CONSECUTIVE HAPPY NUMBERS AND GENERALIZATIONS

By

MR. Kittipong SUBWATTANACHAI

A Thesis Submitted in Partial Fulfillment of the Requirements

for Master of Science (MATHEMATICS)

Department of MATHEMATICS

Graduate School, Silpakorn University

Academic Year 2020

จำนวนแฮปปี้ที่ติดกันและกรณีทั่วไป

โดย

นายกิตติพงศ์ ทรัพย์วัฒนาชัย

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Title	Consecutive happy numbers and generalizations
By	Kittipong SUBWATTANACHAI
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Advisor	Associate Professor Dr. Prapanpong Pongsriiam

Graduate School Silpakorn University in Partial Fulfillment of the Requirements for the Master of Science

....Dean of graduate school

(Associate Professor Jurairat Nunthanid, Ph.D.)

Approved by

....Chair person

(Assistant Professor Dr. Tammatada Khemaratchatakumthorn)

....Advisor

(Associate Professor Dr. Prapanpong Pongsriiam)

....External Examiner

(Assistant Professor Dr. Kantaphon Kuhapatanakul)

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Chase introduced the concept of digit maps generalizing that of happy functions. In this thesis, we extend the investigation further by considering compositions of various dagit maps.

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Kittipong SUBWATTANACHAI

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Chapter 1

Introduction

For integers $e, b \geq 2$, let $S_{e,b} : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ be the function that takes a nonnegative integer x to the sum of the e-th powers of its digits in base b, that is,

$$S_{e,b}(x) = a_k^e + a_{k-1}^e + \cdots + a_0^e,$$

if $x = (a_k a_{k-1} \cdots a_0)_b = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_0$ is the b-adic expansion of x with $a_k \neq 0$ and $a_i \in \{0, 1, \ldots, b-1\}$ for all $i = 0, 1, \ldots, k$. We call $S_{e,b}$ an (e, b)-happy function and if there exists $n \in \mathbb{N}$ such that $S_{e,b}^{(n)}(x) = 1$, then we call x an (e, b)-happy number. Here and throughout this article, $f^{(0)}$ is the identity function mapping x to x and $f^{(n)} = f^{(n-1)} \circ f$ is the n-fold composition of f. In addition, if we write a number without specifying a base, then it is always written in base 10.

It is well-known that [7] for any $x \in \mathbb{N}$, the sequence $(S_{2,10}^{(n)}(x))_{n \geq 0}$ either converges to 1 or eventually becomes the cycle

$$(4, 16, 37, 58, 89, 145, 42, 20).$$

For example, the sequence $(S_{2,10}^{(n)}(13))_{n \geq 0}$ is $(13, 10, 1, 1, \ldots)$ and $(S_{2,10}^{(n)}(2))_{n \geq 0}$ is $(2, 4, 16, \ldots, 20, 4, 16, \ldots)$. Then 13 is $(2, 10)$-happy but 2 is not. As usual, $(a_1, a_2, \ldots, a_k)$ and any of its cyclic permutation are considered the same cycle.

El-Sedy and Siksek [3] were the first to prove that there exist arbitrarily long strings of consecutive integers which are $(2, 10)$-happy. That is, for each

$m \geq 1$, there exists an integer ℓ_0 such that every element of the finite sequence $\ell_0+1, \ell_0+2, \ldots, \ell_0+m$ is a happy number. Pan [11] obtained in 2009 that if $e-1$ is not divisible by $p-1$ for any prime divisor p of $b-1$, then there exist arbitrarily long sequences of consecutive (e,b)-happy numbers.

Let P be the product of all prime divisors p of $b-1$ such that $p-1$ divides $e-1$. It is not difficult to verify that $S_{e,b}(n) \equiv n \pmod{P}$ for every n, and so if $P > 1$, then (e,b)-happy numbers do not contain consecutive integers. Zhou and Cai [24] extended Pan's result by proving that if $P > 1$, then the (e,b)-happy numbers contain arbitrarily long arithmetic progressions with common difference P.

About 9 years later, Chase [1] introduced a concept of digit maps generalizing that of happy functions and obtained a theorem extending those by Pan [11] and El-Sedy and Siksek [3]. Noppakeaw, Phoopha, and Pongsriiam [10] considered compositions of various (e,b)-happy functions. For each $\underline{e} = (e_1, e_2, \ldots, e_k) \in \mathbb{N}^k$ and $\underline{b} = (b_1, b_2, \ldots, b_k) \in \mathbb{N}^k$ with $e_i \geq 1$ and $b_i \geq 2$ for all $i = 1, 2, \ldots, k$, they [10] defined an $(\underline{e}, \underline{b})$-happy function $S_{\underline{e},\underline{b}} : \mathbb{N}\cup\{0\} \to \mathbb{N}\cup\{0\}$ by

$$S_{\underline{e},\underline{b}}(x) = (S_{e_1,b_1} \circ S_{e_2,b_2} \circ \cdots \circ S_{e_k,b_k})(x) \quad \text{for all } x \in \mathbb{N}\cup\{0\}.$$

and showed that for each $x \in \mathbb{N}$, the iteration sequence $\left(S^{(n)}_{\underline{e},\underline{b}}(x)\right)_{n\geq 0}$ either converges to a fixed point or eventually enters into a cycle. Moreover, they [10] proved that the number of all such fixed points and cycles is finite. This implies the possibility of obtaining similar results on $(\underline{e}, \underline{b})$-happy numbers.

For other results on happy numbers and happy functions, we refer the reader to [4, 9, 20, 21]. For results on long arithmetic progressions in other integer sequences, see [2, 5, 6, 8, 14, 22, 23] for example.

In this thesis, we combine the ideas from Chase [1] and Noppakeaw, Phoopha, and Pongsriiam [10] and study the composition of various digit maps. We show that such a composition also has the same property as $S_{\underline{e},\underline{b}}$. For more information, we invite the reader to visit Pongsriiam's ResearchGate website [19] for some freely downloadable articles [12, 13, 14, 15, 16, 17, 18] in related topics of research.

Chapter 2

Preliminaries and Lemmas

We first recall the definition of digit maps and u-integers from [1].

Definition 2.1. Let $b \geq 2$ be an integer. A digit map with respect to b is a function $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ satisfying $\gcd(b, f(b-1)) = 1$, $f(0) = 0$, $f(1) = 1$, and

$$f(x) = f(a_k) + f(a_{k-1}) + \cdots + f(a_0)$$

if $x = (a_k a_{k-1} \cdots a_0)_b = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_0$ is the b-adic expansion of x where $a_i \in \{0, 1, \ldots, b-1\}$ for all $i = 0, 1, \ldots, k$ and $a_k \neq 0$.

If f is a digit map with respect to a base $b \geq 2$ and $x, u \in \mathbb{N}$, then x is called a u-integer if $f^{(n)}(x) = u$ for some $n \geq 0$. When f is an (e, b)-happy function and $u = 1$, the u-integers are the same as (e, b)-happy numbers. So the following theorem extends those of Pan [11] and El-Sedy and Siksek [3].

Theorem 2.2. (Chase [1])*Let $b \geq 2$ be an integer. Suppose f is a digit map with respect to b and there is an $m \in \{0, 1, \ldots, b-1\}$ such that* $\gcd(f(m) - m, f(b-1)) = 1$*. If $u, n \in \mathbb{N}$ and u is a member of a cycle, then there exists $\ell \in \mathbb{N}$ such that $\ell, \ell+1, \ell+2, \ldots, \ell+n-1$ are u-integers.*

To extend Theorem 2.2 in the future, it may be useful to have a function g such that, for each $x \in \mathbb{N}$, the iteration sequence $(g^{(n)}(x))_{n \geq 0}$ converges to a fixed point or eventually enters into a cycle. Noppakeaw, Phoopha, and

Pongsriiam [10] obtained such a function g by considering compositions of happy functions. Our purpose is to extend their result [10, Theorem 1.4] further to the compositions of various digit maps. To do this, we consider the following two conditions for a function $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$.

(A) There exists $N_f \in \mathbb{N}$ such that $f(x) < x$ for all $x \geq N_f$.

(B) For each $x \in \mathbb{N} \cup \{0\}$, the sequence $(f^{(n)}(x))_{n \geq 0}$ converges to a fixed point or eventually enters into a cycle. In addition, the number of all such fixed points and cycles is finite.

We first show that a digit map satisfies the condition (A) and if $f_1, f_2, \ldots, f_k$ satisfy (A), then so does $f_1 \circ f_2 \circ \cdots \circ f_k$. A proof of a similar result was already done in [10, Theorem 1.3] but it was for $f : \mathbb{N} \to \mathbb{N}$. So we need to adjust it for $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$. Recall also that, for $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the largest integer less than or equal to x and $\lceil x \rceil$ is the smallest integer larger than or equal to x.

Chapter 3

Main Results

In this chapter, we study the composition of various digit maps. We prove that if F is such a composition and x is any nonnegative integer, then the sequence $(F^{(n)}(x))_{n\geq 0}$ either converges or eventually becomes a cycle. Furthermore, the number of such fixed points and cycles is finite. We begin with showing that a digit map with respect to a base $b \geq 2$ satisfies the condition (A).

Theorem 3.1. *Let f be a digit map with respect to $b \geq 2$. Then there exists $M \in \mathbb{N}$ such that*

$$f(x) < x \quad \textit{for all} \quad x \geq M. \tag{3.1}$$

Proof. Let $M' = \max\{f(i) \mid i = 0, 1, \ldots, b-1\}$. Then $M' \geq f(1) = 1$. Since $e^x/x \to \infty$ as $x \to \infty$, there exists $c > 1$ such that $e^c/c > bM'/\log b$. This implies

$$c - \log c > \log b + \log M' - \log\log b. \tag{3.2}$$

Let $M = \lceil \frac{cM'}{\log b} \rceil$ and $x \geq M$. Next, we show that $f(x) < x$. We write $x = (a_k a_{k-1} \ldots a_1 a_0)_b$ where $a_k \neq 0$ and $0 \leq a_i < b$ for all $i = 0, 1, 2, \ldots, k$. Then $b^k \leq a_k b^k \leq x$, so $k \leq \frac{\log x}{\log b}$ and

$$f(x) = f(a_k) + f(a_{k-1}) + \cdots + f(a_0) \leq M'(k+1) \leq M'\big(\frac{\log x}{\log b} + 1\big). \tag{3.3}$$

Let $h(y) = \frac{y}{M'} - \frac{\log y}{\log b} - 1$ for all $y > 0$. Then $h'(y) = \frac{1}{M'} - \frac{1}{y \log b} > 0$ for all $y > \frac{M'}{\log b}$. Since $M \geq cM'/\log b > M'/\log b$ and h is increasing on

$[M'/\log b, \infty)$, we obtain that if $y \geq M$, then

$$h(y) \geq h\big(M\big) \geq h\big(cM'/\log b\big) = \frac{c - \log c - \log M' + \log\log b - \log b}{\log b} > 0,$$

where the last inequality is obtained from (3.2). This shows that $h(y) > 0$ for all $y \geq M$. In particular, $h(x) > 0$, and so $1 + \log x/\log b < x/M'$. By (3.3), we obtain $f(x) < x$, as required. □

Theorem 3.2. *If* $f_1, f_2, \ldots, f_k : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ *satisfy the condition* (A), then $f_1 \circ f_2 \circ \cdots \circ f_k$ also satisfies (A).

Proof. We can prove this by induction on k and it is actually the same as that given by Noppakeaw et al. [10, Theorem 1.3], but for completeness, we give the proof again here. When $k = 1$, the result is obvious. Assume that $k \in \mathbb{N}$ and the result holds for k. Suppose $f_1, f_2, \ldots, f_{k+1} : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ satisfy (A). Let $f = f_1 \circ f_2 \circ \cdots \circ f_{k+1}$ and $g = f_1 \circ f_2 \circ \cdots \circ f_k$. Then there are m_1, $m_2 \in \mathbb{N}$ such that

$$g(x) < x \quad \text{for all} \quad x \geq m_1, \quad \text{and} \quad f_{k+1}(x) < x \quad \text{for all} \quad x \geq m_2. \tag{3.4}$$

Let $m_3 = \max\{g(x) \mid 1 \leq x < m_1\}$ and $m = \max\{m_1, m_2, m_3\} + 1$. Let $x \geq m$. We will show that $f(x) < x$. If $f_{k+1}(x) \geq m_1$, then we obtain by (3.4) that

$$f(x) = g(f_{k+1}(x)) < f_{k+1}(x) < x.$$

On the other hand, if $f_{k+1}(x) < m_1$, then $f(x) = g(f_{k+1}(x)) \leq m_3 < m \leq x$. This completes the proof. □

We already have that a digit map is a function on the set of all nonnegative integers and it satisfies the condition (A). Therefore, by Theorem 3.2, we immediately obtain the following corollary.

Corollary 3.3. *A composition of digit maps satisfies the condition* (A).

Proof. This follows immediately from Theorem 3.1 and 3.2. □

Theorem 3.4. *If* $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ *satisfies* (A), then f satisfies (B).

Proof. This is given in [10, Theorem 1.2] for a function $f : \mathbb{N} \to \mathbb{N}$, and we can use the same method in our proof too. However, directly applying [10, Theorem 1.2] does not lead to our desired result for $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$, so we still used to give the proof here. For convenience, we write N instead of N_f and we assert that

$$\text{for every } y \in \mathbb{N} \cup \{0\}, \text{ there exists } n \in \mathbb{N} \cup \{0\} \text{ such that } f^{(n)}(y) < N. \quad (3.5)$$

If $y < N$, then we can choose $n = 0$. If $y \geq N$, then by (A), $f(y) < y$. If $f(y) < N$, then we can choose $n = 1$; otherwise, we obtain by (A) that $f^{(2)}(y) < f(y)$. We can repeat this process and obtain a strictly decreasing sequence of positive integers $f(y), f^{(2)}(y), f^{(3)}(y), \ldots$, and eventually $f^{(n)}(y) < N$ for some n. Hence (3.5) is proved.

Now let $x \in \mathbb{N} \cup \{0\}$ and suppose that $(f^{(n)}(x))_{n \geq 0}$ does not converge to a fixed point of f. By (3.5), there exists $n_1 \in \mathbb{N}$ such that $f^{(n_1)}(x) < N$. Again by (3.5), there exists $n_2 \in \mathbb{N}$ such that $f^{(n_2)}(f^{(n_1)}(x)) < N$. Repeating this process $N + 1$ times, we obtain the set of nonnegative integers

$$f^{(n_1)}(x), f^{(n_1+n_2)}(x), \ldots, f^{(n_1+n_2+\cdots+n_{N+1})}(x),$$

which are less than N. By the pigeonhole principle, some of them are the same, say

$$f^{(n_1+n_2+\cdots+n_j)}(x) = f^{(n_1+n_2+\cdots+n_j+\cdots+n_\ell)}(x) \quad \text{for some } \ell > j \geq 1.$$

Let $y = f^{(n_1+n_2+\cdots+n_j)}(x)$. Then the tail of the sequence $(f^{(n)}(x))_{n \geq 0}$ eventually becomes

$$(y, f(y), f^{(2)}(y), \ldots, f^{(n_{j+1}+n_{j+2}+\cdots+n_\ell-1)}(y), y, \ldots),$$

which is a cycle. This proves the first part of (B). Next we show that the set U_f of fixed points and cycles is finite. More precisely, we will show that

$$U_f := \{x \in \mathbb{N} \cup \{0\} \mid \exists n \in \mathbb{N}, f^{(n)}(x) = x\} \subseteq [0, M], \quad (3.6)$$

where $M = \max\{N, f(0), f(1), f(2), \ldots, f(N)\}$. First of all, by (A), if x is a fixed point of f, then $x < N$ and so $x \in [0, M]$. Suppose that x is an element in a cycle arising from the iteration $(f^{(n)}(y))_{n\geq 0}$ for some $y \in \mathbb{N} \cup \{0\}$. If $x < N$, then $x \in [0, M]$ and we are done. So suppose $x \geq N$. By (3.5), there exists $n \in \mathbb{N}$ such that $f^{(n)}(x) < N$. Since x is in a cycle, after some iterations, it must come back to x. That is, there exists $k \in \mathbb{N}$ such that $f^{(k)}(f^{(n)}(x)) = x$. If $k = 1$ or $f^{(n+k-1)}(x) \leq N$, then $x = f(f^{(n+k-1)}(x)) \leq M$ and we are done. So suppose $k \geq 2$ and $f^{(n+k-1)}(x) > N$. Let ℓ be the smallest positive integer such that $f^{(n+k-\ell)}(x) < N$. Then $1 < \ell \leq k$ and for each $1 \leq i < \ell$, $f^{(n+k-i)}(x) \geq N$. So

$$f^{(n+k-\ell+1)}(x) > f^{(n+k-\ell+2)}(x) > \cdots > f^{(n+k-1)}(x) > f^{(n+k)}(x) = x.$$

So $x < f^{(n+k-\ell+1)}(x) = f(f^{(n+k-\ell)}(x)) \leq M$. Therefore (3.6) is verified and the proof is complete. $\square$

Recall that a composition of digit maps satisfies the condition (A). Then, by Theorem 3.4, it satisfies the condition (B). So we obtain our main result as in the following corollary.

Corollary 3.5. *Let $f_1, f_2, \ldots, f_k$ be digit maps with respect to $b_1, b_2, \ldots, b_k$ respectively, where $b_i \geq 2$ for every i. Let $F : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ be given by $F = f_1 \circ f_2 \circ \cdots \circ f_k$. Then F satisfies* (B).

Proof. This follows immediately from Corollary 3.3 and Theorem 3.4. $\square$

Suppose that $f_1, f_2, \ldots, f_k$ are digit maps with respect to bases $b_1, b_2, \ldots, b_k$, respectively, and $F = f_1 \circ f_2 \circ \cdots \circ f_k$. By Corollary 3.3, there is $N \in \mathbb{N}$ such that $F(x) < x$ for all $x \geq N$. Then all fixed points and cycles can be found by considering the sequence $(F^{(n)}(x))_{n\geq 0}$ where $0 \leq x < N$. We show some explicit calculations for such N in the following example.

Example 3.6. Consider $F = f_1 \circ f_2 \circ f_3$, where f_1, f_2, f_3 are digit maps with respect to $b_1 = 6, b_2 = 5, b_3 = 4$, respectively, and for $0 \leq x < b_i$, they are defined by $f_1(x) = x^4, f_2(x) = x^2, f_3(x) = x^3$.

First, we show that there exists an integer m such that $(f_1 \circ f_2)(x) < x$ for all $x > m$. By following the proof of Theorem 3.1, we consider $M_1' = \max\{f_1(i) \mid i = 0, 1, \ldots, 5\} = 5^4$ and $M_2' = \max\{f_2(i) \mid i = 0, 1, 2, 3, 4\} = 4^2$. Since $\frac{e^{10}}{10} > \frac{6(5)^4}{\log 6}$, $\frac{e^6}{6} > \frac{5(4)^2}{\log 5}$, we let $c_1 = 10$ and $c_2 = 6$. The corresponding M_1 for f_1 and M_2 for f_2 are $M_1 = 3489$ and $M_2 = 60$. Therefore

$$f_1(x) < x \quad \text{for all} \quad x \geq 3489 \quad \text{and} \quad f_2(x) < x \quad \text{for all} \quad x \geq 60.$$

Let $m_1 = 3489$, $m_2 = 60$, and $m_3 = \max\{f_1(x) \mid 1 \leq x < 3489\}$. Since $3489 = (24053)_6$, we see that $m_3 = f_1((15555)_6) = 2501$.

Let $m = \max\{m_1, m_2, m_3\} + 1 = 3490$. By the proof of Theorem 3.2, we have that

$$\big(f_1 \circ f_2\big)(x) < x \quad \text{for all} \quad x \geq 3490.$$

Next, we consider $f_1 \circ f_2 \circ f_3$. Similarly, $M_3' = 3^3$ and $\frac{e^7}{7} > \frac{4(3)^3}{\log 4}$, so we let $c_3 = 7$, and $M_3 = \lceil \frac{7(3)^3}{\log 4} \rceil = 137$ and obtain

$$f_3(x) < x \quad \text{for all} \quad x \geq 137.$$

We let $m_1 = 3490$, $m_2 = 137$, $m_3 = \max\{(f_1 \circ f_2)(x) \mid 1 \leq x < 3490\}$. Then

$$\begin{aligned} m_3 &= \max\{f_1(f_2(x)) \mid 1 \leq x < (102430)_5\} \\ &= \max\{f_1(x) \mid 1 \leq x < 80\} \\ &= \max\{f_1(x) \mid 1 \leq x < (212)_6\} = 1251. \end{aligned}$$

Let $m = \max\{m_1, m_2, m_3\} + 1 = 3491$. So

$$\big(f_1 \circ f_2 \circ f_3\big)(x) < x \quad \text{for all} \quad x \geq 3491.$$

The lower bound 3491 may not be best possible but it is not difficult to search for the best one by using a computer. We can check whether $(f_1 \circ f_2 \circ f_3)(x) < x$ for $x = 1, 2, 3, \ldots, 3490$. If $(f_1 \circ f_2 \circ f_3)(x) < x$ for $x = N, N+1, \ldots, 3490$ and $(f_1 \circ f_2 \circ f_3)(x) \geq x$ for $x = N - 1$, then such the integer N is the optimal lower bound. In fact, by using a computer, we obtain $N = 831$.

We give two more examples to illustrate alternative calculations.

Example 3.7. Let $b_1 = b_2 = 10$ and let f_1, f_2 be digit maps with respect to b_1, b_2 such that $f_1(x) = 2x^2 - x$ and $f_2(x) = 3x^3 - x^2 - x$ for $0 \le x < 10$. Let $F = f_1 \circ f_2$. Then, for each $x \in \mathbb{N}$, the sequence $(F^{(n)}(x))_{n \ge 0}$ either converges to 1 or eventually becomes the cycle $(6, 132, 240, 154, 166, 23, 211)$.

Proof. We first show that

$$F(x) = (f_1 \circ f_2)(x) < x \quad \text{for all } x \ge 10930. \tag{3.7}$$

If $x \in [10930, 99999]$, then $x = (a_4a_3a_2a_1a_0)_{10}$ where $0 \le a_i \le 9$, and so

$$f_2(x) = f_2(a_4) + f_2(a_3) + \cdots + f_2(a_0) \le 5f_2(9) = 10485,$$

and thus

$$F(x) \le \max\{f_1(x) \mid 1 \le x \le 10485\} = f_1(9999) = 4(153) = 612 < x.$$

Next, suppose that $x \ge 10^5$ and write $x = (a_ka_{k-1}\cdots a_1a_0)_{10}$ where $k \ge 5$ and $a_k \ne 0$.

It is easy to prove by induction on k that $2097(k+1) < 10^k$ for all $k \ge 5$. Then,

$$f_2(a_k) + f_2(a_{k-1}) + \cdots + f_2(a_0) \le (k+1)f_2(9) = 2097(k+1) < 10^k.$$

Therefore,

$$F(x) \le \max\{f_1(x) \mid 0 \le x \le 10^k\} = f_1(\underbrace{99\cdots 9}_{k \text{ digits}}) = 153k < 10^k \le a_k10^k \le x.$$

So (3.7) is verified. It only remains to check that, for each $x < 10930$, whether the sequence $(F^{(n)}(x))_{n \ge 0}$ converges to a fixed point or becomes a cycle. This can be done using a computer. We find that for each positive integer $x < 10930$, the sequence $(F^{(n)}(x))_{n \ge 0}$ converges to 1 or becomes the cycle $(6, 132, 240, 154, 166, 23, 211)$. $\square$

The next example is slightly different from Example 3.7 because b_1 and b_2 are different.

Example 3.8. Let $b_1 = 7, b_2 = 5, f_1$ and f_2 digit maps with respect to b_1 and b_2, respectively, $f_1(x) = 2x^2 - x$ for $0 \leq x \leq 6$, and $f_2(x) = 3x^3 - 2x$ for $0 \leq x \leq 4$. Let $F = f_1 \circ f_2$. Then, for each $x \in \mathbb{N}$, the sequence $(F^{(n)}(x))_{n\geq 0}$ contains $1, 6, 43, 56,$ or 61. Moreover, $1, 6,$ and 43 are the only fixed points of F and if the sequence $(F^{(n)}(x))_{n\geq 0}$ does not contain $1, 6,$ or 43, then it eventually enters into the cycles $(56, 16, 82, 112)$ or $(61, 111, 35, 15)$.

Proof. We first show that $F(x) < x$ for all $x \geq 1030$. Let $x \geq 1030$. Since $x > 5^4$, we write $x = (a_k a_{k-1} \ldots a_0)_5$, where $k \geq 4$, $0 \leq a_i \leq 4$ for every i, and $a_k \neq 0$. Then $f_2(x) \leq (k+1)f_2(4) = 184(k+1)$ and it is easy to prove by induction on k that $184(k+1) < 7^k$ for all $k \geq 4$. Then

$$F(x) \leq \max\{f_1(x) \mid 0 \leq x < 7^k\} = f_1((\underbrace{66\cdots6}_{k \text{ digits}})_7) = 66k.$$

Since $5^k \leq a_k 5^k \leq x$, it follows that $k \leq \frac{\log x}{\log 5}$. Since the function $y \to \frac{\log y}{y}$ is decreasing on $[3, \infty)$ and $x \geq 1030$, we obtain

$$F(x) \leq 66k \leq 66\Big(\frac{\log x}{\log 5}\Big) \leq \frac{66}{\log 5}\Big(\frac{\log x}{x}\Big)x \leq \frac{66}{\log 5}\Big(\frac{\log 1030}{1030}\Big)x < x$$

Similar to Example 3.7, the rest can be verified using a computer. □

Other examples of compositions of digit maps and their fixed points and cycles are shown in Table 3.1. For instance, Line 6 of Table 3.1 means that if f_1, f_2, f_3 are digit maps such that $f_1(x) = 2x^2 - x$ for $0 \leq x \leq 3$, $f_2(x) = 3x^3 - x^2 - x$ for $0 \leq x \leq 4$, and $f_3(x) = 3x^4 + 2x^2 - 4x$ for $0 \leq x \leq 6$, then the fixed points of $F = f_1 \circ f_2 \circ f_3$ are 1, 7, and 53, and for any $x \in \mathbb{N}$, $(F^{(n)}(x))_{n\geq 0}$ converges to 1, 7, or 53. Note that zero is also a fixed point of F but we are not interested in this fixed point since in our example $(F^{(n)}(x))_{n\geq 0}$ does not converge to zero for any $x \in \mathbb{N}$.

f_1	f_2	f_3	$\underline{b}$	Fixed points of F or cycles in $(F^{(n)}(x))_{n\geq 0}$
$3x^3-x^2-x$	$2x^2-x$		(10,10)	1, (20,606,88,190,518,1213,87)
$3x^3-x^2-x$	$2x^2-x$		(8,8)	1, 70, 173, (71,242,992,974,1060,1579)
$2x^2-x$	$3x^3-x^2-x$		(10,10)	1,(6,132,240,154,166,23,211)
$2x^2-x$	$3x^3-x^2-x$		(7,5)	1,6,43,(16,82,112,56),(15,61,111,35)
$2x^2-x$	$3x^3-x^2-x$	$3x^4+2x^2-4x$	(4,5,7)	1,7,53
$2x^2-x$	$3x^3-x^2-x$	$3x^4+2x^2-4x$	(5,4,7)	1
2^x-1	4^x-3^x		(10,6)	1, 321, 581, 638, (41, 385)
$\lfloor e^x\rfloor-1$			8	1, 1103 ,(20,59,1114,32,53,549,201,21,153,26,25), (462,1498,1126)
$\lfloor e^x\rfloor-1$	x^2		(8,10)	1,(59,154,153,72,549,1102,402)
$\lfloor e^x\rfloor-1$	x^2	2^x-1	(8,10,10)	1,2

Table 3.1: Fixed points of F or cycles in $(F^{(n)}(x))_{n\geq 0}$

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Appendix

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Composition of Happy Functions and Digit Maps

Kittipong Subwattanachai, Prapanpong Pongsriiam

Department of Mathematics
Faculty of Science
Silpakorn University
Nakhon Pathom, 73000, Thailand

email: subwattanachai.k@gmail.com, prapanpong@gmail.com,
pongsriiam_p@silpakorn.edu

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Abstract

Chase [1] introduced the concept of digit maps generalizing that of happy functions. We extend the investigation further by considering compositions of various digit maps. We prove that if F is such a composition and x is any positive integer, then the sequence $(F^{(n)}(x))_{n\geq 0}$ either converges or eventually becomes a cycle. Furthermore, we show that the number of all possible limits and cycles is finite.

1 Introduction

For integers $e, b \geq 2$, let $S_{e,b} : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ be the function that takes a nonnegative integer x to the sum of the e-th powers of its digits in base b, that is,

$$S_{e,b}(x) = a_k^e + a_{k-1}^e + \cdots + a_0^e,$$

if $x = (a_k a_{k-1} \cdots a_0)_b = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_0$ is the b-adic expansion of x with $a_k \neq 0$ and $a_i \in \{0, 1, \ldots, b-1\}$ for all $i = 0, 1, \ldots, k$. We call $S_{e,b}$ an (e, b)-happy function and if there exists $n \in \mathbb{N}$ such that $S_{e,b}^{(n)}(x) = 1$,

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Prapanpong Pongsriiam is the corresponding author.
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then we call x an (e,b)-happy number. Here and throughout this article, $f^{(0)}$ is the identity function mapping x to x and $f^{(n)} = f^{(n-1)} \circ f$ is the n-fold composition of f. In addition, if we write a number without specifying a base, then it is always written in base 10.

It is well-known that [7, Section E34] for any $x \in \mathbb{N}$, the sequence $(S_{2,10}^{(n)}(x))_{n\geq 0}$ either converges to 1 or eventually becomes the cycle

$$(4, 16, 37, 58, 89, 145, 42, 20).$$

For example, the sequence $(S_{2,10}^{(n)}(13))_{n\geq 0}$ is $(13, 10, 1, 1, \ldots)$ and $(S_{2,10}^{(n)}(2))_{n\geq 0}$ is $(2, 4, 16, \ldots, 20, 4, 16, \ldots)$, so 13 is $(2,10)$-happy but 2 is not. As usual, $(a_1, a_2, \ldots, a_k)$ and any of its cyclic permutation are considered the same cycle.

El-Sedy and Siksek [3] were the first to prove that there exist arbitrarily long strings of consecutive integers which are $(2,10)$-happy. That is, for each $m \geq 1$, there exists an integer ℓ_0 such that every element of the finite sequence $\ell_0+1, \ell_0+2, \ldots, \ell_0+m$ is a happy number. Pan [11] obtained in 2009 that if $e-1$ is not divisible by $p-1$ for any prime divisor p of $b-1$, then there exist arbitrarily long sequences of consecutive (e,b)-happy numbers.

Let P be the product of all prime divisors p of $b-1$ such that $p-1$ divides $e-1$. It is not difficult to verify that $S_{e,b}(n) \equiv n \pmod{P}$ for every n, and so if $P > 1$, then (e,b)-happy numbers do not contain consecutive integers. Zhou and Cai [17] extended Pan's result by proving that if $P > 1$, then the (e,b)-happy numbers contain arbitrarily long arithmetic progressions with common difference P.

About 9 years later, Chase [1] introduced a concept of digit maps generalizing that of happy functions and obtained a theorem extending those by Pan [11] and El-Sedy and Siksek [3]. Noppakeaw, Phoopha, and Pongsriiam [10] consider compositions of various (e,b)-happy functions. For each $\underline{e} = (e_1, e_2, \ldots, e_k) \in \mathbb{N}^k$ and $\underline{b} = (b_1, b_2, \ldots, b_k) \in \mathbb{N}^k$ with $e_i \geq 1$ and $b_i \geq 2$ for all $i = 1, 2, \ldots, k$, they [10] defined an $(\underline{e}, \underline{b})$-happy function $S_{\underline{e},\underline{b}} : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ by

$$S_{\underline{e},\underline{b}}(x) = (S_{e_1,b_1} \circ S_{e_2,b_2} \circ \cdots \circ S_{e_k,b_k})(x) \quad \text{for all } x \in \mathbb{N} \cup \{0\}.$$

and showed that for each $x \in \mathbb{N}$, the iteration sequence $\left(S_{\underline{e},\underline{b}}^{(n)}(x)\right)_{n\geq 0}$ either converges to a fixed point or eventually enters into a cycle. Moreover, they [10] proved that the number of all such fixed points and cycles is finite. This implies the possibility of obtaining similar results on $(\underline{e}, \underline{b})$-happy numbers.

For other results on happy numbers and happy functions, we refer the reader to [4, 9, 13, 14]. For results on long arithmetic progressions in other integer sequences, see [2, 5, 6, 8, 12, 15, 16] for example.

In this article, we combine the ideas from Chase [1] and Noppakeaw, Phoopha, and Pongsriiam [10] and study the composition of various digit maps. We show that such a composition also has the same property as $S_{\underline{e},\underline{b}}$.

2 Results

We first recall the definition of digit maps and u-integers from [1].

Definition 2.1. *Let $b \geq 2$ be an integer. A digit map with respect to b is a function $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ satisfying $\gcd(b, f(b-1)) = 1$, $f(0) = 0$, $f(1) = 1$, and*

$$f(x) = f(a_k) + f(a_{k-1}) + \cdots + f(a_0)$$

if $x = (a_k a_{k-1} \cdots a_0)_b = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_0$ is the b-adic expansion of x where $a_i \in \{0, 1, \ldots, b-1\}$ for all $i = 0, 1, \ldots, k$ and $a_k \neq 0$.

If f is a digit map with respect to a base $b \geq 2$ and $x, u \in \mathbb{N}$, then x is called a u-integer if $f^{(n)}(x) = u$ for some $n \geq 0$. When f is an (e, b)-happy function and $u = 1$, the u-integers are the same as (e, b)-happy numbers. So the following theorem extends those of Pan [11] and El-Sedy and Siksek [3].

Theorem 2.2. (Chase [1])*Let $b \geq 2$ be an integer. Suppose f is a digit map with respect to b and there is an $m \in \{0, 1, \ldots, b-1\}$ such that $\gcd(f(m) - m, f(b-1)) = 1$. If $u, n \in \mathbb{N}$ and u is a member of a cycle, then there exists $\ell \in \mathbb{N}$ such that $\ell, \ell+1, \ell+2, \ldots, \ell+n-1$ are u-integers.*

To extend Theorem 2.2 in the future, it may be useful to have a function g such that, for each $x \in \mathbb{N}$, the iteration sequence $(g^{(n)}(x))_{n \geq 0}$ converges to a fixed point or eventually enters into a cycle. Noppakeaw, Phoopha, and Pongsriiam [10] obtained such a function g by considering compositions of happy functions. Our purpose is to extend their result [10, Theorem 1.4] further to the compositions of various digit maps. To do this, consider the following two conditions for a function $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$.

(A) There exists $N_f \in \mathbb{N}$ such that $f(x) < x$ for all $x \geq N_f$.

(B) For each $x \in \mathbb{N} \cup \{0\}$, the sequence $(f^{(n)}(x))_{n \geq 0}$ converges to a fixed point or eventually enters into a cycle. In addition, the number of all such fixed points and cycles is finite.

We first show that a digit map satisfies the condition (A) and if $f_1, f_2, \ldots, f_k$ satisfy (A), then so does $f_1 \circ f_2 \circ \cdots \circ f_k$. A proof of a similar result was already done in [10, Theorem 1.3] but it was for $f : \mathbb{N} \to \mathbb{N}$. So we need to adjust it for $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$. Recall also that, for $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the largest integer less than or equal to x and $\lceil x \rceil$ is the smallest integer larger than or equal to x.

Theorem 2.3. *Let f be a digit map with respect to $b \geq 2$. Then there exists $M \in \mathbb{N}$ such that*

$$f(x) < x \quad \text{for all } x \geq M. \tag{2.1}$$

Proof. Let $M' = \max\{f(i) \mid i = 0, 1, \ldots, b-1\}$. Then $M' \geq f(1) = 1$. Since $e^x/x \to \infty$ as $x \to \infty$, there exists $c > 1$ such that $e^c/c > bM'/\log b$. This implies

$$c - \log c > \log b + \log M' - \log\log b. \tag{2.2}$$

Let $M = \lceil \frac{cM'}{\log b} \rceil$ and $x \geq M$. Next, we show that $f(x) < x$. We write $x = (a_k a_{k-1} \cdots a_1 a_0)_b$ where $a_k \neq 0$ and $0 \leq a_i < b$ for all $i = 0, 1, 2, \ldots, k$. Then $b^k \leq a_k b^k \leq x$. So $k \leq \frac{\log x}{\log b}$ and

$$f(x) = f(a_k) + f(a_{k-1}) + \cdots + f(a_0) \leq M'(k+1) \leq M'\Big(\frac{\log x}{\log b} + 1\Big). \tag{2.3}$$

Let $h(y) = \frac{y}{M'} - \frac{\log y}{\log b} - 1$ for all $y > 0$. Then $h'(y) = \frac{1}{M'} - \frac{1}{y \log b} > 0$ for all $y > \frac{M'}{\log b}$. Since $M \geq cM'/\log b > M'/\log b$ and h is increasing on $[M'/\log b, \infty)$, we obtain that if $y \geq M$, then

$$h(y) \geq h\big(M\big) \geq h\big(cM'/\log b\big) = \frac{c - \log c - \log M' + \log\log b - \log b}{\log b} > 0,$$

where the last inequality is obtained from (2.2). This shows that $h(y) > 0$ for all $y \geq M$. In particular, $h(x) > 0$, and so $1 + \log x/\log b < x/M'$. By (2.3), we obtain $f(x) < x$, as required. ☐

Theorem 2.4. *If $f_1, f_2, \ldots, f_k : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ satisfy the condition* (A), then $f_1 \circ f_2 \circ \cdots \circ f_k$ also satisfies (A).

Proof. We can prove this by induction on k and it is actually the same as that given by Noppakeaw et al. [10, Theorem 1.3], but for completeness, we give the proof again here. When $k = 1$, the result is obvious. Assume that $k \in \mathbb{N}$ and the result holds for k. Suppose $f_1, f_2, \ldots, f_{k+1} : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$

satisfy (A). Let $f = f_1 \circ f_2 \circ \cdots \circ f_{k+1}$ and $g = f_1 \circ f_2 \circ \cdots \circ f_k$. Then there are $m_1, m_2 \in \mathbb{N}$ such that

$$g(x) < x \quad \text{for all } x \geq m_1, \quad \text{and} \quad f_{k+1}(x) < x \quad \text{for all } x \geq m_2. \tag{2.4}$$

Let $m_3 = \max\{g(x) \mid 1 \leq x < m_1\}$ and $m = \max\{m_1, m_2, m_3\} + 1$. Let $x \geq m$. We will show that $f(x) < x$. If $f_{k+1}(x) \geq m_1$, then we obtain by (2.4) that

$$f(x) = g(f_{k+1}(x)) < f_{k+1}(x) < x.$$

On the other hand, if $f_{k+1}(x) < m_1$, then $f(x) = g(f_{k+1}(x)) \leq m_3 < m \leq x$. This completes the proof. □

Corollary 2.5. *A composition of digit maps satisfies the condition* (A).

Proof. This follows immediately from Theorems 2.3 and 2.4. □

Theorem 2.6. *If* $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ *satisfies* (A), then f satisfies (B).

Proof. This is given in [10, Theorem 1.2] for a function $f : \mathbb{N} \to \mathbb{N}$, and we can use the same method in our proof too. However, directly applying [10, Theorem 1.2] does not lead to our desired result for $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$, so we still used to give the proof here. For convenience, we write N instead of N_f and we assert that

$$\text{for every } y \in \mathbb{N} \cup \{0\}, \text{ there exists } n \in \mathbb{N} \cup \{0\} \text{ such that } f^{(n)}(y) < N. \tag{2.5}$$

If $y < N$, then we can choose $n = 0$. If $y \geq N$, then by (A), $f(y) < y$. If $f(y) < N$, then we can choose $n = 1$; otherwise, we obtain by (A) that $f^{(2)}(y) < f(y)$. We can repeat this process and obtain a strictly decreasing sequence of positive integers $f(y), f^{(2)}(y), f^{(3)}(y), \ldots$, and eventually $f^{(n)}(y) < N$ for some n. Hence (2.5) is proved.

Now let $x \in \mathbb{N} \cup \{0\}$ and suppose that $(f^{(n)}(x))_{n \geq 0}$ does not converge to a fixed point of f. By (2.5), there exists $n_1 \in \mathbb{N}$ such that $f^{(n_1)}(x) < N$. Again by (2.5), there exists $n_2 \in \mathbb{N}$ such that $f^{(n_2)}(f^{(n_1)}(x)) < N$. Repeating this process $N + 1$ times, we obtain the set of nonnegative integers

$$f^{(n_1)}(x), f^{(n_1+n_2)}(x), \ldots, f^{(n_1+n_2+\cdots+n_{N+1})}(x),$$

which are less than N. By the pigeonhole principle, some of them are the same, say

$$f^{(n_1+n_2+\cdots+n_j)}(x) = f^{(n_1+n_2+\cdots+n_j+\cdots+n_\ell)}(x) \quad \text{for some } \ell > j \geq 1.$$

Let $y = f^{(n_1+n_2+\cdots+n_j)}(x)$. Then the tail of the sequence $(f^{(n)}(x))_{n\geq 0}$ eventually becomes

$$(y, f(y), f^{(2)}(y), \ldots, f^{(n_{j+1}+n_{j+2}+\cdots+n_\ell-1)}(y), y, \ldots),$$

which is a cycle. This proves the first part of (B). Next we show that the set U_f of fixed points and cycles is finite. More precisely, we will show that

$$U_f := \{x \in \mathbb{N} \cup \{0\} \mid \exists n \in \mathbb{N}, f^{(n)}(x) = x\} \subseteq [0, M], \tag{2.6}$$

where $M = \max\{N, f(0), f(1), f(2), \ldots, f(N)\}$. First of all, by (A), if x is a fixed point of f, then $x < N$ and so $x \in [0, M]$. Suppose that x is an element in a cycle arising from the iteration $(f^{(n)}(y))_{n\geq 0}$ for some $y \in \mathbb{N} \cup \{0\}$. If $x < N$, then $x \in [0, M]$ and we are done. So suppose $x \geq N$. By (2.5), there exists $n \in \mathbb{N}$ such that $f^{(n)}(x) < N$. Since x is in a cycle, after some iterations, it must come back to x. That is, there exists $k \in \mathbb{N}$ such that $f^{(k)}(f^{(n)}(x)) = x$. If $k = 1$ or $f^{(n+k-1)}(x) \leq N$, then $x = f(f^{(n+k-1)}(x)) \leq M$ and we are done. So suppose $k \geq 2$ and $f^{(n+k-1)}(x) > N$. Let ℓ be the smallest positive integer such that $f^{(n+k-\ell)}(x) < N$. Then $1 < \ell \leq k$ and for each $1 \leq i < \ell$, $f^{(n+k-i)}(x) \geq N$. So

$$f^{(n+k-\ell+1)}(x) > f^{(n+k-\ell+2)}(x) > \cdots > f^{(n+k-1)}(x) > f^{(n+k)}(x) = x.$$

So $x < f^{(n+k-\ell+1)}(x) = f(f^{(n+k-\ell)}(x)) \leq M$. Therefore (2.6) is verified and the proof is complete. □

Corollary 2.7. *Let $f_1, f_2, \ldots, f_k$ be digit maps with respect to $b_1, b_2, \ldots, b_k$ respectively, where $b_i \geq 2$ for every i. Let $F : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ be given by $F = f_1 \circ f_2 \circ \cdots \circ f_k$. Then F satisfies* (B).

Proof. This follows immediately from Corollary 2.5 and Theorem 2.6. □

Suppose that $f_1, f_2, \ldots, f_k$ are digit maps with respect to bases b_1, b_2, …, b_k, respectively, and $F = f_1 \circ f_2 \circ \cdots \circ f_k$. By Corollary 2.5, there is $N \in \mathbb{N}$ such that $F(x) < x$ for all $x \geq N$. Then all fixed points and cycles can be found by considering the sequence $(F^{(n)}(x))_{n\geq 0}$ where $0 \leq x < N$. We show some explicit calculations for such N in the following example.

Example 2.8. *Consider $F = f_1 \circ f_2 \circ f_3$, where f_1, f_2, f_3 are digit maps with respect to $b_1 = 6$, $b_2 = 5$, $b_3 = 4$, respectively, and for $0 \leq x < b_i$, they are defined by $f_1(x) = x^4$, $f_2(x) = x^2$, $f_3(x) = x^3$.*

First, we show that there exists an integer m such that $(f_1 \circ f_2)(x) < x$ for all $x > m$. By following the proof of Theorem 2.3, we consider $M_1' =$

$\max\{f_1(i) \mid i = 0, 1, \ldots, 5\} = 5^4$ and $M_2' = \max\{f_2(i) \mid i = 0, 1, 2, 3, 4\} = 4^2$. Since $\frac{e^{10}}{10} > \frac{6(5)^4}{\log 6}$, $\frac{e^6}{6} > \frac{5(4)^2}{\log 5}$, we let $c_1 = 10$ and $c_2 = 6$. The corresponding M_1 for f_1 and M_2 for f_2 are $M_1 = 3489$ and $M_2 = 60$. Therefore

$$f_1(x) < x \quad \text{for all } x \geq 3489 \quad \text{and} \quad f_2(x) < x \quad \text{for all } x \geq 60.$$

Let $m_1 = 3489$, $m_2 = 60$, and $m_3 = \max\{f_1(x) \mid 1 \leq x < 3489\}$. Since $3489 = (24053)_6$, we see that $m_3 = f_1((15555)_6) = 2501$.

Let $m = \max\{m_1, m_2, m_3\} + 1 = 3490$. By the proof of Theorem 2.4, we have that

$$\big(f_1 \circ f_2\big)(x) < x \quad \text{for all } x \geq 3490.$$

Next, we consider $f_1 \circ f_2 \circ f_3$. Similarly, $M_3' = 3^3$ and $\frac{e^7}{7} > \frac{4(3)^3}{\log 4}$, so we let $c_3 = 7$, and $M_3 = \lceil \frac{7(3)^3}{\log 4} \rceil = 137$ and obtain

$$f_3(x) < x \quad \text{for all } x \geq 137.$$

We let $m_1 = 3490$, $m_2 = 137$, $m_3 = \max\{(f_1 \circ f_2)(x) \mid 1 \leq x < 3490\}$. Then

$$\begin{aligned} m_3 &= \max\{f_1(f_2(x)) \mid 1 \leq x < (102430)_5\} = \max\{f_1(x) \mid 1 \leq x < 80\} \\ &= \max\{f_1(x) \mid 1 \leq x < (212)_6\} = 1251. \end{aligned}$$

Let $m = \max\{m_1, m_2, m_3\} + 1 = 3491$. So

$$\big(f_1 \circ f_2 \circ f_3\big)(x) < x \quad \text{for all } x \geq 3491.$$

The lower bound 3491 may not be best possible but it is not difficult to search for the best one by using a computer. We can check whether $(f_1 \circ f_2 \circ f_3)(x) < x$ for $x = 1, 2, 3, \ldots, 3490$. If $(f_1 \circ f_2 \circ f_3)(x) < x$ for $x = N, N+1, \ldots, 3490$ and $(f_1 \circ f_2 \circ f_3)(x) \geq x$ for $x = N - 1$, then such the integer N is the optimal lower bound. In fact, by using a computer, we obtain $N = 831$.

We give two more examples to illustrate alternative calculations.

Example 2.9. *Let $b_1 = b_2 = 10$ and let f_1, f_2 be digit maps with respect to b_1, b_2 such that $f_1(x) = 2x^2 - x$ and $f_2(x) = 3x^3 - x^2 - x$ for $0 \leq x < 10$. Let $F = f_1 \circ f_2$. Then, for each $x \in \mathbb{N}$, the sequence $(F^{(n)}(x))_{n \geq 0}$ either converges to 1 or eventually becomes the cycle $(6, 132, 240, 154, 166, 23, 211)$.*

Proof. We first show that

$$F(x) = (f_1 \circ f_2)(x) < x \quad \text{for all } x \geq 10930. \tag{2.7}$$

f_1	f_2	f_3	$\underline{b}$	Fixed points of F or cycles in $(F^{(n)}(x))_{n\geq 0}$
$3x^3-x^2-x$	$2x^2-x$		(10,10)	1, (606,88,190,518,1213,87,20)
$2x^2-x$	$3x^3-x^2-x$		(10,10)	1,(6,132,240,154,166,23,211)
$2x^2-x$	$3x^3-x^2-x$		(7,5)	1,6,43,(56,16,82,112),(61,111,35,15)
$2x^2-x$	$3x^3-x^2-x$	$3x^4+2x^2-4x$	(4,5,7)	1,7,53
$2x^2-x$	$3x^3-x^2-x$	$3x^4+2x^2-4x$	(5,4,7)	1
$\lfloor e^x\rfloor-1$			8	1,(1114,32,53,549,201,21,153,26,25, 20,59),1103,(462,1498,1126)
$\lfloor e^x\rfloor-1$	x^2		(8,10)	1,(59,154,153,72,549,1102,402)

Table 1: Fixed points of F (except zero) or cycles in $(F^{(n)}(x))_{n\geq 0}$

If $x\in[10930,99999]$, then $x=(a_4a_3a_2a_1a_0)_{10}$ where $0\leq a_i\leq 9$, and so

$$f_2(x)=f_2(a_4)+f_2(a_3)+\cdots+f_2(a_0)\leq 5f_2(9)=10485,$$

and thus

$$F(x)\leq\max\{f_1(x)\mid 1\leq x\leq 10485\}=f_1(9999)=4(153)=612<x.$$

Next, suppose that $x\geq 10^5$ and write $x=(a_ka_{k-1}\cdots a_1a_0)_{10}$ where $k\geq 5$ and $a_k\neq 0$.

It is easy to prove by induction on k that $2097(k+1)<10^k$ for all $k\geq 5$. Then,

$$f_2(a_k)+f_2(a_{k-1})+\cdots+f_2(a_0)\leq(k+1)f_2(9)=2097(k+1)<10^k.$$

Then, $F(x)\leq\max\{f_1(x)\mid 0\leq x\leq 10^k\}=f_1(\underbrace{99\cdots 9}_{k\text{ digits}})=153k<10^k\leq a_k10^k\leq x$. So (2.7) is verified. It only remains to check that, for each $x<10930$, whether the sequence $(F^{(n)}(x))_{n\geq 0}$ converges to a fixed point or becomes a cycle. This can be done using a computer. We find that for each positive integer $x<10930$, the sequence $(F^{(n)}(x))_{n\geq 0}$ converges to 1 or becomes the cycle $(6,132,240,154,166,23,211)$. □

The next example is slightly different from Example 2.9 because b_1 and b_2 are different.

Example 2.10. *Let $b_1=7$, $b_2=5$, f_1 and f_2 digit maps with respect to b_1 and b_2, respectively, $f_1(x)=2x^2-x$ for $0\leq x\leq 6$, and $f_2(x)=3x^3-2x$ for $0\leq x\leq 4$. Let $F=f_1\circ f_2$. Then, for each $x\in\mathbb{N}$, the sequence $(F^{(n)}(x))_{n\geq 0}$ contains 1, 6, 43, 56, or 61. Moreover, 1, 6, and 43 are the only fixed points of F and if the sequence $(F^{(n)}(x))_{n\geq 0}$ does not contain 1, 6, or 43, then it eventually enters into the cycles $(56,16,82,112)$ or $(61,111,35,15)$.*

Proof. We first show that $F(x) < x$ for all $x \geq 1030$. Let $x \geq 1030$. Since $x > 5^4$, we write $x = (a_k a_{k-1} \cdots a_0)_5$, where $k \geq 4$, $0 \leq a_i \leq 4$ for every i, and $a_k \neq 0$. Then $f_2(x) \leq (k+1)f_2(4) = 184(k+1)$ and it is easy to prove by induction on k that $184(k+1) < 7^k$ for all $k \geq 4$. Then

$$F(x) \leq \max\{f_1(x) \mid 0 \leq x < 7^k\} = f_1((\underbrace{66\cdots 6}_{k \text{ digits}})_7) = 66k.$$

Since $5^k \leq a_k 5^k \leq x$, it follows that $k \leq \frac{\log x}{\log 5}$. Since the function $y \to \frac{\log y}{y}$ is decreasing on $[3, \infty)$ and $x \geq 1030$, we obtain

$$F(x) \leq 66k \leq 66\Big(\frac{\log x}{\log 5}\Big) \leq \frac{66}{\log 5}\Big(\frac{\log x}{x}\Big)x \leq \frac{66}{\log 5}\Big(\frac{\log 1030}{1030}\Big)x < x.$$

Similar to Example 2.9, the rest can be verified using a computer. $\square$

Other examples of compositions of digit maps and their fixed points and cycles are shown in Table 1. For instance, Line 5 of Table 1 means that if f_1, f_2, f_3 are digit maps such that $f_1(x) = 2x^2 - x$ for $0 \leq x \leq 3$, $f_2(x) = 3x^3 - x^2 - x$ for $0 \leq x \leq 4$, and $f_3(x) = 3x^4 + 2x^2 - 4x$ for $0 \leq x \leq 6$, then the fixed points of $F = f_1 \circ f_2 \circ f_3$ are 1, 7, and 53, and for any $x \in \mathbb{N}$, $(F^{(n)}(x))_{n\geq 0}$ converges to 1, 7, or 53. Note that zero is also a fixed point of F but we are not interested in this fixed point since in our example $(F^{(n)}(x))_{n\geq 0}$ does not converge to zero for any $x \in \mathbb{N}$.

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VITA

NAME	Kittipong Subwattanachai
DATE OF BIRTH	3 February 1996
PLACE OF BIRTH	Kanchanaburi
INSTITUTIONS ATTENDED	Bachelor of Science in Mathematics, Silpakorn University in 2017.
HOME ADDRESS	40/18 Swine 2, Wai Neaw Sub-district, Thamaka District, Kanchanaburi, 71120
PUBLICATION	P. Pongsriiam and K. Subwattanachai, Exact formulas for the number of palindromes up to a given positive integer, International Journal of Mathematics and Computer Science, 14(1) (2019), pp. 27-46.