{2b_2+1} + 3^{a_3}) > 8(3^{2\alpha}5^{2\beta-1} + 5 + 3) = 8 \cdot 3^{2\alpha}5^{2\beta-1} + 64.$$

From (3.84), we get

$$\begin{aligned} 8 \cdot 3^{2\alpha}5^{2\beta-1} + 64 &< 3^{2\alpha}5^{2\beta} + 3^{2\alpha+1} + 5^{2\beta+1} \\ 3 \cdot 3^{2\alpha}5^{2\beta-1} - 3^{2\alpha+1} - 5^{2\beta+1} &< -64 \\ 3 \cdot 3^{2\alpha}5^{2\beta-1} - 3^{2\alpha+1} - 5^{2\beta+1} + 25 &< -64 + 25 \\ (3^{2\alpha+1} - 25)(5^{2\beta-1} - 1) &< -39. \end{aligned}$$

As $\alpha,\ \beta \geq 1$, so $(3^{2\alpha+1} - 25)(5^{2\beta-1} - 1) > 0$, which is a contradiction.

Case 11.2.6. $a_1 = 2\alpha$ and $b_1 = 2\beta$. Then $d_1 = n$, which is impossible.

Case 11.3. $(3^{2\alpha+1} - 1)(5^{2\beta+1} - 1) = 16 \cdot 3^{2\alpha}5^{2\beta} - 8(5^{2b_1+1} + 3^{a_2}5^{b_2} + 3^{a_3}5^{b_3})$ where $a_2 \neq 0$, $a_3 \neq 0$.

Case 11.3.1. $b_2 \neq 0$ and $b_3 \neq 0$. Then

$$16 \cdot 3^{2\alpha}5^{2\beta} - 8(5^{2b_1+1} + 3^{a_2}5^{b_2} + 3^{a_3}5^{b_3}) \equiv 0 - 3(0 + 0 + 0) \equiv 0 \pmod{5},$$

which contradicts with (3.82).

Case 11.3.2. $b_2 = 0$ and $b_3 = 0$. From (3.83) we get that

$$\begin{aligned} 1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(5^{2b_1+1} + 3^{a_2} + 3^{a_3}) \\ &\leq \frac{8}{3^{2\alpha}5^{2\beta}}(5^{2\beta-1} + 3^{2\alpha} + 3^{2\alpha-1}) \\ &= 8\left(\frac{1}{3^{2\alpha} \cdot 5} + \frac{1}{5^{2\beta}} + \frac{1}{3 \cdot 5^{2\beta}}\right) \\ &\leq 8\left(\frac{1}{3^2 \cdot 5} + \frac{1}{5^2} + \frac{1}{3 \cdot 5^2}\right) \\ &= 0.6044\ldots, \end{aligned}$$

which is a contradiction.

Case 11.3.3. $b_2 = 0$ and $b_3 \neq 0$. From (3.83) we get that

$$\begin{aligned} 1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(5^{2b_1+1} + 3^{a_2} + 3^{a_3}5^{b_3}) \\ &< \frac{8}{3^{2\alpha}5^{2\beta}}(5^{2\beta-1} + 3^{2\alpha} + 3^{2\alpha}) \\ &= 8\left(\frac{1}{3^{2\alpha}\cdot 5} + \frac{1}{5^{2\beta}} + \frac{1}{5^{2\beta}}\right) \\ &\leq 8\left(\frac{1}{3^{2}\cdot 5} + \frac{1}{5^{2}} + \frac{1}{5^{2}}\right) \\ &= 0.8177\ldots, \end{aligned}$$

which is not possible.

Case 11.3.4. $b_2 \neq 0$ and $b_3 = 0$. From (3.83) we get that

$$\begin{aligned} 1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(5^{2b_1+1} + 3^{a_2}5^{b_2} + 3^{a_3}) \\ &< \frac{8}{3^{2\alpha}5^{2\beta}}(5^{2\beta-1} + 5^{2\beta-1} + 3^{2\alpha}) \\ &= 8\left(\frac{1}{3^{2\alpha}\cdot 5} + \frac{1}{3^{2\alpha}\cdot 5} + \frac{1}{5^{2\beta}}\right) \\ &\leq 8\left(\frac{1}{3^{2}\cdot 5} + \frac{1}{3^{2}\cdot 5} + \frac{1}{5^{2}}\right) \\ &= 0.6755\ldots, \end{aligned}$$

we get a contradiction.

Case 11.4. $(3^{2\alpha+1} - 1)(5^{2\beta+1} - 1) = 16 \cdot 3^{2\alpha}5^{2\beta} - 8(5^{2b_1} + 5^{2b_2} + 3^{a_3}5^{b_3})$ where $a_3 \neq 0$. As $d_1 > d_2 > d_3$, so $b_1 > b_2 \geq 1$. Since $\min\{b_1, b_2, b_3\} = 0$, so $b_3 = 0$.

If $\alpha \geq 2$, then from (3.83) we get that

$$\begin{aligned} 1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(5^{2b_1} + 5^{2b_2} + 3^{a_3}) \\ &\leq \frac{8}{3^{2\alpha}5^{2\beta}}(5^{2\beta} + 5^{2\beta-2} + 3^{2\alpha}) \\ &= 8\left(\frac{1}{3^{2\alpha}} + \frac{1}{3^{2\alpha}\cdot 5^{2}} + \frac{1}{5^{2\beta}}\right) \\ &\leq 8\left(\frac{1}{3^{4}} + \frac{1}{3^{4}\cdot 5^{2}} + \frac{1}{5^{2}}\right) \\ &= 0.4227\ldots, \end{aligned}$$

which is not true. So $\alpha = 1$. As $a_3 \neq 0$, thus $a_3 = 1$ or 2.

If $a_3 = 1$, then

$$16 \cdot 3^{2\alpha}5^{2\beta} - 8(5^{2b_1} + 5^{2b_2} + 3) \equiv 0 - 3(0 + 0 + 3) \equiv 1 \pmod 5,$$

which contradicts with (3.82).

If $a_3 = 2$, then

$$16 \cdot 3^{2\alpha}5^{2\beta} - 8(5^{2b_1} + 5^{2b_2} + 3^2) \equiv 0 - 3(0 + 0 + 4) \equiv 3 \pmod 5,$$

which contradicts with (3.82).

Case 11.5. $(3^{2\alpha+1} - 1)(5^{2\beta+1} - 1) = 16 \cdot 3^{2\alpha}5^{2\beta} - 8(5^{2b_1} + 3^{a_2}5^{b_2} + 5^{2b_3})$ where $a_2 \neq 0$.

Case 11.5.1. $b_2 \neq 0$ and $b_3 \neq 0$. Then from $d_1 > d_2 > d_3$ we get that $\min\{b_1, b_2, b_3\} > 0$, a contradiction.

Case 11.5.2. $b_2 = 0$ and $b_3 = 0$. If $\alpha \geq 2$, then from (3.83)

$$\begin{aligned}
1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(5^{2b_1} + 3^{a_2} + 1) \\
&\leq \frac{8}{3^{2\alpha}5^{2\beta}}(5^{2\beta} + 3^{2\alpha} + 1) \\
&= 8\left(\frac{1}{3^{2\alpha}} + \frac{1}{5^{2\beta}} + \frac{1}{3^{2\alpha}5^{2\beta}}\right) \\
&\leq 8\left(\frac{1}{3^4} + \frac{1}{5^2} + \frac{1}{3^4 \cdot 5^2}\right) \\
&= 0.4227\ldots,
\end{aligned}$$

which is false. So $\alpha = 1$. As $a_2 \neq 0$, thus $a_2 = 1$ or 2.

If $a_2 = 1$, then

$$16 \cdot 3^{2\alpha}5^{2\beta} - 8(5^{2b_1} + 3^{a_2} + 1) \equiv 0 - 3(0 + 3 + 1) \equiv 3 \pmod 5,$$

which contradicts with (3.82).

If $a_2 = 2$, then

$$16 \cdot 3^{2\alpha}5^{2\beta} - 8(5^{2b_1} + 3^{a_2} + 1) \equiv 0 - 3(0 + 4 + 1) \equiv 0 \pmod 5,$$

which contradicts with (3.82).

Case 11.5.3. $b_2 = 0$ and $b_3 \neq 0$. Then $d_1 = 5^{2b_1}$, $d_2 = 3^{a_2}$, $d_3 = 5^{2b_3}$. As $d_1 > d_2 > d_3$, so $a_1 \geq 3$, i.e., $\alpha \geq 2$. From (3.83), we get that

$$\begin{aligned}
1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(5^{2b_1} + 3^{a_2} + 5^{2b_3}) \\
&\leq \frac{8}{3^{2\alpha}5^{2\beta}}(5^{2\beta} + 3^{2\alpha} + 5^{2\beta-2}) \\
&= 8\left(\frac{1}{3^{2\alpha}} + \frac{1}{5^{2\beta}} + \frac{1}{3^{2\alpha}5^2}\right) \\
&\leq 8\left(\frac{1}{3^4} + \frac{1}{5^2} + \frac{1}{3^4 \cdot 5^2}\right) \\
&= 0.4227\ldots.
\end{aligned}$$

Case 11.5.4. $b_2 \neq 0$ and $b_3 = 0$. Then

$$16 \cdot 3^{2\alpha}5^{2\beta} - 8(5^{2b_1} + 3^{a_2}5^{b_2} + 5^{2b_3}) \equiv 0 - 3(0 + 0 + 1) \equiv 2 \pmod 5,$$

which contradicts with (3.82).

Case 11.6. $(3^{2\alpha+1} - 1)(5^{2\beta+1} - 1) = 16 \cdot 3^{2\alpha}5^{2\beta} - 8(3^{a_1}5^{b_1} + 5^{2b_2} + 5^{2b_3})$ where $a_1 \neq 0$.

Case 11.6.1. $b_1 \neq 0$ and $b_3 \neq 0$. Since $d_1 > d_2 > d_3$, we get that $\min\{b_1, b_2, b_3\} > 0$, which is a contradiction.

Case 11.6.2. $b_1 = 0$ and $b_3 = 0$. Then $d_1 = 3^{a_1}$, $d_2 = 5^{2b_2}$, $d_3 = 1$. As $d_1 > d_2 > d_3$, so $b_2 \geq 1$ and $a_1 \geq 3$, i.e., $\alpha \geq 2$. From (3.83), we get that

$$\begin{aligned}
1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(3^{a_1} + 5^{2b_2} + 1) \\
&\leq \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha} + 5^{2\beta} + 1) \\
&= 8\left(\frac{1}{5^{2\beta}} + \frac{1}{3^{2\alpha}} + \frac{1}{3^{2\alpha}5^{2\beta}}\right) \\
&\leq 8\left(\frac{1}{5^2} + \frac{1}{3^4} + \frac{1}{3^4 \cdot 5^2}\right) \\
&= 0.4227\ldots,
\end{aligned}$$

which is false.

Case 11.6.3. $b_1 = 0$ and $b_3 \neq 0$. Then $d_1 = 3^{a_1}$, $d_2 = 5^{2b_2}$, $d_3 = 5^{2b_3}$. As

$d_1 > d_2 > d_3$, so $b_2 \geq 2$ and $a_1 \geq 6$, i.e., $\alpha \geq 3$. From (3.83), we get that

$$\begin{aligned} 1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(3^{a_1} + 5^{2b_2} + 5^{2b_3}) \\ &\leq \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha} + 5^{2\beta} + 5^{2\beta-2}) \\ &= 8\left(\frac{1}{5^{2\beta}} + \frac{1}{3^{2\alpha}} + \frac{1}{3^{2\alpha}5^2}\right) \\ &\leq 8\left(\frac{1}{5^2} + \frac{1}{3^6} + \frac{1}{3^6 \cdot 5^2}\right) \\ &= 0.3314\ldots, \end{aligned}$$

which is impossible.

Case 11.6.4. $b_1 \neq 0$ and $b_3 = 0$. Then

$$16 \cdot 3^{2\alpha}5^{2\beta} - 8(3^{a_1}5^{b_1} + 5^{2b_2} + 1) \equiv 0 - 3(0 + 0 + 1) \equiv 2 \pmod 5,$$

which contradicts with (3.82). □

Remark 3.4. In addition, we determine all odd exactly 2-near-perfect number with at most three distinct prime factors. To better understand, let n be a positive integer with at most three distinct prime factors. Then n is an odd exactly 2-near-perfect number if and only if n is one of the following numbers:

(i) $n = 2205 = 3^2 \cdot 5 \cdot 7^2$ with two redundant divisors $d_1 = 35$ and $d_2 = 1$ or $d_1 = 21$ and $d_2 = 15$;

(ii) $n = 15435 = 3^2 \cdot 5 \cdot 7^3$ with two redundant divisors $d_1 = 315$ and $d_2 = 15$;

(iii) $n = 945 = 3^3 \cdot 5 \cdot 7$ with two redundant divisors $d_1 = 27$ and $d_2 = 3$ or $d_1 = 21$ and $d_2 = 9$;

(iv) $n = 6615 = 3^3 \cdot 5 \cdot 7^2$ with two redundant divisors $d_1 = 441$ and $d_2 = 9$ or $d_1 = 315$ and $d_2 = 135$;

(v) $n = 23625 = 3^3 \cdot 5^3 \cdot 7$ with two redundant divisors $d_1 = 2625$ and $d_2 = 45$;

(vi) $n = 2835 = 3^4 \cdot 5 \cdot 7$ with two redundant divisors $d_1 = 135$ and $d_2 = 3$;

(vii) $n = 7425 = 3^3 \cdot 5^2 \cdot 11$ with two redundant divisors $d_1 = 27$ and $d_2 = 3$ or $d_1 = 25$ and $d_2 = 5$;

(viii) $n = 37125 = 3^3 \cdot 5^3 \cdot 11$ with two redundant divisors $d_1 = 495$ and $d_2 = 135$;

(ix) $n = 22275 = 3^4 \cdot 5^2 \cdot 11$ with two redundant divisors $d_1 = 297$ and $d_2 = 165$;

(x) $n = 2695275 = 3^4 \cdot 5^2 \cdot 11^3$ with two redundant divisors $d_1 = 99825$ and $d_2 = 1089$;

(xi) $n = 570375 = 3^3 \cdot 5^3 \cdot 13^2$ with two redundant divisors $d_1 = 1125$ and $d_2 = 45$ or $d_1 = 975$ and $d_2 = 195$ or $d_1 = 845$ and $d_2 = 325$;

(xii) $n = 7414875 = 3^3 \cdot 5^3 \cdot 13^3$ with two redundant divisors $d_1 = 21125$ and $d_2 = 325$ or $d_1 = 12675$ and $d_2 = 8775$;

(xiii) $n = 14259375 = 3^3 \cdot 5^5 \cdot 13^2$ with two redundant divisors $d_1 = 73125$ and $d_2 = 45$;

(xiv) $n = 131625 = 3^4 \cdot 5^3 \cdot 13$ with two redundant divisors $d_1 = 975$ and $d_2 = 39$;

(xv) $n = 78975 = 3^5 \cdot 5^2 \cdot 13$ with two redundant divisors $d_1 = 25$ and $d_2 = 1$;

(xvi) $n = 394875 = 3^5 \cdot 5^3 \cdot 13$ with two redundant divisors $d_1 = 4875$ and $d_2 = 351$.

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Appendix

ON EXACTLY 3-DEFICIENT-PERFECT NUMBERS

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Abstract. Let n and k be positive integers and $\sigma(n)$ the sum of all positive divisors of n. We call n an exactly k-deficient-perfect number with deficient divisors $d_1, d_2, \ldots, d_k$ if $d_1, d_2, \ldots, d_k$ are distinct proper divisors of n and $\sigma(n) = 2n - (d_1 + d_2 + \cdots + d_k)$. In this article, we show that the only odd exactly 3-deficient-perfect number with at most two distinct prime factors is $1521 = 3^2 \cdot 13^2$.

1. Introduction

Throughout this article, let n be a positive integer, $\sigma(n)$ the sum of all positive divisors of n, and $\omega(n)$ the number of distinct prime factors of n. We say that n is perfect if $\sigma(n) = 2n$. It is well-known that n is even and perfect if and only if $n = 2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are primes. It has also been a longstanding conjecture that there are infinitely many even perfect numbers and that an odd perfect number does not exist. Attempting to understand perfect numbers, mathematicians have studied other closely related concepts. Recall that if $\sigma(n) < 2n$, then n is said to be deficient; if $\sigma(n) > 2n$, then n is abundant; if $\sigma(n) = 2n + 1$, then n is quasiperfect; if $\sigma(n) = 2n - 1$, then n is almost perfect. For more information on this topic, see for example the work of Cohen [5, 6], Hagis and Cohen [11], Kishore [14], Ochem and Rao [18], Yamada [36], and the online databases GIMPS [10] and OEIS [30].

Sierpiński [29] called n pseudoperfect if n can be written as a sum of some of its proper divisors. Pollack and Shevelev [21] have recently initiated the study of a subclass of pseudoperfect numbers leading to an active investigation. We summarize this work in the following definition.

Definition 1.1. *Let n and k be positive integers. We say that n is near-perfect if n is the sum of all of its proper divisors except one of them. In addition, n is k-near-perfect if n can be written as a sum of all of its proper divisors with at most k exceptions. Moreover, n is exactly k-near-perfect if n is expressible as a sum of all of its proper divisors with exactly k exceptions. The exceptional divisors are said to be redundant. In other words,*

n is near-perfect with a redundant divisor $d \Leftrightarrow 1 \le d < n$, $d \mid n$, and $\sigma(n) = 2n + d$;

n is 1-near-perfect $\Leftrightarrow$ n is perfect or n is near-perfect;

n is exactly k-near-perfect with redundant divisors $d_1, d_2, \ldots, d_k \Leftrightarrow$

$d_1, d_2, \ldots, d_k$ are distinct proper divisors of n and $\sigma(n) = 2n + d_1 + d_2 + \cdots + d_k$.

Motivated by the concept of near-perfect numbers, Tang, Ren, and Li [35] define the notion of deficient-perfect numbers, which also leads to an interesting research problem.

Definition 1.2. *Let $n, k \in \mathbb{N}$. Then, n is called a deficient-perfect number with a deficient divisor d if d is a proper divisor of n and $\sigma(n) = 2n - d$. Furthermore, n is exactly k-deficient-perfect with deficient divisors $d_1, d_2, \ldots, d_k$ if $d_1, d_2, \ldots, d_k$ are distinct proper divisors of n*

*Prapanpong Pongsriiam is the corresponding author.

and $\sigma(n) = 2n - (d_1 + d_2 + \cdots + d_k)$. *In addition,* n *is* k*-deficient-perfect if* n *is perfect or* n *is exactly* ℓ*-deficient-perfect for some* $\ell = 1, 2, \ldots, k$.

In 2012, Pollack and Shevelev [21] showed that the number of near-perfect numbers not exceeding x is $\ll x^{5/6+o(1)}$ as $x \to \infty$, and that if k is fixed and is large enough, then there are infinitely many exactly k-near-perfect numbers. A year later, Ren and Chen [27] determined all near-perfect numbers n that have $\omega(n) = 2$, and we can see from this classification that all such n are even. In the same year, Tang, Ren, and Li [35] proved that there is no odd near-perfect number n with $\omega(n) = 3$ and found all deficient-perfect numbers m with $\omega(m) \leq 2$. After that, Tang and Feng [33] extended this result by showing that there is no odd deficient-perfect number n with $\omega(n) = 3$. In 2016, Tang, Ma, and Feng [34] found the only odd near-perfect number with $\omega(n) = 4$, namely, $n = 3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2$, whereas in 2019, Sun and He [32] asserted that the only odd deficient-perfect number n with $\omega(n) = 4$ is $n = 3^2 \cdot 7^2 \cdot 11^2 \cdot 13^2$. Cohen, et al. [7] have recently improved the estimate of Pollack and Shevelev [21] on the number of near-perfect numbers $\leq x$. Hence, most results in the literature are devoted to characterizing, only when $k = 1$, the exactly k-near-perfect or exactly k-deficient-perfect numbers. Chen [4] started a slightly new direction by determining all 2-deficient-perfect numbers n with $\omega(n) \leq 2$.

In this article, we continue the investigation on odd 3-deficient-perfect numbers n with $\omega(n) \leq 2$. We found that the only such n is $n = 1521 = 3^2 \cdot 13^2$. For other articles related to the divisor functions or divisibility problems, see examples in [1, 2, 3, 8, 9, 12, 13, 15, 16, 17, 19, 20, 22, 23, 24, 25, 26, 28, 31, 36].

2. Main Results

By the definition, n is deficient-perfect if and only if n is exactly 1-deficient-perfect. Tang and Feng [33, Lemma 2.1] showed that if n is deficient-perfect and n is odd, then n is a square. We can extend their result to the following form.

Lemma 2.1. *Let* n *and* k *be positive integers. Suppose that* n *is exactly* k*-deficient-perfect and* n *is odd. Then,* n *is a square if and only if* k *is odd. In particular, if* n *is odd and exactly* 3*-deficient-perfect, then* n *is a square.*

Proof. Because 1 has no proper divisor, we can assume that $n > 1$ and write $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where $p_1, \ldots, p_r$ are distinct odd primes and $\alpha_1, \alpha_2, \ldots, \alpha_r$ are positive integers. Let $d_1, d_2, \ldots, d_k$ be distinct proper divisors of n such that

$$2n - d_1 - d_2 - \cdots - d_k = \sigma(n) = \prod_{i=1}^{r} \sigma(p_i^{\alpha_i}) = \prod_{i=1}^{r} (1 + p_i + \cdots + p_i^{\alpha_i}). \tag{2.1}$$

Because n is odd, d_i and p_j are odd for every $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, r$. Reducing (2.1) mod 2, we obtain $k \equiv \prod_{i=1}^{r} (\alpha_i + 1) \pmod 2$. From this, we have the equivalence k is odd $\Leftrightarrow$ α_i is even for all i $\Leftrightarrow$ n is a square, which proves our lemma. □

Tang, Ren, and Li [35] determine all deficient-perfect numbers n with $\omega(n) \leq 2$. In particular, they show that if $\omega(n) = 1$ and n is deficient-perfect, then n is a power of 2. We can extend this for exactly k-deficient-perfect numbers as follows.

Lemma 2.2. *Let* $n \geq 2$, $k \geq 1$ *be integers. If* n *is exactly* k*-deficient-perfect and* $\omega(n) = 1$*, then* $k = 1$ *and* n *is a power of* 2*. Consequently, if* n *is exactly* k*-deficient-perfect and* $k \geq 2$*, then* n *has at least two distinct prime divisors. In particular, every exactly* 3*-deficient-perfect number* n *has* $\omega(n) \geq 2$.

Proof. Suppose $n = p^\alpha$ and the deficient divisors of n are $d_1 = p^{\beta_1}$, $d_2 = p^{\beta_2}$, ..., $d_k = p^{\beta_k}$, where $\alpha > \beta_1 > \beta_2 > \cdots > \beta_k \geq 0$. Because $(p^{\alpha+1} - 1)/(p-1) = \sigma(n) = 2n - d_1 - \cdots - d_k$, we obtain

$$(d_1 + d_2 + \cdots + d_k)(p-1) - 1 = p^\alpha(p-2). \tag{2.2}$$

If $p \geq 3$, then

$$\begin{aligned} p^\alpha &\leq p^\alpha(p-2) = (d_1 + d_2 + \cdots + d_k)(p-1) - 1 \\ &\leq (p^{\alpha-1} + p^{\alpha-2} + \cdots + p^{\alpha-k})(p-1) - 1 = p^\alpha - p^{\alpha-k} - 1, \end{aligned}$$

which is impossible. Therefore, $p = 2$ and n is a power of 2. By (2.2), we also obtain $d_1 + \cdots + d_k = 1$, which implies $k = 1$ and $\beta_1 = 0$. □

We now give the main result of this paper.

Theorem 2.3. *The only odd exactly* 3*-deficient-perfect number that has* $\omega(n) = 2$ *is* $1521 = 3^2 \cdot 13^2$, *with three deficient divisors* $d_1 = 507$, $d_2 = 117$, *and* $d_3 = 39$.

Proof. It is easy to check that if $n = 1521$ and d_1, d_2, d_3 are as above, then $\omega(n) = 2$, n is odd, d_1, d_2, d_3 are proper divisors of n, $\sigma(n) = 2n - d_1 - d_2 - d_3$, and so n is exactly 3-deficient-perfect. For the other direction, assume that n is odd, $\omega(n) = 2$, and n is exactly 3-deficient-perfect. By Lemma 2.1, n is a square, so we can write $n = p_1^{2\alpha} p_2^{2\beta}$, where $2 < p_1 < p_2$ and $\alpha, \beta \geq 1$. In addition, let $d_1 > d_2 > d_3$ be the deficient divisors of n, and let $D_1 = n/d_1$, $D_2 = n/d_2$, $D_3 = n/d_3$. Then $p_1 \leq D_1 < D_2 < D_3 \leq n$. Because $\sigma(n) = 2n - d_1 - d_2 - d_3$, we obtain

$$\begin{aligned} 2 &= \frac{\sigma(n)}{n} + \frac{d_1}{n} + \frac{d_2}{n} + \frac{d_3}{n} \\ &= \frac{({p_1}^{2\alpha+1} - 1)({p_2}^{2\beta+1} - 1)}{(p_1 - 1)(p_2 - 1)p_1^{2\alpha} p_2^{2\beta}} + \frac{1}{D_1} + \frac{1}{D_2} + \frac{1}{D_3} \\ &< \frac{p_1 p_2}{(p_1 - 1)(p_2 - 1)} + \frac{1}{D_1} + \frac{1}{D_2} + \frac{1}{D_3}. \end{aligned} \tag{2.3}$$

If $p_1 \geq 5$, then $p_1/(p_1 - 1) \leq 5/4$, $p_2 \geq 7$, $p_2/(p_2 - 1) \leq 7/6$, $D_1 \geq 5$, $D_2 \geq 7$, $D_3 \geq 25$, and (2.3) implies that

$$2 < \frac{5}{4} \cdot \frac{7}{6} + \frac{1}{5} + \frac{1}{7} + \frac{1}{25} = 1.8411\ldots,$$

which is a contradiction. So, $p_1 = 3$. For convenience, let $p_2 = p$. Then, $n = 3^{2\alpha} p^{2\beta}$ and (2.3) becomes

$$2 < \frac{3p}{2(p-1)} + \frac{1}{D_1} + \frac{1}{D_2} + \frac{1}{D_3}. \tag{2.4}$$

If $p \geq 83$, then (2.4) leads to $2 < (3/2)(83/82) + 1/3 + 1/9 + 1/27 = 1.9997\ldots$, which is impossible. So, $5 \leq p \leq 79$. Recall that the primes in $[5, 79]$ are 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79. If $p \geq 11$ and $D_1 > 3$, then $D_1 \geq 9$, $D_2 \geq 11$, $D_3 \geq 27$, and (2.4) gives $2 < (3/2)(11/10) + 1/9 + 1/11 + 1/27 = 1.8890\ldots$, which is false. Therefore,

$$\text{if } p \geq 11, \text{ then } D_1 = 3. \tag{2.5}$$

Similarly, if $p \geq 23$ and $D_2 > 9$, then $2 < (3/2)(23/22) + 1/3 + 1/23 + 1/27 = 1.9820\ldots$, which is not true. Thus,

$$\text{if } p \geq 23, \text{ then } D_2 = 9. \tag{2.6}$$

Next, we divide our calculations into 11 cases according to the value of p. In addition, we write the possible values of D_1, D_2, D_3 in increasing order.

Case 1. $47 \le p \le 79$. By (2.5) and (2.6), we have $D_1 = 3$, $D_2 = 9$, and the possible values of D_3 in increasing order are $D_3 = 27, p, 81, \ldots$. If $D_3 \ge p$, then (2.4) implies $2 < (3/2)(47/46) + 1/3 + 1/9 + 1/47 = 1.9983\ldots$, which is false. So, $D_3 = 27$. Then, $2\alpha \ge 3$, $d_1 = n/D_1 = 3^{2\alpha-1}p^{2\beta}$, $d_2 = 3^{2\alpha-2}p^{2\beta}$, $d_3 = 3^{2\alpha-3}p^{2\beta}$, and

$$\begin{aligned}\frac{\left(3^{2\alpha+1}-1\right)\left(p^{2\beta+1}-1\right)}{2(p-1)} &= \sigma(3^{2\alpha}p^{2\beta}) = 2\cdot 3^{2\alpha}p^{2\beta} - d_1 - d_2 - d_3 \\ &= 3^{2\alpha-3}p^{2\beta}(2\cdot 3^3 - 3^2 - 3 - 1) = 41\cdot 3^{2\alpha-3}p^{2\beta}.\end{aligned}$$

This leads to

$$3^{2\alpha-3} = \frac{p^{2\beta+1}-1}{(82-p)p^{2\beta}-81}. \tag{2.7}$$

The left side of (2.7) is an integer, and we get a contradiction by showing that the right side of (2.7) is not an integer. From this point on, let A be the number on the right side of (2.7). If $p = 47$, then A is equal to

$$\frac{47\cdot 47^{2\beta}-1}{35\cdot 47^{2\beta}-81} = 1 + \frac{12\cdot 47^{2\beta}+80}{35\cdot 47^{2\beta}-81} = 1 + \frac{12 + (80/47^{2\beta})}{35 - (81/47^{2\beta})} \in (1,2),$$

and so $A \notin \mathbb{Z}$. Similarly,

$$\begin{aligned}&\text{if } p = 53, \text{ then } A = 1 + \frac{24p^{2\beta}+80}{29p^{2\beta}-81} \in (1,2);\\ &\text{if } p = 59, \text{ then } A = 2 + \frac{13p^{2\beta}+161}{23p^{2\beta}-81} \in (2,3);\\ &\text{if } p = 61, \text{ then } A = 2 + \frac{19p^{2\beta}+161}{21p^{2\beta}-81} \in (2,3);\\ &\text{if } p = 67, \text{ then } A = 4 + \frac{7p^{2\beta}+323}{15p^{2\beta}-81} \in (4,5).\end{aligned}$$

The remaining cases $p = 71, 73, 79$ lead to $A \in (6,7)$, $A \in (8,9)$, and $A \in (26,27)$, respectively. In any case, $A \notin \mathbb{Z}$ and we have a contradiction. Hence, this case does not lead to a solution.

Case 2. $p \in \{37, 41, 43\}$. By (2.5) and (2.6), we have $D_1 = 3$, $D_2 = 9$, and $D_3 = 27, p, 81, \ldots$. If $D_3 \ge 81$, then (2.4) implies $2 < (3/2)(37/36) + 1/3 + 1/9 + 1/81 = 1.9984\ldots$, which is not possible. So, $D_3 = \{27, p\}$.

Case 2.1. $D_1 = 3$, $D_2 = 9$, and $D_3 = 27$. Then $2\alpha \ge 3$, (2.7) holds, and the calculations in Case 1 work in this case too. Because (2.7) holds, we still let A be the right side of (2.7). Therefore, if $p = 37$, then $A \in (0,1)$ and if $p \in \{41, 43\}$, then $A \in (1,2)$, which is a contradiction.

Case 2.2. $D_1 = 3$, $D_2 = 9$, and $D_3 = p$. Then,

$$\begin{aligned}\frac{\left(3^{2\alpha+1}-1\right)\left(p^{2\beta+1}-1\right)}{2(p-1)} &= \sigma(3^{2\alpha}p^{2\beta}) = \sigma(n) = 2n - d_1 - d_2 - d_3 \\ &= 2\cdot 3^{2\alpha}p^{2\beta} - 3^{2\alpha-1}p^{2\beta} - 3^{2\alpha-2}p^{2\beta} - 3^{2\alpha}p^{2\beta-1} \\ &= 3^{2\alpha-2}p^{2\beta-1}(14p-9),\end{aligned}$$

which implies

$$3^{2\alpha-2} = \frac{p^{2\beta+1}-1}{(46p-p^2-18)p^{2\beta-1}-27}. \tag{2.8}$$

Equality (2.8) can be used in the same way as (2.7). So, let B be the number on the right side of (2.8). Similar to the previous computation, we see that if $p = 37$, then $B \in (4, 5)$ and if $p = 43$, then $B \in (16, 17)$, which contradicts that $B = 3^{2\alpha-2} \in \mathbb{Z}$. Suppose $p = 41$. Then, $B \in (8, 10)$, which implies $B = 9$. Equating the right side of (2.8) with $B = 9$, substituting $p = 41$, and performing a straightforward manipulation leads to $41^{2\beta-1} = 121$, which is not possible. Hence, there is no solution in this case.

Remark 2.4. *Before going further, we note that the calculations similar to* (2.7) *and* (2.8) *and their applications occur throughout the proof, and we give less details than those in* (2.7) *and* (2.8).

Case 3. $p \in \{29, 31\}$. Then by (2.5) and (2.6), $D_1 = 3$, $D_2 = 9$, and $D_3 = 27, p, 81, 3p, 243, 9p, 729, \ldots$. If $p = 31$ and $D_3 \geq 243$, then (2.4) implies $2 < (3/2)(31/30)+1/3+1/9+1/243 = 1.9985\ldots$, which is false. Similarly, assuming $p = 29$ and $D_3 \geq 729$ leads to a false inequality. Therefore,

$$\text{if } p = 31, \text{ then } D_3 \in \{27, 31, 81, 93\}, \tag{2.9}$$

$$\text{if } p = 29, \text{ then } D_3 \in \{27, 29, 81, 87, 243, 261\}. \tag{2.10}$$

Next, we divide our calculations according to the value of D_3.

Case 3.1. $D_3 = 27$. Then, (2.7) holds and the same method still works. We obtain

$$\text{if } p = 29, \text{ then } A = \left(29p^{2\beta} - 1\right) / \left(53p^{2\beta} - 81\right) \in (0, 1);$$

$$\text{if } p = 31, \text{ then } A = \left(31p^{2\beta} - 1\right) / \left(51p^{2\beta} - 81\right) \in (0, 1).$$

So, $A \notin \mathbb{Z}$ and we get a contradiction.

Case 3.2. $D_3 = p \in \{29, 31\}$. Then, (2.8) holds and

$$\text{if } p = 29, \text{ then } B = \left(841p^{2\beta-1} - 1\right) / \left(475p^{2\beta-1} - 27\right) \in (1, 2);$$

$$\text{if } p = 31, \text{ then } B = \left(961p^{2\beta-1} - 1\right) / \left(447p^{2\beta-1} - 27\right) \in (1, 2),$$

which is a contradiction.

Case 3.3. $D_3 = 81$. Similar to the calculations for (2.7) and (2.8), we write $\sigma(n) = 2n - d_1 - d_2 - d_3$, where d_1, d_2 are the same as before, but $d_3 = n/D_3 = 3^{2\alpha-4}p^{2\beta}$ and $2\alpha \geq 4$. After a similar algebraic manipulation, we get

$$3^{2\alpha-4} = \frac{p^{2\beta+1} - 1}{(250 - 7p)p^{2\beta} - 243}. \tag{2.11}$$

When $p = 29$ or 31, the right side of (2.11) is in the interval $(0, 1)$, which is impossible.

Case 3.4. $D_3 = 93$. By (2.9) and (2.10), we know that $p = 31$. Similar to Case 3.3 but with $d_3 = n/D_3 = 3^{2\alpha-1}p^{2\beta-1}$, we start with $\sigma(n) = 2n - d_1 - d_2 - d_3$ and perform an algebraic manipulation to obtain

$$3^{2\alpha-2} = \frac{p^{2\beta+1} - 1}{(34p - p^2 - 6)p^{2\beta-1} - 27} = \frac{961p^{2\beta-1} - 1}{87p^{2\beta-1} - 27} \in (11, 12),$$

which is false.

Case 3.5. $D_3 \in \{87, 243, 261\}$. By (2.9) and (2.10), we have $p = 29$. Similar to Case 3.3 but with different values of $d_3 = n/D_3 = 3^{2\alpha-1}p^{2\beta-1}$, $3^{2\alpha-5}p^{2\beta}$, or $3^{2\alpha-2}p^{2\beta-1}$ when $D_3 = 87$, 243, or 261, respectively. These lead to

$$2\alpha \geq 2 \text{ and } 3^{2\alpha-2} = \frac{p^{2\beta+1}-1}{(34p-p^2-6)p^{2\beta-1}-27} = \frac{841p^{2\beta-1}-1}{139p^{2\beta-1}-27} \in (6,7), \text{ if } D_3 = 87;$$

$$2\alpha \geq 5 \text{ and } 3^{2\alpha-5} = \frac{p^{2\beta+1}-1}{(754-25p)p^{2\beta}-729} = \frac{29^{2\beta+1}-1}{29^{2\beta+1}-729} \in (1,2), \text{ if } D_3 = 243;$$

$$2\alpha \geq 2 \text{ and } 3^{2\alpha-2} = \frac{p^{2\beta+1}-1}{(30p-p^2-2)p^{2\beta-1}-27} = \frac{841p^{2\beta-1}-1}{27p^{2\beta-1}-27} \in (31,33), \text{ if } D_3 = 261.$$

In any case, we get a contradiction.

Case 4. $p = 23$. By (2.5) and (2.6), we have $D_1 = 3$ and $D_2 = 9$. We start from

$$\begin{aligned}(3^{2\alpha+1}-1)(p^{2\beta+1}-1) &= 2(p-1)\sigma(n) = 2(p-1)(2n-d_1-d_2-d_3)\\ &= 28(p-1)3^{2\alpha-2}p^{2\beta} - 2(p-1)d_3.\end{aligned}$$

Writing $(3^{2\alpha+1}-1)(p^{2\beta+1}-1) = 27p3^{2\alpha-2}p^{2\beta} - 3^{2\alpha+1} - p^{2\beta+1} + 1$, the above leads to

$$(28-p)3^{2\alpha-2}p^{2\beta} - 3^{2\alpha+1} - p^{2\beta+1} + 1 + 2(p-1)d_3 = 0. \tag{2.12}$$

Multiplying both sides of (2.12) by $28-p$ and factoring a part of it gives us

$$\left((28-p)3^{2\alpha-2}-p\right)\left((28-p)p^{2\beta}-27\right) = 28(p-1) - 2(28-p)(p-1)d_3. \tag{2.13}$$

Substituting $p = 23$, the equation (2.13) becomes

$$(5 \cdot 3^{2\alpha-2} - 23)(5 \cdot 23^{2\beta} - 27) = 616 - 220d_3. \tag{2.14}$$

Let A_1 and A_2 be the expressions on the left and the right side of (2.14), respectively. If $\alpha \geq 2$, then $A_1 > 616$, while $A_2 < 616$, which is not the case. So, $\alpha = 1$ and $A_1 = -18(5 \cdot 23^{2\beta} - 27)$. Because $3 \mid A_1$ and $3 \nmid 616$, we see that $3 \nmid d_3$. Because $d_3 \mid n$ and $n = 3^{2\alpha}23^{2\beta}$, we obtain $d_3 = 23^{b_3}$ for some $b_3 \geq 0$. If $b_3 = 0$, then $A_2 = 616 - 220 \equiv 5 \pmod{23}$; if $b_3 \geq 1$, then $A_2 \equiv 18 \pmod{23}$. But, $A_1 \equiv 3 \pmod{23}$, and so $A_1 = A_2$ and $A_1 \not\equiv A_2 \pmod{23}$, which is not possible.

Case 5. $p = 19$. By (2.5), $D_1 = 3$. So, $\{D_2, D_3\} \subseteq \{9, 19, 27, 57, \ldots\}$. If $D_2 \geq 19$ and $D_3 \geq 57$, then (2.4) implies that $2 < (3/2)(19/18) + 1/3 + 1/19 + 1/57 = 1.9868\ldots$, which is not true. Therefore, ($D_2 = 9$) or ($D_2 = 19$ and $D_3 = 27$).

Case 5.1. $D_2 = 9$. Then, the computation in Case 4 still works and (2.13) holds. Substituting $p = 19$ in (2.13) and dividing both sides by 9, we obtain

$$(3^{2\alpha} - 19)(19^{2\beta} - 3) = 56 - 36d_3. \tag{2.15}$$

Let A_3, A_4 be the expressions on the left and the right side of (2.15), respectively. If $\alpha \geq 2$, then $A_3 > 56$, while $A_4 < 56$, which is not true. Therefore, $\alpha = 1$. Then, $11 \equiv A_3 \equiv A_4 \equiv -1 + 2d_3 \pmod{19}$, and so $19 \nmid d_3$. Because $d_3 \mid n$ and $n = 3^{2\alpha}p^{2\beta} = 3^2 \cdot 19^{2\beta}$, we see that $d_3 = 1, 3, 9$. Substituting $d_3 = 1, 3, 9$ in (2.15) leads to $5 \cdot 19^{2\beta} = 5, 41, 149$, respectively, which has no solution.

Case 5.2. $D_2 = 19$ and $D_3 = 27$. Similar to the calculations for (2.7) and (2.14) but with different values of d_2 and d_3, we obtain, after an algebraic manipulation, that

$$3^{2\alpha-3} = \frac{361 \cdot 19^{2\beta-1} - 1}{117 \cdot 19^{2\beta-1} - 81} \in (3,4),$$

which is not possible.

Case 6. $p \in \{11, 13, 17\}$. Then by (2.5), we have $D_1 = 3$. The possible values of D_2 and D_3 listed in increasing order are 9, p, 27, $3p$, 81, $9p$, $\min\{p^2, 243\}$, $\max\{p^2, 243\}$, ... We can eliminate some cases by using (2.4) as before. If $p = 17$ and $D_2 \geq 27$, then (2.4) implies $2 < (3/2)(17/16) + 1/3 + 1/27 + 1/51 < 2$; if $p = 17, D_2 \geq 17$, and $D_3 \geq 81$, then (2.4) leads to $2 < (3/2)(17/16) + 1/3 + 1/17 + 1/81 < 2$. Similarly, if $p = 13$, then we must have $D_2 < 39$; if $p = 13$ and $D_2 \geq 27$, then it forces $D_3 < 243$; if $p = 11$, then $D_2 < 81$ or $D_3 < 243$. Therefore, we obtain

$$\text{if } p = 17, \text{ then } (D_2 = 9) \text{ or } (D_2 = 17 \text{ and } D_3 \in \{27, 51\}); \tag{2.16}$$

$$\text{if } p = 13, \text{ then } (D_2 \in \{9, 13\}) \text{ or } (D_2 = 27 \text{ and } D_3 \in \{39, 81, 117, 169\}); \tag{2.17}$$

$$\begin{aligned} &\text{if } p = 11, \text{ then } (D_2 \in \{9, 11, 27, 33\}) \text{ or } (D_2 = 81 \text{ and } D_3 \in \{99, 121\}) \text{ or} \\ &\qquad (D_2 = 99 \text{ and } D_3 = 121). \end{aligned} \tag{2.18}$$

We divide our calculations according to the values of D_2 and D_3 listed in (2.16), (2.17), and (2.18).

Case 6.1. $D_2 = 9$ (so p can be any of 11, 13, or 17). Because $D_1 = 3$ and $D_2 = 9$, equation (2.13) holds. Substituting $p = 11, 13, 17$ in (2.13), we obtain, respectively

$$(17 \cdot 3^{2\alpha-2} - 11)(17 \cdot 11^{2\beta} - 27) = 280 - 340d_3 \text{ (if } p = 11), \tag{2.19}$$

$$(15 \cdot 3^{2\alpha-2} - 13)(15 \cdot 13^{2\beta} - 27) = 336 - 360d_3 \text{ (if } p = 13), \tag{2.20}$$

$$(11 \cdot 3^{2\alpha-2} - 17)(11 \cdot 17^{2\beta} - 27) = 448 - 352d_3 \text{ (if } p = 17), \tag{2.21}$$

where d_3 in (2.19) is a proper divisor of $3^{2\alpha}11^{2\beta}$, d_3 in (2.20) is a proper divisor of $3^{2\alpha}13^{2\beta}$, and d_3 in (2.21) is a proper divisor of $3^{2\alpha}17^{2\beta}$. Because $\alpha, \beta \geq 1$, the left side of (2.19) and (2.20) are positive, whereas the right side of (2.19) and (2.20) are negative. So, (2.19) and (2.20) do not lead to a solution. For (2.21), we have $448 - 352d_3 \leq 96$, which implies $\alpha = 1$. Then, (2.21) reduces to $3 \cdot 17^{2\beta} + 13 - 16d_3 = 0$. Reducing this mod 3 and mod 17, we see that $d_3 \equiv 1 \pmod 3$ and $d_3 \equiv 4 \pmod{17}$. Because $d_3 \mid 3^{2\alpha}17^{2\beta}$, $3 \nmid d_3$, and $17 \nmid d_3$, we obtain $d_3 = 1$, which contradicts that $d_3 \equiv 4 \pmod{17}$. Thus, there is no solution in this case.

Case 6.2. $D_2 = p$, where $p \in \{11, 13\}$. Similar to the calculation for (2.13), we have

$$\begin{aligned} (3^{2\alpha+1} - 1)(p^{2\beta+1} - 1) &= 2(p-1)\sigma(n) = 2(p-1)(2n - d_1 - d_2 - d_3) \\ &= 2(p-1)(2 \cdot 3^{2\alpha}p^{2\beta} - 3^{2\alpha-1}p^{2\beta} - 3^{2\alpha}p^{2\beta-1} - d_3). \end{aligned}$$

Let $B_p = 16p - p^2 - 6$. Following a straightforward algebraic manipulation and multiplying both sides by B_p, the above leads to

$$(B_p 3^{2\alpha-1} - p^2)(B_p p^{2\beta-1} - 9) = 9p^2 - B_p - 2B_p(p-1)d_3. \tag{2.22}$$

Substituting $p = 11$ in (2.22), we obtain

$$(49 \cdot 3^{2\alpha-1} - 121)(49 \cdot 11^{2\beta-1} - 9) = 1040 - 980d_3. \tag{2.23}$$

Because $\alpha, \beta \geq 1$, the left side of (2.23) is larger than 60, whereas the right side of (2.23) is at most 60, so (2.23) does not give a solution. Next, substituting $p = 13$ in (2.22) and dividing both sides by 3, we obtain

$$(33 \cdot 3^{2\alpha-1} - 169)(11 \cdot 13^{2\beta-1} - 3) = 496 - 264d_3. \tag{2.24}$$

Because the right side of (2.24) is at most 232, we obtain $\alpha = 1$ and (2.24) reduces to

$$35 \cdot 13^{2\beta-1} - 12d_3 + 13 = 0. \tag{2.25}$$

Recall that $d_3 \mid n$ and $n = 3^{2\alpha}p^{2\beta} = 3^2 \cdot 13^{2\beta}$. So, $d_3 = 3^{a_3}13^{b_3}$ for some $a_3 \in \{0, 1, 2\}$ and $b_3 \geq 0$. Reducing (2.25) modulo 7, we see that $2d_3 \equiv 1 \pmod 7$. If $a_3 = 0$, then $2d_3 = 2 \cdot 13^{b_3} \equiv$

$2(-1)^{b_3} \equiv 2, -2 \not\equiv 1 \pmod 7$. If $a_3 = 2$, then $2d_3 = 18 \cdot 13^{b_3} \equiv 4(-1)^{b_3} \equiv 4, -4 \not\equiv 1 \pmod 7$. Therefore, $a_3 = 1$ and (2.25) becomes

$$35 \cdot 13^{2\beta-1} - 36 \cdot 13^{b_3} + 13 = 0. \tag{2.26}$$

Suppose, for a contradiction, that $\beta \geq 2$. Reducing (2.26) modulo 13^2, we obtain $36 \cdot 13^{b_3} \equiv 13 \pmod{13^2}$. If $b_3 \geq 2$, then $36 \cdot 13^{b_3} \equiv 0 \not\equiv 13 \pmod{13^2}$. If $b_3 = 1$, then $36 \cdot 13^{b_3} - 13 = 35 \cdot 13 \not\equiv 0 \pmod{13^2}$. If $b_3 = 0$, then $36 \cdot 13^{b_3} = 36 \not\equiv 13 \pmod{13^2}$. In any case, we reach a contradiction. Therefore, $\beta = 1$. Substituting $\beta = 1$ in (2.26), we obtain $b_3 = 1$, and so $d_3 = 3^{a_3}13^{b_3} = 39$. This leads to $n = 3^{2\alpha}p^{2\beta} = 3^2 \cdot 13^2$ with the deficient divisors $d_1 = n/D_1 = 3 \cdot 13^2 = 507$, $d_2 = n/D_2 = 3^2 \cdot 13 = 117$, and $d_3 = 39$, which we already verified at the beginning of the proof that this is indeed a solution to our problem. The elimination for the other cases can be done in a similar way to the previous cases, so we give less details. Recall that $D_1 = 3$. The other cases are as follows:

(i) $p = 17$, $D_2 = 17$, and $D_3 \in \{27, 51\}$ (this is the remaining case from (2.16)).
(ii) $p = 13$, $D_2 = 27$, and $D_3 \in \{39, 81, 117, 169\}$ (this is the remaining case from (2.17)).
(iii) $p = 11$, $D_2 \in \{27, 33\}$.
(iv) $p = 11$, $D_2 = 81$, and $D_3 \in \{99, 121\}$.
(v) $p = 11$, $D_2 = 99$, and $D_3 = 121$.

In (i), (ii), (iv), and (v), we know the values of D_1, D_2, D_3, and so we have the values of d_1, d_2, d_3. We start from the equality $\sigma(n) = 2n - d_1 - d_2 - d_3$, perform the usual algebraic manipulation, and try to write the minimum nonnegative power of 3 appearing among d_1, d_2, d_3 in terms of the other variables. We obtain the following results. For (i), we have $p = 17$, $D_1 = 3$, $D_2 = 17$, and

$$\text{if } D_3 = 27, \text{ then } 2\alpha \geq 3 \text{ and } 3^{2\alpha-3} = \frac{289 \cdot 17^{2\beta-1} - 1}{337 \cdot 17^{2\beta-1} - 81} \in (0,1);$$

$$\text{if } D_3 = 51, \text{ then } 3^{2\alpha-1} = \frac{289 \cdot 17^{2\beta-1} - 1}{9 \cdot 17^{2\beta-1} - 9} \in (32, 35),$$

which is a contradiction. For (ii), we have $p = 13$, $D_1 = 3$, $D_2 = 27$, $2\alpha \geq 3$, and

$$\text{if } D_3 = 39, \text{ then } 3^{2\alpha-3} = \frac{169 \cdot 13^{2\beta-1} - 1}{177 \cdot 13^{2\beta-1} - 81} \in (0,1);$$

$$\text{if } D_3 = 81, \text{ then } 2\alpha \geq 4 \text{ and } 3^{2\alpha-4} = \frac{13 \cdot 13^{2\beta} - 1}{15 \cdot 13^{2\beta} - 243} \in (0,1);$$

$$\text{if } D_3 = 117, \text{ then } 3^{2\alpha-3} = \frac{169 \cdot 13^{2\beta-1} - 1}{33 \cdot 13^{2\beta-1} - 81} \in (5,7);$$

$$\text{if } D_3 = 169, \text{ then } 3^{2\alpha-3} = \frac{2197 \cdot 13^{2\beta-2} - 1}{141 \cdot 13^{3\beta-2} - 81} \in (15, 37).$$

The first three cases above give a contradiction. The last case implies that

$$2197 \cdot 13^{2\beta-2} - 1 = 27(141 \cdot 13^{2\beta-2} - 81),$$

which leads to $1610 \cdot 13^{2\beta-2} = 2186$, which is impossible. For (iv), we have $p = 11$, $D_1 = 3$, $D_2 = 81$, $2\alpha \geq 4$, and

$$\text{if } D_3 = 99, \text{ then } 3^{2\alpha-4} = \frac{121 \cdot 11^{2\beta-1} - 1}{103 \cdot 11^{2\beta-1} - 243} \in (1,2);$$

$$\text{if } D_3 = 121, \text{ then } 3^{2\alpha-4} = \frac{1331 \cdot 11^{2\beta-2} - 1}{773 \cdot 11^{2\beta-2} - 243} \in (1,3),$$

which is false. For (v), we have $p = 11$, $D_1 = 3$, $D_2 = 99$, $D_3 = 121$, which leads to

$$3^{2\alpha-2} = \frac{1331 \cdot 11^{2\beta-2} - 1}{37 \cdot 11^{2\beta-2} - 27} \in (35, 37) \cup \{133\},$$

which is not possible. We now consider (iii). We have $p = 11$, $D_1 = 3$, $D_2 \in \{27, 33\}$. We know the values of d_1, d_2 but not d_3. We start with $\sigma(n) = 2n - d_1 - d_2 - d_3$ and write d_3 in terms of the product of the other variables. Similar to the calculation for (2.13), we obtain

$$\text{if } D_2 = 27, \text{ then } 2\alpha \geq 3 \text{ and } (3^{2\alpha-3} - 1)(11^{2\beta+1} - 81) = 80 - 20d_3; \tag{2.27}$$

$$\text{if } D_2 = 33, \text{ then } (3^{2\alpha+1} - 121)(11^{2\beta-1} - 1) = 120 - 20d_3. \tag{2.28}$$

In (2.27), 2α is an even integer ≥ 3, so $2\alpha \geq 4$, and thus, the left side of (2.27) is larger than 80, whereas the right side of (2.27) is less than 80, which is a contradiction. Because the right side of (2.28) is less than 120, we see that $\alpha = 1$ and (2.28) reduces to $47 \cdot 11^{2\beta+1} - 10d_3 + 13 = 0$. Reducing this modulo 11, we see that $10d_3 \equiv 2 \pmod{11}$, and therefore, $d_3 \equiv 9 \pmod{11}$. So, $11 \nmid d_3$. Because $d_3 \mid n$ and $n = 3^{2\alpha}p^{2\beta} = 3^2 \cdot 11^{2\beta}$, we have $d_3 = 1, 3, 9$. Because $d_3 \equiv 9 \pmod{11}$, $d_3 = 9$ only. Then $47 \cdot 11^{2\beta+1} - 90 + 13 = 0$. This leads to $47 \cdot 11^{2\beta+1} = 77$, which has no solution.

Case 7. $p = 7$. Then, $\{D_1, D_2, D_3\} \subseteq \{3, 7, 9, 21, \ldots\}$. If $D_1 \geq 7$ and $D_2 \geq 21$, then (2.4) implies $2 < (3/2)(7/6) + 1/7 + 1/21 + 1/21 < 2$, which is impossible. So, ($D_1 = 3$) or ($D_1 = 7$ and $D_2 = 9$). If $D_1 = 3$, then $d_1 = 3^{2\alpha-1}7^{2\beta}$ and we have

$$\begin{aligned}
0 &= 12\left(\sigma(n) - 2n + d_1 + d_2 + d_3\right) \\
&= (3^{2\alpha+1} - 1)(7^{2\beta+1} - 1) - 24n + 12(d_1 + d_2 + d_3) \\
&= 3^{2\alpha}7^{2\beta}\left(21 - 3/7^{2\beta} - 7/3^{2\alpha} - 24\right) + 1 + 12(d_1 + d_2 + d_3) \\
&= 1 + 12d_1\left(1 + d_2/d_1 + d_3/d_1\right) - 3^{2\alpha}7^{2\beta}(3 + 3/7^{2\beta} + 7/3^{2\alpha}) \\
&> 1 + 12d_1 - 3^{2\alpha}7^{2\beta}(3 + 3/7^2 + 7/3^2) \\
&> 12d_1 - 3^{2\alpha}7^{2\beta}(4) = 0,
\end{aligned}$$

which is a contradiction. So, $D_1 = 7$ and $D_2 = 9$. We start with $\sigma(n) = 2n - d_1 - d_2 - d_3$, substitute $d_1 = 3^{2\alpha}7^{2\beta-1}$, $d_2 = 3^{2\alpha-2}7^{2\beta}$, and do the usual algebraic manipulation to obtain

$$(3^{2\alpha-1} - 49)(7^{2\beta-1} - 9) = 440 - 12d_3. \tag{2.29}$$

If $\alpha \geq 3$ and $\beta \geq 2$, then the left side of (2.29) is larger than 440, whereas the right side of (2.29) is smaller than 440. Therefore, ($\alpha \in \{1, 2\}$) or ($\alpha \geq 3$ and $\beta = 1$). Because $d_3 \mid n$ and $n = 3^{2\alpha}7^{2\beta}$, $d_3 = 3^{a_3}7^{b_3}$ for some $a_3, b_3 \geq 0$.

Case 7.1. $\alpha \geq 3$ and $\beta = 1$. Then, (2.29) reduces to

$$3^{2\alpha-1} + 171 = 6 \cdot 3^{a_3}7^{b_3}. \tag{2.30}$$

Because $3^{2\alpha-1} + 171 = 3^2(3^{2\alpha-3} + 19)$, we obtain $3^2 \parallel 6d_3$, which implies $a_3 = 1$. Dividing both sides of (2.30) by 9, we obtain $3^{2\alpha-3} + 19 = 2 \cdot 7^{b_3}$. Reducing this modulo 3, we have a contradiction.

Case 7.2. $\alpha \in \{1, 2\}$. If $\alpha = 2$, then (2.29) leads to $d_3 \equiv 0 \pmod{11}$, which contradicts that $d_3 = 3^{a_3}7^{b_3}$. So, $\alpha = 1$. Then, $a_3 \in \{0, 1, 2\}$ and (2.29) reduces to $23 \cdot 7^{2\beta-1} - 6d_3 + 13 = 0$. From this, we see that $7 \nmid d_3$. So, $b_3 = 0$, $d_3 = 3^{a_3}$, and the above equation becomes $23 \cdot 7^{2\beta-1} - 6 \cdot 3^{a_3} + 13 = 0$. Substituting $a_3 = 0, 1, 2$, we obtain $23 \cdot 7^{2\beta-1} = -7, 5, 41$, which is not possible. Hence, there is no solution in this case.

Case 8. $p = 5$. Then, the possible values of D_1, D_2, D_3 listed in increasing order are $3, 5, 9, 15, 25, \ldots$. If $D_1 \geq 25$, then (2.4) implies $2 < (3/2)(5/4) + 1/25 + 1/25 + 1/25 < 2$, which is false. Therefore, $D_1 \in \{3, 5, 9, 15\}$. It is possible to obtain bounds for D_2 and D_3 as in the other cases, but the same method will lead to a longer calculation. In this case, it is better to get a bound only for D_1 and go back to d_1, d_2, d_3. Let $d_1 = 3^{a_1}5^{b_1}$, $d_2 = 3^{a_2}5^{b_2}$, and $d_3 = 3^{a_3}5^{b_3}$, where $a_i, b_i \geq 0$, and recall that $n > d_1 > d_2 > d_3 \geq 1$ and d_1, d_2, d_3 are the deficient divisors of $n = 3^{2\alpha}5^{2\beta}$. In addition, from $\sigma(n) = 2n - (d_1 + d_2 + d_3)$, we get

$$\begin{aligned}(3^{2\alpha+1} - 1)(5^{2\beta+1} - 1) &= 16 \cdot 3^{2\alpha}5^{2\beta} - 8(d_1 + d_2 + d_3) \\ &= 16 \cdot 3^{2\alpha}5^{2\beta} - 8(3^{a_1}5^{b_1} + 3^{a_2}5^{b_2} + 3^{a_3}5^{b_3}).\end{aligned} \tag{2.31}$$

From (2.31), we see that $8(d_1 + d_2 + d_3) = 3^{2\alpha}5^{2\beta} + 3^{2\alpha+1} + 5^{2\beta+1} - 1$, which implies

$$1 < \frac{8}{3^{2\alpha}5^{2\beta}}(d_1 + d_2 + d_3) < 1 + \frac{3}{5^2} + \frac{5}{3^2} < 2. \tag{2.32}$$

Because $D_1 \in \{3, 5, 9, 15\}$ and $d_1 = n/D_1$, we see that

$$(a_1, b_1) = (2\alpha - 1, 2\beta), (2\alpha, 2\beta - 1), (2\alpha - 2, 2\beta), \text{ or } (2\alpha - 1, 2\beta - 1). \tag{2.33}$$

Observe that $3^4 \equiv 1 \pmod 5$, $5^2 \equiv 1 \pmod 3$, and the exponents 4 and 2 are the smallest positive integers satisfying each congruence. From this, it is not difficult to verify that the left side of (2.31) satisfies

$$(3^{2\alpha+1} - 1)(5^{2\beta+1} - 1) \equiv \begin{cases} 3 \pmod 5, & \text{if } \alpha \text{ is even;} \\ 4 \pmod 5, & \text{if } \alpha \text{ is odd,} \end{cases} \tag{2.34}$$

$$(3^{2\alpha+1} - 1)(5^{2\beta+1} - 1) \equiv 2 \pmod 3. \tag{2.35}$$

Because 5 does not divide the left side of (2.31), at least one of d_1, d_2, d_3 is not divisible by 5, that is, at least one of b_1, b_2, b_3 is zero. By (2.33), we see that $b_1 \neq 0$. Thus,

$$b_1 \neq 0 \text{ and } \min\{b_2, b_3\} = 0. \tag{2.36}$$

Suppose, for a contradiction, that $a_1 = a_2 = a_3 = 0$. That is, $d_1 = 5^{b_1}$, $d_2 = 5^{b_2}$, $d_3 = 5^{b_3}$. Because $d_1 > d_2 > d_3$, we have $b_1 > b_2 > b_3$. So by (2.36), $b_3 = 0$ and $b_1 > b_2 > 0$. Then, the right side of (2.31) is $\equiv 2 \pmod 5$, contradicting (2.34). So, one of a_1, a_2, a_3 is not zero. By (2.35) and (2.31), one of d_1, d_2, d_3 is not divisible by 3, and so one of a_1, a_2, a_3 is zero. We conclude that

$$\max\{a_1, a_2, a_3\} \geq 1 \text{ and } \min\{a_1, a_2, a_3\} = 0. \tag{2.37}$$

The right side of (2.31) is congruent to

$$\begin{cases} (0 + 0 + 5^{b_3}) \pmod 3, & \text{if } a_1 \neq 0, a_2 \neq 0, \text{ and } a_3 = 0; \\ (0 + 5^{b_2} + 0) \pmod 3, & \text{if } a_1 \neq 0, a_2 = 0, \text{ and } a_3 \neq 0; \\ (5^{b_1} + 0 + 0) \pmod 3, & \text{if } a_1 = 0, a_2 \neq 0, \text{ and } a_3 \neq 0; \\ (5^{b_1} + 5^{b_2} + 0) \pmod 3, & \text{if } a_1 = a_2 = 0, \text{ and } a_3 \neq 0; \\ (5^{b_1} + 0 + 5^{b_3}) \pmod 3, & \text{if } a_1 = a_3 = 0, \text{ and } a_2 \neq 0; \\ (0 + 5^{b_2} + 5^{b_3}) \pmod 3, & \text{if } a_2 = a_3 = 0, \text{ and } a_1 \neq 0. \end{cases} \tag{2.38}$$

By comparing (2.31), (2.35), and (2.38), we obtain the parities of b_1, b_2, b_3 as follows. If $5^b \equiv 2 \pmod 3$, then b is odd. If $5^x + 5^y \equiv 2 \pmod 3$, then x and y are even. For convenience, for each $i \in \{1, 2, 3\}$, if b_i is odd, we write b_i' for b_i; if b_i is even, then we replace b_i by b_i''. Therefore, for each $i \in \{1, 2, 3\}$, b_i', $b_i'' \geq 0$, $b_i' = b_i$ is odd, and $b_i'' = b_i$ is even, and there are six cases to consider as follows:

Case 8.1. $d_1 = 3^{a_1}5^{b_1}$, $d_2 = 3^{a_2}5^{b_2}$, $d_3 = 5^{b_3'}$, $a_1 \neq 0$, $a_2 \neq 0$, and $a_3 = 0$,
Case 8.2. $d_1 = 3^{a_1}5^{b_1}$, $d_2 = 5^{b_2'}$, $d_3 = 3^{a_3}5^{b_3}$, $a_1 \neq 0$, $a_2 = 0$, and $a_3 \neq 0$,
Case 8.3. $d_1 = 5^{b_1'}$, $d_2 = 3^{a_2}5^{b_2}$, $d_3 = 3^{a_3}5^{b_3}$, $a_1 = 0$, $a_2 \neq 0$, and $a_3 \neq 0$,
Case 8.4. $d_1 = 5^{b_1''}$, $d_2 = 5^{b_2''}$, $d_3 = 3^{a_3}5^{b_3}$, $a_1 = a_2 = 0$, and $a_3 \neq 0$,
Case 8.5. $d_1 = 5^{b_1''}$, $d_2 = 3^{a_2}5^{b_2}$, $d_3 = 5^{b_3''}$, $a_1 = a_3 = 0$, and $a_2 \neq 0$,
Case 8.6. $d_1 = 3^{a_1}5^{b_1}$, $d_2 = 5^{b_2''}$, $d_3 = 5^{b_3''}$, $a_2 = a_3 = 0$, and $a_1 \neq 0$.
Some cases are shorter, but we will begin with Case 8.1.

Case 8.1. Because $b_3' \neq 0$, we obtain, by (2.36), that $b_1 \neq 0$ and $b_2 = 0$. By (2.33), there are four cases to consider. If $a_1 = 2\alpha - 1$ and $b_1 = 2\beta$, then

$$8(d_1 + d_2 + d_3)/(3^{2\alpha}5^{2\beta}) = 8\left(3^{2\alpha-1}5^{2\beta} + 3^{a_2} + 5^{b_3'}\right) / \left(3^{2\alpha}5^{2\beta}\right) > 8/3 > 2,$$

which contradicts (2.32). Next, suppose that $a_1 = 2\alpha$ and $b_1 = 2\beta - 1$. Because $3^{a_2} = d_2 > d_3 = 5^{b_3'} \geq 5$, we obtain $a_2 \geq 2$. Thus,

$$\begin{aligned} 0 &= 8\left(\sigma(n) - 2n + d_1 + d_2 + d_3\right) = 8(d_1 + d_2 + d_3) - 3^{2\alpha}5^{2\beta} - 3^{2\alpha+1} - 5^{2\beta+1} + 1 \\ &> 8(3^{2\alpha}5^{2\beta-1} + 3^2 + 5) - 3^{2\alpha}5^{2\beta} - 3^{2\alpha+1} - 5^{2\beta+1} \\ &= (3^{2\alpha+1} - 25)(5^{2\beta-1} - 1) + 87 > 0, \end{aligned}$$

which is false. Next, consider the case $(a_1, b_1) = (2\alpha - 2, 2\beta)$. Because $a_1 \neq 0$, $\alpha \geq 2$. If $\beta \geq 2$, then (2.32) implies that

$$\begin{aligned} 1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha-2}5^{2\beta} + 3^{a_2} + 5^{b_3'}) \leq \frac{8}{3^{2\alpha}5^{2\beta}}\left(3^{2\alpha-2}5^{2\beta} + 3^{2\alpha} + 5^{2\beta-1}\right) \\ &= 8\left(\frac{1}{3^2} + \frac{1}{5^{2\beta}} + \frac{1}{3^{2\alpha}\cdot 5}\right) \leq 8\left(\frac{1}{3^2} + \frac{1}{5^4} + \frac{1}{3^4\cdot 5}\right) < 1, \end{aligned}$$

which is a contradiction. So, $\beta = 1$. Then, $d_3 = 5$.

Starting with $0 = 8\left(\sigma(n) - 2n + d_1 + d_2 + d_3\right)$, and then simplifying leads to $2 \cdot 3^{a_2} = 13 \cdot 3^{2\alpha-2} + 21$. Because $13 \cdot 3^{2\alpha-2} + 21 > 2 \cdot 3^{2\alpha-1}$, we obtain $a_2 = 2\alpha$. But, then $21 = 2 \cdot 3^{a_2} - 13 \cdot 3^{2\alpha-2} = 5 \cdot 3^{2\alpha-2} \equiv 0 \pmod 5$, a contradiction. Next, we consider the last case: $(a_1, b_1) = (2\alpha - 1, 2\beta - 1)$. If $\alpha \geq 2$ or $\beta \geq 2$, then (2.32) implies

$$\begin{aligned} 1 &< \frac{8}{3^{2\alpha}5^{2\beta}}\left(3^{2\alpha-1}5^{2\beta-1} + 3^{a_2} + 5^{b_3'}\right) \\ &\leq 8\left(\frac{1}{15} + \max\left\{\frac{1}{25} + \frac{1}{3^4\cdot 5}, \frac{1}{5^4} + \frac{1}{3^2\cdot 5}\right\}\right) < 1, \end{aligned}$$

which is impossible. So, $\alpha = 1 = \beta$. Then, $a_1 = 1 = b_1$. Because $15 = d_1 > 3^{a_2} = d_2 > d_3 = 5^{b_3'} = 5$, we have $d_2 = 9$. Now, it is easy to verify that $\sigma(n) - 2n + d_1 + d_2 + d_3 = -18 \neq 0$. So, there is no solution in this case.

Case 8.2. Because $b_2 = b_2' \neq 0$, we obtain, by (2.36), that $b_3 = 0$. Similar to Case 8.1, we divide our calculation into four cases according to the values of a_1 and b_1 as given in (2.33). If $(a_1, b_1) = (2\alpha - 1, 2\beta)$, then $8(d_1 + d_2 + d_3)/(3^{2\alpha}5^{2\beta}) > 8d_1/(3^{2\alpha}5^{2\beta}) > 8/3 > 2$, contradicting (2.32). If $(a_1, b_1) = (2\alpha, 2\beta - 1)$, then $d_2 \geq 5$, $d_3 \geq 3$, and

$$\begin{aligned} 0 &= 8\left(\sigma(n) - 2n + d_1 + d_2 + d_3\right) = 8(d_1 + d_2 + d_3) - 3^{2\alpha}5^{2\beta} - 3^{2\alpha+1} - 5^{2\beta+1} + 1 \\ &\geq 3^{2\alpha+1}5^{2\beta-1} - 3^{2\alpha+1} - 5^{2\beta+1} + 65 \\ &= (3^{2\alpha+1} - 25)(5^{2\beta-1} - 1) + 40 > 0, \end{aligned}$$

which is not possible. Suppose $(a_1, b_1) = (2\alpha - 2, 2\beta)$. Because $a_1 \neq 0$, we have $\alpha \geq 2$. If $\beta \geq 2$, then (2.32) implies

$$\begin{aligned} 1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha-2}5^{2\beta} + 5^{b_2'} + 3^{a_3}) \\ &\leq \frac{8}{3^{2\alpha}5^{2\beta}}\left(3^{2\alpha-2}5^{2\beta} + 5^{2\beta-1} + 3^{2\alpha}\right) \leq 8\left(\frac{1}{9} + \frac{1}{3^4 \cdot 5} + \frac{1}{5^4}\right) < 1, \end{aligned}$$

which is false. So, $\beta = 1$. Then, $d_2 = 5$ and $d_3 = 3$.

Starting from $8(\sigma(n) - 2n + d_1 + d_2 + d_3) = 0$ and then simplifying leads to $13 \cdot 3^{2\alpha-2} + 15 = 0$, which is impossible. The last case of (2.33) is $(a_1, b_1) = (2\alpha - 1, 2\beta - 1)$. If $\alpha \geq 2$ or $\beta \geq 2$, then (2.32) implies

$$\begin{aligned} 1 &< \frac{8}{3^{2\alpha}5^{2\beta}}\left(3^{2\alpha-1}5^{2\beta-1} + 5^{b_2'} + 3^{a_3}\right) \\ &\leq 8\left(\frac{1}{15} + \frac{1}{3^{2\alpha} \cdot 5} + \frac{1}{5^{2\beta}}\right) \\ &\leq 8\left(\frac{1}{15} + \max\left\{\frac{1}{3^4 \cdot 5} + \frac{1}{5^2}, \frac{1}{3^2 \cdot 5} + \frac{1}{5^4}\right\}\right) < 1, \end{aligned}$$

which is not true. Thus, $\alpha = \beta = 1$. So, $a_1 = b_1 = 1$, $d_2 = 5$, and $d_3 = 3$. Now, it is easy to verify that $\sigma(n) - 2n + d_1 + d_2 + d_3 = -24 \neq 0$. So, there is no solution in this case.

Case 8.3. By (2.32), we obtain $1 < 8(3d_1)/(3^{2\alpha}5^{2\beta}) \leq 24 \cdot 5^{2\beta-1}/(3^{2\beta}5^{2\beta}) \leq 24/45 < 1$, a contradiction.

Case 8.4. By (2.32), we obtain

$$\begin{aligned} 1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(d_1 + 2d_2) = \frac{8}{3^{2\alpha}5^{2\beta}}(5^{b_1''} + 2 \cdot 5^{b_2''}) \\ &\leq \frac{8}{3^{2\alpha}5^{2\beta}}(5^{2\beta} + 2 \cdot 5^{2\beta-2}) \leq 8\left(\frac{1}{9} + \frac{2}{9 \cdot 25}\right) < 1, \end{aligned}$$

which is not possible.

Case 8.5. If $\alpha \geq 2$, then (2.32) implies that

$$1 < \frac{8}{3^{2\alpha}5^{2\beta}}(2d_1 + d_3) \leq \frac{8}{3^{2\alpha}5^{2\beta}}(2 \cdot 5^{2\beta} + 5^{2\beta-2}) \leq 8\left(\frac{2}{3^4} + \frac{1}{3^4 \cdot 5^2}\right) < 1,$$

which is false. Therefore, $\alpha = 1$. Then the left side of (2.31) is $\equiv 4 \pmod 5$, whereas the right side is $\equiv 2(d_1 + d_2 + d_3) \equiv 2(3^{a_2}5^{b_2} + 5^{b_3''}) \pmod 5$. By (2.36), $b_2 = 0$ or $b_3 = 0$. If $b_2 = 0$ and $b_3 \neq 0$, then $5^2 \leq d_3 < d_2 = 3^{a_2}$, and so $a_2 \geq 3$, contradicting that $d_2 \mid n$ and $n = 3^{2\alpha}5^{2\beta} = 3^2 \cdot 5^{2\beta}$. If $b_2 \neq 0$ and $b_3 = 0$, then $2(3^{a_2}5^{b_2} + 5^{b_3''}) \equiv 2 \pmod 5$, which is not the case. Because $\alpha = 1$, $a_2 \in \{1, 2\}$. So if $b_2 = b_3 = 0$, then $2(3^{a_2}5^{b_2} + 5^{b_3''}) \equiv 3, 0 \pmod 5$, which is not true. So there is no solution in this case.

Case 8.6. Because $5^{b_2''} = d_2 > d_3 \geq 1$, we have $b_2'' \neq 0$. By (2.36), we see that $b_3 = 0$. Then, the right side of (2.31) is $\equiv 2(3^{a_1}5^{b_1} + 5^{b_2''} + 5^{b_3''}) \equiv 2 \pmod 5$, contradicting (2.34).

This completes the proof of this theorem. □

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