SOME MATCHING PROPERTIES IN COMPLEMENTARY PRISM OF GRAPHS

By

Sqn.Ldr. Pongthep Janseana

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of
Doctor of Philosophy Program in Mathematics
Department of Mathematics
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สมบัติการจับคู่บางประการในกราฟปริซึมเติมเต็ม

โดย

นาวาอากาศตรี พงค์เทพ จันทน์เสนะ

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปรัชญาดุษฎีบัณฑิต

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Let $\overline{G}$ denote the complement of a simple graph G. The complementary prism of G denoted by $G\overline{G}$ can be obtained by taking a copy of G and a copy of $\overline{G}$ and then joining corresponding vertices by an edge. A connected graph G of order at least $2k+2$ is k-extendable if for every matching M of size k in G, there is a perfect matching in G containing all edges of M.

In this thesis, we establish some sufficient conditions for the complementary prism of regular graphs to be 2-extendable. We also show that for positive integers l_1 and l_2, there exists a non-bipartite graph G such that G is l_1-extendable and $\overline{G}$ is l_2-extendable. Finally, we show that if G is l_1-extendable and $\overline{G}$ is l_2-extendable non-bipartite graphs for $l_1 \geq 2$ and $l_2 \geq 2$, then $G\overline{G}$ is $(l+1)$-extendable where $l = min\{l_1, l_2\}$.

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พงค์เทพ จันทน์เสนะ, นาวาอากาศตรี : สมบัติการจับคู่บางประการในกราฟปริซึมเติมเต็ม อาจารย์ที่ปรึกษาวิทยานิพนธ์ : รศ.ดร.นวรัตน์ อนันต์ชื่น. 30 หน้า.

ให้ $\overline{G}$ แทนกราฟเติมเต็มของกราฟ G กราฟปริซึมเติมเต็มของกราฟ G เขียนแทนด้วย $G\overline{G}$ สามารถสร้างได้จากกราฟ G และกราฟ $\overline{G}$ โดยเชื่อมจุดที่สมนัยกันด้วยเส้นเชื่อม จะกล่าวว่ากราฟเชื่อมโยง G ที่มีอันดับอย่างน้อย $2k+2$ มีการขยายการจับคู่ขนาด k ถ้าทุก ๆ การจับคู่ M ขนาด k ในกราฟ G มีการจับคู่สมบูรณ์ใน G ที่มี M เป็นสับเซต

ในวิทยานิพนธ์นี้ เราศึกษาเงื่อนไขเพียงพอบางประการที่ทำให้กราฟปริซึมเติมเต็มของกราฟปรกติมีการขยายการจับคู่ขนาด 2 เรายังแสดงว่าสำหรับจำนวนเต็ม l_1 และ l_2 จะมีกราฟ G ที่ไม่ใช่กราฟสองส่วนซึ่ง G มีการขยายการจับคู่ขนาด l_1 และ $\overline{G}$ มีการขยายการจับคู่ขนาด l_2 และท้ายสุด เราแสดงว่า ถ้า G มีการขยายการจับคู่ขนาด l_1 และ $\overline{G}$ มีการขยายการจับคู่ขนาด l_2 โดยที่ G และ $\overline{G}$ ไม่ใช่กราฟสองส่วน สำหรับ $l_1 \geq 2$ และ $l_2 \geq 2$ แล้ว $G\overline{G}$ มีการขยายการจับคู่ขนาด $l+1$ เมื่อ $l = min\{l_1, l_2\}$

ภาควิชาคณิตศาสตร์ บัณฑิตวิทยาลัย มหาวิทยาลัยศิลปากร

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ลายมือชื่ออาจารย์ที่ปรึกษาวิทยานิพนธ์ ....

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Chapter 1

Introduction

In this chapter, we introduce some definitions and notations used in this thesis. Most of them follows that of Bondy and Murty ([3]).

A **graph** is a triple $G = (V(G), E(G), \omega_G)$, where $V(G)$ is a finite set of **vertices**, $E(G)$ is a set of **edges** and an incidence function ω_G that associates with each edge of G and an unordered pair of vertices of G. If e is an edge and x and y are vertices such that $\omega_G(e) = \{x, y\}$, then e is said to be **incident** to x and y. Further, the vertices x and y are called **end** vertices of e and we say that x and y are **adjacent**. The **order** of G is the cardinality of $V(G)$. Two or more edges that join the same pair of vertices are called **parallel** edges. An edge that joins itself is a **loop**. A graph G is **simple** if G has no loops and parallel edges. If G is simple and $\omega_G(e) = \{x, y\}$, then we simply denote e by xy.

A **complete** graph is a simple graph in which every pair of vertices are adjacent. A complete graph of order n is denoted by K_n. The **complement** $\overline{G}$ of a graph G is that graph with $V(\overline{G}) = V(G)$ and $xy \in E(\overline{G})$ if and only if $xy \notin E(G)$. A graph H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. H is called an **induced subgraph** of G, denoted by $G[H]$, if, for every pair of $x, y \in V(H)$, $xy \in E(H)$ if and only if $xy \in E(G)$. For graphs H and G, G is called H**-free** if G does not contain H as an induced subgraph. A subset of vertices $S \subseteq V(G)$ is called a **clique** if $G[S] \cong K_r$, for some r. A **bipartite graph** is a graph whose vertices can be divided into two disjoint sets X and Y such that every edge connects a vertex in X to a vertex in Y.

The **neighbor set** of a vertex v in G, denoted by $N_G(v)$, is defined by $\{u \in V(G) | uv \in E(G)\}$. For $v \in V(G)$ and $T \subseteq V(G)$, a neighbor set of a vertex v in T is denoted by $N_T(v) = \{u \in T | uv \in E(G)\}$ and if $X \subseteq V(G)$, $N_G(X)$ denotes $\bigcup_{v \in X} N_G(v)$. Observe that $N_T(v) = N_G(v) \cap T$. The **degree** of a vertex u in G is denoted by $deg_G(u) = |N_G(u)|$. The **minimum degree** and **maximum degree** in a graph G is denoted by $\delta(G)$ and $\Delta(G)$, respectively. A **regular graph** is a graph which each vertex has the same degree and a k**-regular graph** is a regular graph with degree of a vertex is k.

A **walk** in a graph G is a finite, non-empty alternating sequence $W = v_0 e_1 v_1 e_2 ... e_n v_n$ of vertices and edges such that for $1 \leq i \leq n$, the ends of edge e_i are v_{i-1} and v_i. W is said to be a walk from v_o to v_n. A **path** is a walk with distinct vertices. Two vertices x and y of G are connected if there is a path from

x to y. The **distance** between two vertices x, y in G, denoted by $d_G(x, y)$, is the length of a shortest xy-path in G. A graph G is **connected** if every pair of vertices of G are connected otherwise G is **disconnected**. A maximal connected subgraph of G is called a **component** of G. A graph G is called k**-connected** if removing less than k vertices from G, the resulting graph is connected. An odd(even) component is a component of odd (even) order. The **number of odd components** of G is denoted by $c_o(G)$.

A set $S \subseteq V(G)$ is called an **independent vertex** set if no two vertices of S are adjacent. The maximum cardinality of an independent set of G is denoted by $\alpha(G)$. A subset M of $E(G)$ is called a **matching** if no two edges of M have common end vertex. A vertex u is **saturated** by M if there is an edge in M incident with u. For simplicity, a set of all vertices saturated by M is denoted by $V(M)$. M is called a **maximum matching** in G if G contains no matching of size greater than $|M|$. A **perfect matching** in G is a matching that saturates all vertices of G. If M_1 and M_2 are matching in a graph G, then a **symmetric different** of M_1 and M_2, denoted by $M_1 \triangle M_2$, is an induced subgraph $G[(M_1 - M_2) \cup (M_2 - M_1)]$.

A set $S \subseteq V(G)$ is a **dominating set** of G, if $N_G(S) \cup S = V(G)$ and is a **total dominating set** if $N_G(S) = V(G)$. The **domination number** of G, denoted by $\gamma(G)$, (respectively, **total domination number** of G, denoted by $\gamma_t(G)$) is the number of vertices in a smallest dominating set (respectively, total dominating set) of G. A set $S \subseteq V(G)$ is an **independent dominating set** of G, if S is a dominating set and S is independent. The **independent domination number** of G, denoted by $\gamma_i(G)$, is the number of vertices in a smallest independent dominating set of G. A set $S \subseteq V(G)$ is a **connected dominating set** of G, if S is a dominating set and the induced subgraph $G[S]$ is connected. The **connected domination number** of G, denoted by $\gamma_c(G)$, is the number of vertices in a smallest connected dominating set of G. A set $D \subseteq V(G)$ is a **locating-dominating set** of G if for every $u \in V(G) - D$, its neigborhood $N_G(u) \cap D$ is non-empty and distinct from $N_G(v) \cap D$ for all $v \in V(G) - D$ where $v \neq u$. The **locating-domination number** of G, denoted by $\gamma_L(G)$, is the number of vertices in a smallest locating-dominating set of G. A set $D \subseteq V(G)$ is a **double dominating set** if D dominates every vertex of G twice or $|N_G(u) \cap D| \geq 2$ for all $u \in V(G)$. The **double domination number** of G, denoted by $\gamma_{\times 2}(G)$, is the number of vertices in a smallest double dominating set of G. A set $S \subseteq V(G)$ is a **restrained dominating set** of G, if for every vertex $v \in V(G) - S$, v is adjacent to a vertex in S and to a vertex in $V(G) - S$. The **restrained domination number** of G, denoted by $\gamma_r(G)$, is the number of vertices in a smallest restrained dominating set of G.

For a positive integer k, a connected graph G of order at least $2k+2$ is k**-extendable** if for every matching M of size k in G, there is a perfect matching in G containing all edges of M. A graph G is k**-factor-critical** if, for every set $S \subseteq V(G)$ with $|S| = k$, the graph $G - S$ contains a perfect matching. For $k = 1$ and $k = 2$, k-factor-critical graph is also called **factor-critical** and **bicritical**, respectively. For simplicity, a graph with a perfect matching is called 0-extendable and 0-factor-critical. Observe that if G is k-extendable, then $|V(G)|$ is even and if G is k-factor-critical, then $|V(G)| \equiv k \pmod 2$.

For graphs H_1 and H_2 with disjoint vertex sets V_1 and V_2, **the join** of H_1 and H_2, denoted by $H_1 + H_2$ is the graph with vertex set $V_1 \cup V_2$ and edge set $E(H_1) \cup E(H_2) \cup \{uv|u \in V_1 \text{ and } v \in V_2\}$. The **cartesian product** $G \times H$ of two graphs G and H has the vertex set $V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent whenever $u_1u_2 \in E(G)$ and $v_1 = v_2$, or $u_1 = u_2$ and $v_1v_2 \in E(H)$. The **lexicographic product** $G \circ H$ of two graphs G and H has the vertex set $V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent either $u_1u_2 \in E(G)$, or $u_1 = u_2$ and $v_1v_2 \in E(H)$.

The **complementary prism** of G, denoted by $G\overline{G}$, is the graph obtained by taking a copy of G and a copy of $\overline{G}$ and then joining corresponding vertices by an edge. The graph $C_5\overline{C}_5$ in Figure 1.1 is a complementary prism of C_5. Note that $C_5\overline{C}_5$ is isomorphic to the Petersen graph.

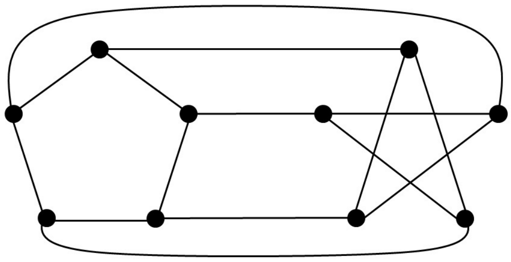

Figure 1.1: $C_5\overline{C}_5$

In this thesis, all graphs are simple and finite. Chapter 2 provides some basic background and preliminaries results on extendability and factor-criticality of graphs that we make use of in establishing our results. In Chapter 3, we establish a sufficient condition for the complementary prism of regular graphs to be 2-extendable. Chapter 4 provides some constructions of a graph G such that G and $\overline{G}$ are l_1-extendable and l_2-extendable non-bipartite, respectively, where l_1 and l_2 are positive integers. We then establish that if G and $\overline{G}$ are l_1-extendable and l_2-extendable non-bipartite graphs, respectively for $l_1 \geq 2$ and $l_2 \geq 2$, then $G\overline{G}$ is $(l+1)$-extendable where $l = min\{l_1, l_2\}$.

Chapter 2

Literature Review

In this chapter, we provide some background and preliminaries related to our work. In 1980, Plummer [16] introduced the concept of matching extension and established a fundamental theorem on k-extendable graphs (see Theorem 2.2). Since then it has been well studied, see surveys by Plummer [18, 19, 20] and a book by Yu and Liu [26]. One of main topics in studying matching extension is to establish some sufficient conditions for a graph to be k-extendable. These conditions include degree sum[15], minimum degree [16], forbidden subgraph [17], genus of graph [21], etc. Moreover there are some results concerning the extendability of a graph obtained from a product of two graphs such as cartesian product [10], lexicographic product [2] and strong product [9]. A reader is directed to references in Bibliography([18], [19], [20] and [26]) for more detailed. We shall provide only some results that used of in our work.

Our first result is a well known theorem for studying an existence of a perfect matching in graphs established by Tutte.

Theorem 2.1. *[3] (Tutte's Theorem) A graph G has a perfect matching if and only if for any $S \subseteq V(G)$, $c_o(G - S) \leq |S|$.* □

In 1980, Plummer [16] established a fundamental theorem on k-extendable graphs as following.

Theorem 2.2. *[16] Let G be a graph of order $p \geq 2k + 2$ and $k \geq 1$. If G is k-extendable, then*

(a) G is $(k - 1)$-extendable, and

(b) G is $(k + 1)$-connected. □

He also gave a sufficient condition for a graph to be k-extendable in terms of minimum degree.

Theorem 2.3. *[16] Let G be a graph of order $2p$. If $\delta(G) \geq p + k$, for a non-negative integer k, then G is k-extendable.* □

Ananchuen and Caccetta [1] gave a necessary condition for a neighbor set of a vertex having minimum degree in extendable graphs. They showed that:

Theorem 2.4. *[1] If G is a k-extendable graph on $p \geq 2k+2$ vertices with $\delta(G) = k+t$, $1 \leq t \leq k \leq p$. If $d_G(u) = \delta(G)$, then the induced subgraph $G[N_G(u)]$ has at most $t-1$ independent edges.* □

A neccessary and sufficient condition for a graph to be k-extendable and to be k-factor-critical were provided by Yu [24] and Favaron [7], respectively.

Theorem 2.5. *[24] A graph G is k-extendable ($k \geq 1$) if and only if for any $S \subseteq V(G)$,*

(a) $c_o(G-S) \leq |S|$ and

(b) $c_o(G-S) = |S|-2t, (0 \leq t \leq k-1)$ implies that $F(S) \leq t$, where $F(S)$ is the size of a maximum matching in $G[S]$. □

Theorem 2.6. *[7] A graph G is k-factor-critical if and only if $|V(G)| \equiv k \pmod 2$ and for $S \subseteq V(G)$ with $|S| \geq k, c_o(G-S) \leq |S|-k$.* □

Some following properties of k-factor-critical graphs were proved in [7].

Theorem 2.7. *[7] Let G be a k-factor-critical graph. Then G is $(k-2)$-factor-critical.* □

Theorem 2.8. *[7] If G is a $2k$-extendable non-bipartite graph for $2k \geq 2$, then G is a $2k$-factor-critical graph.* □

Maschlanka and Volkmann [14] gave a relationship between k-extendable non-bipartite graph and the independence number.

Theorem 2.9. *[14] Let G be a k-extendable non-bipartite graph of order p. Then $\alpha(G) \leq \frac{1}{2}p - k$.* □

In Phd. Thesis of Yu [25], he gave the following observation.

Observation 2.10. *A graph G is k-extendable if and only if for any matching M of size i $(1 \leq i \leq k)$, $G - V(M)$ is a $(k-i)$-extendable graph.* □

An observation on k-factor-critical graphs which is similar to Observation 2.10 can be stated as following.

Observation 2.11. *Let G be a k-factor-critical graph and $S \subseteq V(G)$ where $|S| \leq k$. Then $G-S$ is $(k-|S|)$-factor-critical.* □

A following lemma follows from Theorem 2.9.

Lemma 2.12. *Let G be a k-extendable non-bipartite graph and $S \subseteq V(G)$ where $|S| \leq 2k-2$. Then $G-S$ is a non-bipartite graph.*

Proof. Suppose to the contrary that $G-S$ is a bipartite graph. Then $\alpha(G) \geq \alpha(G-V(S)) \geq \frac{1}{2}(|V(G)|-(2k-2)) = \frac{1}{2}|V(G)|-k+1$. But this contradicts Theorem 2.9 and completes the proof of our lemma. □

Our next corollary follows immediately by Observation 2.10 and Lemma 2.12

Corollary 2.13. *Let G be a k-extendable non-bipartite graph and let $M \subseteq E(G)$ where $|M| = l \leq k-1$. Then $G - V(M)$ is $(k-l)$-extendable non-bipartite.* □

Note that the upper bound on $|M|$ in Corollary 2.13 is best possible. Let $G = K_{2k} + K_{t,t}$ for some positive integers $k, t \geq 2$. It is easy to see that G is k-extendable. Clearly, there is a matching M of size k in $G[K_{2k}]$ such that $G - V(M)$ is a bipartite graph.

The following results concern the extendability of graphs obtained from a cartesian product, established by Györi and Plummer [10], Liu and Yu [13] and Wu et al. [23] and lexicographic product established by Bai et al. [2].

Theorem 2.14. *[10, 13] If G is a k-extendable graph, then $G \times K_2$ is $(k+1)$-extendable.* □

Theorem 2.15. *[13] If G is a k-extendable graph and H is a connected graph, then $G \times H$ is $(k+1)$-extendable.* □

Theorem 2.16. *[10] For non-negative integers l_1 and l_2, let G_i be a l_i -extendable graph for $1 \leq i \leq 2$. Then $G_1 \times G_2$ is $(l_1 + l_2 + 1)$-extendable.* □

Theorem 2.17. *[23] Let G_1 be an m-factor-critical graph and G_2 an n-factor-critical graph. Then $G_1 \times G_2$ is $(m+n+\epsilon)$-factor-critical, when $\epsilon = 0$, if both m and n are even; $\epsilon = 1$, otherwise.* □

Theorem 2.18. *[2] For non-negative integers l_1 and l_2, let G_i be a l_i -extendable graph for $1 \leq i \leq 2$. Then $G_1 \circ G_2$ is $2(l_1+1)(l_2+1)$-factor-critical. In particular, $G_1 \circ G_2$ is $(l_1+1)(l_2+1)$-extendable.* □

We now turn our attention to complementary prism of graphs. A complementary prism is a specific case of complementary product of graphs introduced by Haynes et al.[5] in 2007. Haynes et al. [5, 6] studied some parameters of complementary prism of graphs such as the vertex independence number, the chromatic number and the domination number. Some of them are stated in the following theorems.

Theorem 2.19. *[6] For any graph G, $\alpha(G) + \alpha(\overline{G}) - 1 \leq \alpha(G\overline{G}) \leq \alpha(G) + \alpha(\overline{G})$, and both these bounds are sharp.* □

Theorem 2.20. *[6] For any graph G, $max\{\gamma(G), \gamma(\overline{G})\} \leq \gamma(G\overline{G}) \leq \gamma(G) + \gamma(\overline{G})$.* □

The bound on various domination number of complementary prism of graphs have been studied in Desormeaux [4], Haynes et al. [6], Holmes [11], Góngara, Desormeaux [8] and Vaughan [22]. These results are stated in the next six theorems.

Theorem 2.21. *[4] For any graph G, $max\{\gamma(G), \gamma(\overline{G})\} \leq \gamma_r(G\overline{G}) \leq \gamma_r(G) + \gamma_r(\overline{G})$ and these bounds are sharp.* □

Theorem 2.22. *[6] If G and $\overline{G}$ have no isolated vertices, then $max\{\gamma_t(G), \gamma_t(\overline{G})\} \leq \gamma_t(G\overline{G}) \leq \gamma_t(G) + \gamma_t(\overline{G})$.* □

Theorem 2.23. *[8] For any graph* G, $max\{\gamma_i(G), \gamma_i(\overline{G})\} \leq \gamma_i(G\overline{G}) \leq 2(n-1) - max\{\Delta(G), \Delta(\overline{G})\}$. $\square$

Theorem 2.24. *[11] For any graph* G, $max\{\gamma_L(G), \gamma_L(\overline{G})\} \leq \gamma_L(G\overline{G}) \leq \gamma_L(G) + \gamma_L(\overline{G}) + 1$. $\square$

Theorem 2.25. *[11] For any graph* G, $max\{\gamma(G), \gamma(\overline{G})\} \leq \gamma_c(G\overline{G}) \leq \gamma_c(G) + \gamma_c(\overline{G}) + 1$. $\square$

Theorem 2.26. *[22] For any graph* G *with no isolated vertices,* $max\{\gamma_{\times 2}(G), \gamma_{\times 2}(\overline{G})\} \leq \gamma_{\times 2}(G\overline{G}) \leq \gamma_{\times 2}(G) + \gamma_{\times 2}(\overline{G})$. $\square$

We now conclude this chapter by pointing out that matching extension in complementary prism of graphs has been studied recently. The only known results are the last two theorems established by Janseana et al. [12], in 2014.

Theorem 2.27. *[12] For positive integers* l *and* i *where* $1 \leq i \leq l$, *let* $G_1, \ldots, G_l$ *be components of* G. *If* $G_i\overline{G_i}$ *is* k*-extendable of order* $p_i \geq 2k+2$ *for some positive integer* k, *then* $G\overline{G}$ *is* k*-extendable.* $\square$

Theorem 2.28. *[12] Let* G *be a 2-regular* H*-free graph where* $H \in \{C_3, C_4, C_5\}$, *then* $G\overline{G}$ *is 2-extendable.* $\square$

Chapter 3

Matching extension in complementary prism of regular graphs

We begin this chapter by establishing some lemmas concerning complementary prism of graphs and of regular graphs. These results are essential for establishing Theorem 3.10, a main result of this chapter. To simplify our discussion of complementary prisms, G and $\overline{G}$ are referred to subgraph copies of G and $\overline{G}$, respectively, in $G\overline{G}$. For a vertex v of G, there is exactly one vertex of $\overline{G}$ which is adjacent to v in $G\overline{G}$. This vertex is denoted by $\overline{v}$. That is, $\{\overline{v}\} = N_{G\overline{G}}(v) \cap V(\overline{G})$. Conversely, v is the only vertex of G which is adjacent to $\overline{v}$. Similarly, for $\phi \neq X = \{x_1, x_2, \ldots, x_k\} \subseteq V(G)$, $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k\} \subseteq V(\overline{G})$ is denoted by $\overline{X}$ and vice versa. Clearly, $|X| = |\overline{X}|$.

Lemma 3.1. *Let G be a graph. Then $G\overline{G}$ is even and connected.*

Proof. Clearly, $G\overline{G}$ is even. Let $u, v \in V(G\overline{G})$. It is easy to see that if $u, v \in V(G)(V(\overline{G}))$, then either $uv \in E(G)$ or $u\bar{u}\bar{v}v$ is a $u - v$ path. We may now assume that $u \in V(G)$ and $v \in V(\overline{G})$. Clearly, $uv \in E(G\overline{G})$ if $v = \bar{u}$. So suppose that $v = \bar{w}$ for some $w \in V(G) - \{u\}$. Then either $u\bar{u}\bar{w}$ or $uw\bar{w}$ is a $u - v$ path. This proves that $G\overline{G}$ is connected and completes the proof of our lemma. $\square$

For a graph G, it is easy to see that $G\overline{G}$ has a perfect matching. It then follows by Theorem 2.1 that for a cutset $S \subseteq V(G\overline{G})$, $c_o(G\overline{G} - S) \leq |S|$. The next lemma provides a relationship of a cutset and the number of odd components in a complementary prism.

Lemma 3.2. *Let G be a graph and let $S = A \cup \overline{B}$ be a cutset of $G\overline{G}$, where $A \subseteq V(G)$ and $\overline{B} \subseteq V(\overline{G})$. Then*

a) $c_o(G\overline{G} - S) = |S| - 2t = |A| + |B| - 2t$, for some $t \geq 0$.

b) $c_o(G\overline{G} - S) \leq c_o(G[B - A]) + c_o(\overline{G}[\overline{A} - \overline{B}]) \leq |A| + |B| - 2|A \cap B|$. Consequently, $|A \cap B| \leq t$.

c) If $c_o(G[B - A]) + c_o(\overline{G}[\overline{A} - \overline{B}]) = |A| + |B| - 2|A \cap B|$, then each component of $G[B - A] \cup \overline{G}[\overline{A} - \overline{B}]$ is singleton. Consequently, $A - B$ is a clique.

Proof. a) Since $G\overline{G}$ contains a perfect matching and is of even order, it follows by Theorem 2.1 that there is a non-negative integer t such that $c_o(G\overline{G} - S) = |S| - 2t$, for any cutset $S \subseteq V(G\overline{G})$. Clearly, $|S| = |A| + |B|$. Thus $c(G\overline{G} - S) = |S| - 2t = |A| + |B| - 2t$ as required.

We first observe that $|B - A| + |\overline{A} - \overline{B}| = |B - A| + |A - B| = |A| + |B| - 2|A \cap B|$ since $|A| = |A - B| + |A \cap B|$ and $|B| = |B - A| + |A \cap B|$.

b) Let $C = V(G) - (A \cup B)$. It is easy to see that if $C = \phi$, then $c_o(G\overline{G} - S) = c_o(G[B - A]) + c_o(\overline{G}[\overline{A} - \overline{B}]) \leq |B - A| + |\overline{A} - \overline{B}| = |A| + |B| - 2|A \cap B|$. We now suppose that $C \neq \phi$. Then, by Lemma 3.1, $G\overline{G}[C \cup \overline{C}]$ is even and connected. Thus $c_o(G\overline{G} - S) \leq c_o(G\overline{G} - (S \cup C \cup \overline{C})) = c_o(G[B - A]) + c_o(\overline{G}[\overline{A} - \overline{B}]) \leq |B - A| + |\overline{A} - \overline{B}| = |A| + |B| - 2|A \cap B|$ as required.

c) follows by the fact that $|B - A| + |A - B| = |A| + |B| - 2|A \cap B|$. □

For an induced subgraph H of G, Com_H denotes the set of all components in H. If $X \subseteq V(G)$, then we use Com_X for $Com_{G[X]}$. For a cutset S of $G\overline{G}$, put $A = S \cap V(G)$, $\overline{B} = S \cap V(\overline{G})$ and $C = V(G) - (A \cup B)$. Thus $S = A \cup \overline{B}$. Further, let $T_{B-A} = \{F | F$ is an odd component of $G[B - A]$ and $N_G(u) - V(F) \subseteq A$ for all $u \in V(F)\}$. $T_{\overline{A}-\overline{B}} = \{F | F$ is an odd component of $\overline{G}[\overline{A} - \overline{B}]$ and $N_{\overline{G}}(\bar{u}) - V(F) \subseteq \overline{B}$ for all $\bar{u} \in V(F)\}$. Finally, let $L = L_G \cup L_{\overline{G}}$, where $L_G = \{F | F$ is an odd component in $G[B - A]$ and $N_{G\overline{G}}(V(F)) \cap C \neq \phi\}$ and $L_{\overline{G}} = \{F | F$ is an odd component in $\overline{G}[\overline{A} - \overline{B}]$ and $N_{G\overline{G}}(V(F)) \cap \overline{C} \neq \phi\}$. Note that if $C = \phi$, then $L = \phi$. Clearly, $T_{B-A} \cap L_G = \phi$ and $T_{\overline{A}-\overline{B}} \cap L_{\overline{G}} = \phi$. It is easy to see that, if G is connected and $G[B - A]$ contains only odd components, then $Com_{B-A} = T_{B-A} \cup L_G$. Similarly, if $\overline{G}$ is connected and $\overline{G}[\overline{A} - \overline{B}]$ contains only odd components, then $Com_{\overline{A}-\overline{B}} = T_{\overline{A}-\overline{B}} \cup L_{\overline{G}}$. In what follows, the symbols Com_H, S, A, $\overline{B}$, C, T_{B-A}, $T_{\overline{A}-\overline{B}}$, L, L_G and $L_{\overline{G}}$ are referred to these set up.

The next lemma follows from our set up.

Lemma 3.3. *Let G be an r-regular connected graph of order $p \geq 2r + 1$ and $G\overline{G}$ a complementary prism. If $|A| < r$, then T_{B-A} contains no singleton components. Similarly, if $|\overline{B}| < p - r - 1$, then $T_{\overline{A}-\overline{B}}$ contains no singleton components.* □

Lemma 3.4. *For $r \geq 3$, let G be a connected r-regular graph of order $p \geq 2r + 1$. Let $A, B, T_{B-A}, T_{\overline{A}-\overline{B}}$ be defined as above. Then*

a) If $G[A] = K_r$, then each component of T_{B-A} is of order at least 3.

b) If $|A \cap B| = 1$ and $G[A - B] \cong K_r$, then the number of singleton components in T_{B-A} is at most 1.

c) If $|A \cap B| = 1$ and $G[A - B] \cong K_{r-1}$, then the number of singleton components in T_{B-A} is at most 2.

Proof. a) It follows by the fact that G is connected r-regular of order $p \geq 2r + 1$.

b) Suppose to the contrary that T_{B-A} contains two singleton components, say F_1 and F_2 where $V(F_1) = \{y_1\}$ and $V(F_2) = \{y_2\}$. Because $|A \cap B| = 1$, y_1 and y_2 are adjacent to at least $r - 1$ vertices of $A - B$. Since $G[A - B] = K_r$ and $r \geq 3$, it follows that there exists a vertex of $A - B$, say y_3, such that $\{y_1, y_2\} \cup (A - B) \subseteq N_G(y_3)$. Thus $d_G(y_3) \geq r + 1$, a contradiction

c) By applying similar arguments as in the proof of (b), (c) follows. □

Let a be a real number, $\lfloor a \rfloor_e$ is denoted a greatest even integer less than or equal to a, that is, $\lfloor a \rfloor_e = 2\lfloor a/2 \rfloor$. Note that if a is an integer and $\lfloor a \rfloor_e = k$ then $a = k$ or $a = k + 1$.

Lemma 3.5. *Let G be a graph and $L = L_G \cup L_{\overline{G}}$ be defined as above. Then $c_o(G\overline{G} - S) = c_o(G[B - A]) + c_o(\overline{G}[\overline{A} - \overline{B}]) - \lfloor |L| \rfloor_e$. Consequently $c_o(G[B - A]) + c_o(\overline{G}[\overline{A} - \overline{B}]) - c_o(G\overline{G} - S) \leq |L| \leq c_o(G[B - A]) + c_o(\overline{G}[\overline{A} - \overline{B}]) - c_o(G\overline{G} - S) + 1$.*

Proof. If $C = \phi$, then $|L| = 0$ and thus $c_o(G\overline{G} - S) = c_o(G[B - A]) + c_o(\overline{G}[\overline{A} - \overline{B}])$ as required. We now suppose that $C \neq \phi$. By Lemmas 3.2(a) and (b), $c_o(G\overline{G} - S) \leq c_o(G[B - A]) + c_o(\overline{G}[\overline{A} - \overline{B}])$. By Lemma 3.1, $G\overline{G}[C \cup \overline{C}]$ is even and connected. So it must be contained in some component of $G\overline{G} - S$, say F. If $x \in V(F) - (C \cup \overline{C})$, then x is in some component of $G[B - A] \cup \overline{G}[\overline{A} - \overline{B}]$, say M. So $V(M) \subseteq V(F)$. If M is odd, then $M \in L$. Note that each odd component of L is a subgraph of F. Hence, $|V(F)|$ has the same parity with $|L|$ and $c_o(G\overline{G} - S) = c_o(G[B - A] \cup \overline{G}[\overline{A} - \overline{B}]) - |L| + \epsilon$, where $\epsilon = 1$ if $|L|$ is odd and $\epsilon = 0$ if $|L|$ is even. So $c_o(G\overline{G} - S) = c_o(G[B - A] \cup \overline{G}[\overline{A} - \overline{B}]) - \lfloor |L| \rfloor_e$. Thus $\lfloor |L| \rfloor_e = c_o(G[B - A] \cup \overline{G}[\overline{A} - \overline{B}]) - c_o(G\overline{G} - S)$. By properties of $\lfloor x \rfloor_e$, our result follows. This proves our lemma. □

Lemma 3.6. *If G is an r-regular graph of order $p \geq 2r + 1$, then $\overline{G}$ is connected.*

Proof. Note that $\overline{G}$ is $(p - r - 1)$-regular graph of order p. Suppose $\overline{G}$ is disconnected. Then each component must have order at least $p - r$. So $p \geq 2(p - r)$ and thus $p \leq 2r$, a contradiction. This proves our lemma. □

Lemma 3.7. *Let G be a connected r-regular graph of order $p \geq 2r + 1$. Let S be a cutset of $G\overline{G}$. Then $S \cap V(G) \neq \phi$ and $S \cap V(\overline{G}) \neq \phi$.*

Proof. By Lemma 3.6, $\overline{G}$ is connected. Hence, G and $\overline{G}$ are connected. Suppose without loss of generality that $S \cap V(G) = \phi$. So $S \subseteq V(\overline{G})$. Since $G = G\overline{G} - V(\overline{G})$ is connected and each vertex $\overline{u}$ of $V(\overline{G}) - S$ is adjacent to a vertex u in G, it follows that $G\overline{G} - S$ is connected, a contradiction. Hence, $S \cap V(G) \neq \phi$. By similar arguments, $S \cap V(\overline{G}) \neq \phi$. This proves our lemma. □

Theorem 3.8. *Let G be a connected r-regular graph of order $p \geq 2r + 1$, for some $r \geq 2$. Then $G\overline{G}$ is bicritical. Consequently, $G\overline{G}$ is 1-extendable.*

Proof. Suppose $G\overline{G}$ is not bicritical. By Theorem 2.6, there is a cutset $S \subseteq V(G\overline{G})$, where $|S| \geq 2$ such that $c_o(G\overline{G} - S) > |S| - 2$. It follows by Lemmas 3.2(a) that $c_o(G\overline{G} - S) = |S|$ for $|S| \geq 2$. Note that, by Lemma 3.7, $A = S \cap V(G)$ and $\overline{B} = S \cap V(\overline{G})$ are not empty. Thus $\overline{A}$ and B are not empty. By Lemma 3.2 (b), $A \cap B = \phi$ and thus $c_o(G[B - A]) + c_o(\overline{G}[\overline{A} - \overline{B}]) = c_o(G[B]) + c_o(\overline{G}[\overline{A}])) = c_o(G\overline{G} - S) = |S| = |B| + |\overline{A}|$. By Lemma 3.2(c), each component of $G[B]$ and $\overline{G}[\overline{A}]$ is singleton. Hence, $G[A] \cong K_{|A|}$. Since G is r-regular of order $p \geq 2r + 1$, $|A| \leq r + 1$. If $|A| = r + 1$, then $G[A] \cong K_{r+1}$ is a disconnected component in G, a contradiction. So $1 \leq |A| \leq r$. By Lemmas 3.3 and 3.4(a), no singleton component in $G[B]$ belongs to T_{B-A}. Since each component of $G[B]$ is singleton, $T_{B-A} = \phi$. Because $c_o(G\overline{G} - S) = c_o(\overline{G}[\overline{A}]) + c_o(G[B])$, it follows by Lemma 3.5

that $0 \leq |L| \leq 1$. Since $B \neq \phi$ and $G[B]$ contains only singleton components, it follows that $1 \leq |B| = |T_{B-A}| + |L_G| \leq 1$. Hence, $|B| = |L_G| = 1$. Therefore, $|\overline{B}| = 1 < r \leq p - r - 1$. By Lemma 3.3, $T_{\overline{A}-\overline{B}}$ contains no singleton components. Hence, $T_{\overline{A}-\overline{B}} = \phi$. Since each component of $\overline{G}[\overline{A}]$ is singleton, it is contained in $L_{\overline{G}}$. So $|L_{\overline{G}}| = |\overline{A}| = |A| \geq 1$. Therefore, $|L| = |L_G| + |L_{\overline{G}}| \geq 2$, a contradiction. Hence, $G\overline{G}$ is bicritical. It then follows that $G\overline{G}$ is 1-extendable. This proves our theorem. □

The next lemma follows by Theorem 2.4.

Lemma 3.9. *Let G be a connected r-regular graph of order $p \geq 2r + 1$, for some $r \geq 2$. If G contains a triangle, then $G\overline{G}$ is not r-extendable.* □

By Lemma 3.9, if G is a 3-regular graph of order $p \geq 8$ containing a triangle, then $G\overline{G}$ is not 3-extendable. The next theorem provides a sufficient condition for a connected r-regular graph G which $G\overline{G}$ is 2-extendable, for $r \geq 4$.

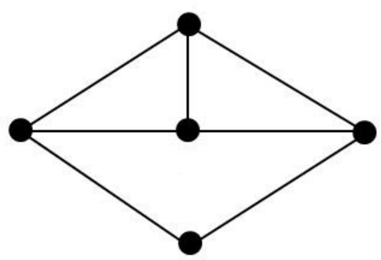

Figure 3.1: the graph F

In case $r = 3$, a graph G in Figure 3.2 contains the graph F in Figure 3.1 as an induced subgraph. It is easy to see that $G\overline{G}$ is not 2-extendable since $\{yz, \bar{x}\bar{w}\}$ cannot be extended to a perfect matching in $G\overline{G}$. We next show that the complementary prism of connected 3-regular F-free graphs and connected r-regular graphs for $r \geq 4$ are 2-extendable.

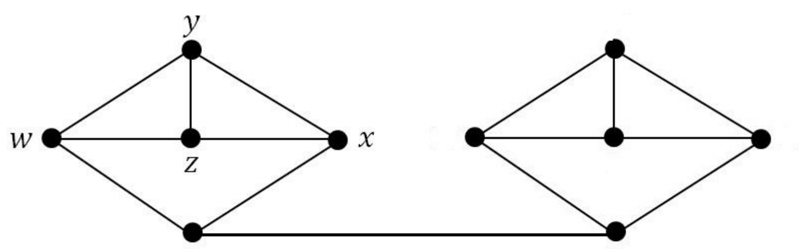

Figure 3.2: a 3-regular graph G which $G\overline{G}$ is not 2-extendable

Theorem 3.10. *Suppose G is a connected graph of order p. If G is either 3-regular F-free where $p \geq 8$ and F is the graph in Figure 3.1 or r_0-regular where $p \geq 2r_0 + 1 \geq 9$, then $G\overline{G}$ is 2-extendable.*

Proof. Observe that $\overline{G}$ is $(p-r-1)$-regular where $r \in \{3, r_0\}$ and $p - r - 1 \geq 4$. By Theorem 3.8, $G\overline{G}$ is bicritical. Suppose to the contrary that $G\overline{G}$ is not 2-extendable. Then there is a matching $M \subseteq E(G\overline{G})$ of size two such that $G\overline{G} - V(M)$ contains no perfect matching. By Theorem 2.1, there is a cutset $T \subseteq V(G\overline{G}) - V(M)$ such that $c_o(G\overline{G} - (V(M) \cup T)) > |T|$. Let $S = T \cup V(M)$. Clearly, $|S| \geq 4$. Thus $c_o(G\overline{G} - S) > |S| - 4$. Because $G\overline{G}$ is bicritical, by Theorem 2.6, $c_o(G\overline{G} - S) \leq |S| - 2$. It follows by parity that $c_o(G\overline{G} - S) = |S| - 2$ and $G\overline{G}[S]$ contains a matching of size at least two. Let $A = S \cap V(G)$ and $\overline{B} = S \cap V(\overline{G})$. By Lemma 3.2 (b), $|A \cap B| \leq 1$. Further, by Lemma 3.7, $A \neq \phi$ and $\overline{B} \neq \phi$. So $\overline{A} \neq \phi$ and $B \neq \phi$. We distinguish 2 cases according to $|A \cap B|$.

Case 1: $|A \cap B| = 1$. Put $\{u\} = A \cap B$. By Lemma 3.2(b) $c_o(G\overline{G} - S) = c_o(G[B - A]) + c_o(\overline{G}[\overline{A} - \overline{B}]) = |S| - 2$. By Lemma 3.5, $|L| \leq 1$. Further, by Lemma 3.2(c), each component of $\overline{G}[\overline{A} - \overline{B}] \cup G[B - A]$ is singleton. Thus, $A - B$ is a clique, $|Com_{\overline{A}-\overline{B}}| = |\overline{A} - \overline{B}|$ and $|Com_{B-A}| = |B - A|$. Since G is connected, it is easy to see that if $|A - B| \geq r + 1$, then $G[A - B] \cong K_{|A-B|}$ contains a vertex of degree greater than r or $G \cong K_{r+1}$ is a graph of order less than p, a contradiction. Hence, $|A - B| \leq r$.

We first show that $|T_{B-A}| \geq 2$. Suppose to the contrary that $|T_{B-A}| \leq 1$. Since $G[B - A]$ contains only singleton components and $|L_G| \leq |L| \leq 1$, it follows that $|B - A| = |Com_{B-A}| = |T_{B-A}| + |L_G| \leq 2$. Thus $|\overline{B}| = |B| = |B - A| + |B \cap A| \leq 3 < 4 \leq p - r - 1$. By Lemma 3.3, $T_{\overline{A}-\overline{B}}$ contains no singleton components. Thus $T_{\overline{A}-\overline{B}} = \phi$. Consequently, $Com_{\overline{A}-\overline{B}} = T_{\overline{A}-\overline{B}} \cup L_{\overline{G}} = L_{\overline{G}}$. Therefore, $|\overline{A} - \overline{B}| = |L_{\overline{G}}| \leq 1$ since $\overline{G}[\overline{A} - \overline{B}]$ contains only singleton components. So $|A| = |\overline{A}| = |\overline{A} - \overline{B}| + |\overline{A} \cap \overline{B}| \leq 2 < r$. By Lemma 3.3, T_{B-A} contains no singleton components. So $T_{B-A} = \phi$. Since $T_{\overline{A}-\overline{B}} = \phi$ and $T_{B-A} = \phi$, it follows that every odd component of $\overline{G}[\overline{A} - \overline{B}] \cup G[B - A]$ is in L. Because $|L| \leq 1$ and $\overline{G}[\overline{A} - \overline{B}] \cup G[B - A]$ contains only singleton components, it follows that $|\overline{A} - \overline{B}| + |B - A| \leq 1$. Hence, $|S| = |A - B| + |A \cap B| + |\overline{A} \cap \overline{B}| + |\overline{B} - \overline{A}| = |A - B| + 2|A \cap B| + |\overline{B} - \overline{A}| \leq 3 < 4$, contradicting the fact that $|S| \geq 4$. Therefore, $|T_{B-A}| \geq 2$.

Let $D_1, D_2 \in T_{B-A}$. Since $G[B - A]$ contains only singleton components, $D_i \cong K_1$, for $1 \leq i \leq 2$. Put $\{v_i\} = V(D_i)$. By Lemma 3.3, $|A| \geq r$. Consequently, $|A - B| \geq r - 1$. Because $|A - B| \leq r$, $r - 1 \leq |A - B| \leq r$. Since $A - B$ is a clique, $|A \cap B| = 1$ and $|T_{B-A}| \geq 2$, it follows by Lemmas 3.4 (b) and (c) that $|A - B| = r - 1$ and $|T_{B-A}| = 2$. Thus $|A| = |A - B| + |A \cap B| = r$. Because $r - 1 = |A - B| = |\overline{A} - \overline{B}| = |Com_{\overline{A}-\overline{B}}| = |T_{\overline{A}-\overline{B}}| + |L_{\overline{G}}| \leq |T_{\overline{A}-\overline{B}}| + 1$, it follows that $|T_{\overline{A}-\overline{B}}| \geq r - 2 \geq 1$. Thus $T_{\overline{A}-\overline{B}}$ contains a singleton component. By Lemma 3.3, $|\overline{B}| \geq p - r - 1 \geq 4$. Therefore, $|\overline{B} - \overline{A}| = |\overline{B}| - |\overline{B} \cap \overline{A}| \geq p - r - 2 \geq 3$. On the other hand, $|B - A| = |Com_{B-A}| = |T_{B-A}| + |L_G| \leq 3$. Then $|B - A| = |\overline{B} - \overline{A}| = 3$. Thus $3 = |T_{B-A}| + |L_G| = 2 + |L_G|$. It follows that $L = L_G = \{K_1\}$ and consequently $L_{\overline{G}} = \phi$. Since $|A| = r$, $deg_G v_1 = deg_G v_2 = r$ and $N_G(v_1) = N_G(v_2) \subseteq A$, it follows that $N_G(v_1) = N_G(v_2) = A$.

We now put $\{\bar{w}\} = V(K_1)$ where $K_1 \in T_{\overline{A}-\overline{B}}$. Clearly, $N_{\overline{G}}(\bar{w}) \subseteq \overline{B} - \{\bar{v}_1, \bar{v}_2\}$ since v_1 and v_2 are adjacent to every vertex in A. Because $|\overline{B}| = |\overline{B} - \overline{A}| + |\overline{A} \cap \overline{B}| = 3 + 1 = 4$, $|N_{\overline{G}}(\bar{w})| \leq |\overline{B}| - |\{\bar{v}_1, \bar{v}_2\}| = 2$ thus $\overline{G}$ is t-regular where $t \leq 2$. This contradicts the fact that $\overline{G}$ is $(p-r-1)$-regular where $p-r-1 \geq 4$. Therefore, Case 1 cannot occur.

Case 2: $|A \cap B| = 0$. By Lemmas 3.2(a) and (b), $|S| - 2 = c_o(G\overline{G} - S) \leq c_o(\overline{G}[\overline{A}]) + c_o(G[B]) \leq |\overline{A}| + |B| = |S|$. By parity, $c_o(\overline{G}[\overline{A}]) + c_o(G[B]) = |S|$ or $c_o(\overline{G}[\overline{A}]) + c_o(G[B]) = |S| - 2$. We distinguish 2 cases.

Case 2.1 : $c_o(\overline{G}[\overline{A}]) + c_o(G[B]) = |S| = |\overline{A}| + |B|$. Clearly, each component of $\overline{G}[\overline{A}] \cup G[B]$ is singleton. So $G[A] \cong K_{|A|}$. It is easy to see that if $|A| \geq r+1$, then $G[A]$ contains a vertex of degree greater than r or $G[A]$ is a disconnected component in G, a contradiction. Hence, $|A| \leq r$. By Lemmas 3.3 and 3.4(a), T_{B-A} contains no singleton components. Therefore, $T_{B-A} = \phi$. Thus $|L_G| = |B|$. Because $c_o(G[B]) + c_o(\overline{G}[\overline{A}]) - c_o(G\overline{G} - S) = |S| - (|S| - 2) = 2$, by Lemma 3.5, $2 \leq |L| \leq 3$. Since $B \neq \phi$ and $|B| = |L_G| \leq |L|$, it follows that $1 \leq |B| \leq 3$. Because $|\overline{B}| = |B| \leq 3 < 4 \leq p - r - 1$, by Lemma 3.3, $T_{\overline{A}-\overline{B}}$ contains no singleton components. Thus $T_{\overline{A}-\overline{B}} = \phi$. Hence, $|L_{\overline{G}}| = |\overline{A}| = |A|$. Therefore, $|L| = |L_G| + |L_{\overline{G}}| = |B| + |\overline{A}| = |S|$ and thus $2 \leq |S| \leq 3$ since $2 \leq |L| \leq 3$, contradicting the fact that $|S| \geq 4$. Hence, Case 2.1 cannot occur.

Case 2.2 : $c_o(\overline{G}[\overline{A}]) + c_o(G[B]) = |S| - 2 = |\overline{A}| + |B| - 2$. Put $s = |S|$. It is easy to see that $\overline{G}[\overline{A}] \cup G[B]$ contains all singleton components except exactly one non-singleton component which is of order 2 or 3. Hence, $\overline{G}[\overline{A}] \cup G[B]$ is isomorphic to a graph in $\{(s-2)K_1 \cup K_2, (s-3)K_1 \cup P_3, (s-3)K_1 \cup K_3\}$. If $|\overline{A}| \geq r+2 \geq 5$, then $\overline{G}[\overline{A}]$ must contain a singleton component, say F, where $V(F) = \{\bar{u}\}$. It follows that $deg_G u \geq r+1$, a contradiction. Hence, $|A| = |\overline{A}| \leq r+1$. Since $c_o(\overline{G}[\overline{A}]) + c_o(G[B]) - c_o(G\overline{G} - S) = (|S| - 2) - (|S| - 2) = 0$, by Lemma 3.5, $|L| \leq 1$. We distinguish 2 subcases according to the non-singleton component.

Subcase 2.2.1 : The only non-singleton component in $\overline{G}[\overline{A}] \cup G[B]$ is contained in $G[B]$. So $\overline{G}[\overline{A}] \cong |\overline{A}|K_1$ and $G[A] \cong K_{|\overline{A}|} \cong K_{|A|}$. Clearly, $|A| \leq r$ otherwise $G[A]$ is a disconnected component in G. By Lemmas 3.3 and 3.4(a), T_{B-A} contains no singleton components. So every singleton component in $G[B]$ is contained in L_G. Since $|L_G| \leq |L| \leq 1$, $G[B]$ contains at most 1 singleton component. We first show that $T_{\overline{A}-\overline{B}} = \phi$. Suppose this is not the case. Then there is $K_1 \in T_{\overline{A}-\overline{B}}$ since $\overline{G}[\overline{A}]$ contains only singleton components. By Lemma 3.3, $|B| = |\overline{B}| \geq p - r - 1 \geq 4$. Because $G[B]$ contains a non-singleton component of order either 2 or 3 and at most 1 singleton component, it follows that $G[B]$ is isomorphic to a graph in $\{K_1 \cup P_3, K_1 \cup K_3\}$. Thus $|B| = 4$ and either $T_{B-A} = \{P_3\}$ or $T_{B-A} = \{K_3\}$, and $L_G = \{K_1\}$. Thus $L_{\overline{G}} = \phi$. So $Com_{\overline{A}} = T_{\overline{A}-\overline{B}} \cup L_{\overline{G}} = T_{\overline{A}-\overline{B}}$. Therefore, each vertex of $\overline{A}$ is adjacent to every vertex of $\overline{B}$ since $\overline{G}$ is $(p-r-1)$-regular and $p-r-1 \geq 4$. It follows that there is no edge joining vertices of A and B. But this contradicts the fact that $T_{B-A} \neq \phi$. Hence, $T_{\overline{A}-\overline{B}} = \phi$ as required.

Therefore, $Com_{\overline{A}} = L_{\overline{G}}$. Since $|L_{\overline{G}}| \leq |L| \leq 1$ and $|\overline{A}| = |A| \neq 0$, it follows that $|Com_{\overline{A}}| = |L_{\overline{G}}| = 1$. Further, $L_G = \phi$ and $\overline{G}[\overline{A}] = K_1$. Thus $Com_B = T_{B-A}$. Because $|A| = |\overline{A}| = 1 < r \leq 3$, by Lemma 3.3, T_{B-A} contains no singleton components. So $G[B]$ contains no singleton components and $G[B]$ is isomorphic to a graph in $\{P_3, K_3\}$ since $|B| = |S| - |A| \geq 3$. Then $G\overline{G}[S] = G[A] \cup \overline{G}[\overline{B}]$ contains a matching of size less than two, contradicting the fact that $G\overline{G}[S]$ contains a matching of size at least two. Hence, Subcase 2.2.1 cannot occur.

Subcase 2.2.2 : The only non-singleton component in $\overline{G}[\overline{A}] \cup G[B]$ is contained in $\overline{G}[\overline{A}]$. So $G[B] \cong |B|K_1$. We first show that $T_{B-A} \neq \phi$. Suppose this is not the case. Then $T_{B-A} = \phi$ and thus $Com_B = T_{B-A} \cup L_G = L_G$. Since $B \neq \phi$ and $|L_G| + |L_{\overline{G}}| = |L| \leq 1$, it follows that $|L_G| = 1$ and $|L_{\overline{G}}| = 0$. Consequently, $|B| = 1$ since $G[B] \cong |B|K_1$. Because $|\overline{B}| = |B| = 1 < r$, $T_{\overline{A}-\overline{B}}$ contains no singleton components by Lemma 3.3. Hence, $G[\overline{A}]$ contains exactly one non-singleton component of order 2 or 3. Thus $|A| = |\overline{A}| \leq 3$. It is easy to see that $G\overline{G}[S] = G[A] \cup \overline{G}[\overline{B}]$ contains a matching of size at most one since $|\overline{B}| = 1$. This contradicts the fact that $G\overline{G}[S]$ contains a matching of size at least two. Hence, $T_{B-A} \neq \phi$. Further, $|T_{B-A}| \geq |B| - 1$ since $|L_G| \leq |L| \leq 1$ and $|T_{B-A}| + |L_G| = |B|$.

Because $G[B] \cong |B|K_1$, there exists $K_1 \in T_{B-A}$. By Lemma 3.3, $|A| \geq r$. So $r \leq |A| \leq r+1$. We first suppose that $|A| = r+1$. Let F_t be the non-singleton component of order t in $\overline{G}[\overline{A}]$ and let $\overline{A}_1 = V(F_t)$. Then $2 \leq t \leq 3$ and $\overline{G}[\overline{A}] \cong (r+1-t)K_1 \cup F_t$. It is easy to see that $G[A]$ contains $r+1-t$ vertices of degree r and each vertex of $A_1 = \overline{\overline{A}}_1$ has degree, in $G[A]$, at least $r+1-t$ and at most $r-1$. Let $\{w\} = V(K_1)$ where $K_1 \in T_{B-A}$, then $N_G(w) \subseteq A_1$ and thus $3 \leq r = deg_G(w) \leq t \leq 3$. It then follows that $N_G(w) = A_1$ and $t = r = 3$. Thus $\bar{w}$ is not adjacent to any vertex of $\overline{A}_1$ and $\overline{G}[\overline{A}] \cong K_1 \cup F_3$. Further, each vertex of A_1 has degree at least $|T_{B-A}| + 1 = |B| - |L_G| + 1 \geq |B|$ since $|L_G| \leq 1$. Thus $|B| \leq 3$ since G is now 3-regular. Because $\overline{G}$ is $(p-r-1)$-regular where $p-r-1 \geq 4$ and each vertex of $V(F_3) = \overline{A}_1$ has degree at most 3 in $\overline{G}[\overline{A} \cup \overline{B}]$ since it must be adjacent to at most one vertex in $\overline{B}$, it follows that $F_3 \in L_{\overline{G}}$. Since $|L_{\overline{G}}| \leq |L| \leq 1$, the only singleton component, K_1, of $\overline{G}[\overline{A}]$ must be in $T_{\overline{A}-\overline{B}}$. By Lemma 3.3, $|\overline{B}| \geq p-r-1 \geq 4$. But this contradicts the fact that $|\overline{B}| = |B| \leq 3$. Therefore, $|A| = r$.

Consequently, for each $w \in V(K_1)$ where $K_1 \in T_{B-A}$, $N_G(w) = A$. Now let $\bar{v} \in \overline{A}$. Then $deg_{\overline{B}}(\bar{v}) \leq |\overline{B}| - |T_{B-A}| = |B| - |T_{B-A}| = |L_G| \leq 1$. Further, $deg_{\overline{A}}(\bar{v}) \leq 2$ since each component of $\overline{G}[\overline{A}]$ has order at most 3. Because $\overline{G}$ is $(p-r-1)$-regular where $p-r-1 \geq 4$, $\bar{v}$ is adjacent to some vertex of $\overline{C}$. Consequently, each odd component of $\overline{G}[\overline{A}]$ is contained in $L_{\overline{G}}$. Because $|\overline{A}| = |A| = r \geq 3$, $\overline{G}[\overline{A}]$ contains a non-singleton component of order either 2 or 3 and $|L_{\overline{G}}| \leq |L| \leq 1$, it follows that $c_o(\overline{G}[\overline{A}]) = 1$. Therefore, $\overline{G}[\overline{A}]$ is isomorphic to a graph in $\{K_1 \cup K_2, P_3, K_3\}$. Hence, $r = |A| = 3$, $|L| = |L_{\overline{G}}| = 1$, $Com_B = T_{B-A} = \{|B|K_1\}$. Further, for $x \in B, y \in A$, $N_G(x) = A$ and $deg_G(y) = r = 3 \geq |B| = |\overline{B}|$.

We first suppose that $\overline{G}[\overline{A}] \cong K_3$. Then $G[A]$ is independent and thus $\overline{G}[\overline{B}]$ must contain a matching of size at least two since $G\overline{G}[S]$ contains a matching of size at least two. So $|B| = |\overline{B}| \geq 4$. But this contradicts the fact that $|B| = |\overline{B}| \leq 3$. Hence, $\overline{G}[\overline{A}] \neq K_3$. Therefore, $\overline{G}[\overline{A}]$ is isomorphic to a graph in $\{P_3, K_1 \cup K_2\}$. In either case, $G[A]$ contains a maximum matching of size one. Then $2 \leq |\overline{B}| \leq 3$ since $G\overline{G}[A \cup \overline{B}]$ contains a matching of size at least two.

We now suppose that $\overline{G}[\overline{A}] \cong K_1 \cup K_2$. Then $G[A] \cong P_3$ and then the vertex of degree two in P_3 has degree, in G, greater than $r = 3$, again a contradiction. Hence, $\overline{G}[\overline{A}] \neq K_1 \cup K_2$. Consequently, $\overline{G}[\overline{A}] \cong P_3$ and then $G[A] \cong K_1 \cup K_2$. Clearly, $|B| \neq 3$ otherwise $G[A]$ contains a vertex of degree greater than $r = 3$. So $|B| = 2$ and thus $G[A \cup B]$ contains the graph F in Figure 3.1 as an induced subgraph. But this contradicts our hypothesis that G is 3-regular F-free graph. This completes the proof of our theorem. □

It is clear that a connected 3-regular graph containing F, in Figure 3.1, as an induced subgraph contains v as a cut vertex. So 2-connected 3-regular graphs are F-free. The next corollary follows by this fact and Theorem 3.10.

Corollary 3.11. *If G is a 2-connected r-regular graph of order $p \geq 2r + 1$, for $r \geq 3$, then $G\overline{G}$ is 2-extendable.* □

Note that for positive integers r and s where $r+s \geq 6$, a graph $G = K_r + \overline{K}_s$ is a non-regular graph with minimum degree of $G\overline{G}$ is 1. Thus $G\overline{G}$ cannot be 2-extendable by Theroem 2.2(b). Hence, the hypothesis of regularity in Theorem 3.10 cannot be dropped.

According to Theorems 2.27 and 3.10, we have the following theorem.

Theorem 3.12. *If each component G_i of G is 3-regular F-free of order at least 8 where F is the graph in Figure 3.1 or r_0-regular of order at least $2r_0 + 1 \geq 9$, then $G\overline{G}$ is 2-extendable.*

We conclude our chapter by posing following problem.

Problem : Establish sufficient condition for a complementary prism of r-regular graphs to be k-extendable for $r \geq k \geq 3$.

Chapter 4

Extendability of complementary prism of extendable graphs

In this chapter, we show, in Section 4.2, that if G and $\overline{G}$ are l_1-extendable and l_2-extendable non-bipartite graphs for $l_1 \geq 2$ and $l_2 \geq 2$, then $G\overline{G}$ is $(l+1)$-extendable where $l = min\{l_1, l_2\}$. One might ask whether there exist such graphs G and $\overline{G}$. We affirm this in Section 4.1 by providing some constructions of a non-bipartite graph G such that G and $\overline{G}$ are l_1-extendable and l_2-extendable non-bipartite graphs, respectively, where l_1 and l_2 are positive integers.

4.1 Some constructions of extendable non-bipartite graphs

In this section, we provide two constructions of extendable non-bipartite graphs in which their complement graphs are also extendable by using cartesian and lexicographic products of two extendable graphs.

Theorem 4.1. *For non-negative integers l_1, l_2, $p_1 \geq 2l_1 + 2$ and $p_2 \geq 2l_2 + 2$ and $1 \leq i \leq 2$, let H_i be l_i-extendable of order p_i. Further, let $G = H_1 \times H_2$. If $\Delta(H_1) = p_1 - 1 - t_1$ and $\Delta(H_2) = p_2 - 1 - t_2$ for some non-negative integers t_1 and t_2, then G is $(l_1 + l_2 + 1)$-extendable and $\overline{G}$ is $(\frac{1}{2}(p_1 - 2)(p_2 - 2) + t_1 + t_2 - 1)$-extendable.*

Proof. By Theorem 2.16, $G = H_1 \times H_2$ is $(l_1 + l_2 + 1)$-extendable as required. We need only to show that $\overline{G}$ is $(\frac{1}{2}(p_1 - 2)(p_2 - 2) + t_1 + t_2 - 1)$-extendable. Clearly, G and $\overline{G}$ are of order p_1p_2. Since $N_G((u, v)) = \{(x, v)|xu \in E(H_1)\} \cup \{(u, y)|vy \in E(H_2)\}$, $deg_G((u, v)) = deg_{H_1}(u) + deg_{H_2}(v)$. Thus $\Delta(G) = \Delta(H_1) + \Delta(H_2) = p_1 + p_2 - 2 - t_1 - t_2$. Therefore, $\delta(\overline{G}) = p_1p_2 - p_1 - p_2 + 2 + t_1 + t_2 - 1 = \frac{1}{2}p_1p_2 + \frac{1}{2}p_1p_2 - p_1 - p_2 + t_1 + t_2 + 1 = \frac{1}{2}p_1p_2 + \frac{1}{2}(p_1 - 2)(p_2 - 2) + t_1 + t_2 - 1$. By Theorem 2.3, $\overline{G}$ is $(\frac{1}{2}(p_1 - 2)(p_2 - 2) + t_1 + t_2 - 1)$-extendable as required. This proves our theorem. □

Corollary 4.2. *Let H_1, H_2 and G be graphs defined in Theorem 4.1. If either H_1 or H_2 is non-bipartite, then G and $\overline{G}$ are also non-bipartite.* □

Theorem 4.3. *For non-negative integers $h_1, h_2, \bar{h}_1, \bar{h}_2$, let H_i be a h_i-extendable and let $\overline{H}_i$ be a $\bar{h}_i$-extendable for $1 \leq i \leq 2$. Then $G = H_1 \circ H_2$ is $(h_1+1)(h_2+1)$-extendable graph and $\overline{G}$ is $(\bar{h}_1 + 1)(\bar{h}_2 + 1)$-extendable graph.*

Proof. By Theorem 2.18, $G = H_1 \circ H_2$ is $(h_1 + 1)(h_2 + 1)$-extendable. We first show that $\overline{G} = \overline{H}_1 \circ \overline{H}_2$. Clearly, $V(\overline{G}) = V(\overline{H_1 \circ H_2}) = V(H_1) \times V(H_2) = V(\overline{H}_1) \times V(\overline{H}_2) = V(\overline{H}_1 \circ \overline{H}_2)$. Let $(u_1, v_1), (u_2, v_2) \in V(H_1) \times V(H_2)$ and let $(u_1, v_1)(u_2, v_2) \in E(\overline{H_1 \circ H_2})$. Thus $(u_1, v_1)(u_2, v_2) \notin E(H_1 \circ H_2)$.

If $u_1 = u_2$, then $v_1v_2 \notin E(H_2)$ and thus $(u_1, v_1)(u_2, v_2) \in E(\overline{H}_1 \circ \overline{H}_2)$. Further, if $u_1 \neq u_2$, then $u_1u_2 \notin E(H_1)$. And again $(u_1, v_1)(u_2, v_2) \in E(\overline{H}_1 \circ \overline{H}_2)$. Hence, $E(\overline{H_1 \circ H_2}) \subseteq E(\overline{H}_1 \circ \overline{H}_2)$.

We now suppose that $(u_1, v_1)(u_2, v_2) \in E(\overline{H}_1 \circ \overline{H}_2)$. If $u_1 = u_2$, then $v_1v_2 \in E(\overline{H}_2)$. Thus $(u_1, v_1)(u_2, v_2) \notin E(H_1 \circ H_2)$. Further, if $u_1 \neq u_2$, then $u_1u_2 \in E(\overline{H}_1)$ and thus $(u_1, v_1)(u_2, v_2) \notin E(H_1 \circ H_2)$. In either case $(u_1, v_1)(u_2, v_2) \in E(\overline{H_1 \circ H_2})$. Hence, $E(\overline{H}_1 \circ \overline{H}_2) \subseteq E(\overline{H_1 \circ H_2})$. Therefore, $E(\overline{H}_1 \circ \overline{H}_2) = E(\overline{H_1 \circ H_2})$. Thus $\overline{H_1 \circ H_2} = \overline{H}_1 \circ \overline{H}_2$. It follows by Theorem 2.18 that $\overline{G}$ is $(\bar{h}_1+1)(\bar{h}_2+1)$-extendable graph as required. This proves our theorem. □

Corollary 4.4. *For $1 \leq i \leq 2$, let $H_i, \overline{H}_i$ and G be graphs defined in Theorem 4.3. If H_1 is connected, $E(H_2) \neq \phi$ and $E(\overline{H}_2) \neq \phi$ then G and $\overline{G}$ are non-bipartite.* □

According to Theorems 4.1 and 4.3, we have shown that there exists a graph G such that G is l_1-extendable and $\overline{G}$ is l_2-extendable for some integers l_1 and l_2. Theorem 4.6 establishes that for any positive integers l_1 and l_2, there is a graph G such that G is l_1-extendable and $\overline{G}$ is l_2-extendable.

Lemma 4.5. *Let P_t be a path of order t. If $t \geq 4$ is an even integer, then P_t is 0-extendable and $\overline{P}_t$ is $(t-4)$-factor-critical. Further, $\overline{P}_t$ is $\frac{1}{2}(t-4)$-extendable.*

Proof. Clearly, P_t contains a perfect matching. We only show that $\overline{P}_t$ is $(t-4)$-factor-critical. Let $T \subseteq V(\overline{P}_t)$ such that $|T| = t-4$. Clearly, $\overline{P}_t - T$ is connected and contains P_4 as a subgraph. Thus $\overline{P}_t - T$ is one of a graph in $\{K_4, K_4 - e, C_4, K_4 - \{e_1, e_2\}, P_4\}$, where e_1 and e_2 have a common end vertex. In either case, $\overline{P}_t - T$ contains a perfect matching. Thus $\overline{P}_t - T$ is $(t-4)$-factor-critical as required. It then follows by definition of k-extendable that $\overline{P}_t$ is $\frac{1}{2}(t-4)$-extendable. This proves our lemma. □

Theorem 4.6. *For positive integers l_1 and l_2, there is a non-bipartite graph G such that G is l_1-extendable and $\overline{G}$ is l_2-extendable non-bipartite.*

Proof. Let $H_1 = \overline{P}_{2l_1+2}$ and $H_2 = P_{2l_2+2}$. By Lemma 4.5, H_1 is (l_1-1)-extendable, $\overline{H}_1$ is 0-extendable, H_2 is 0-extendable and $\overline{H}_2$ is (l_2-1)-extendable. Let $G = H_1 \circ H_2$. By Theorem 4.3, G is l_1-extendable and $\overline{G}$ is l_2-extendable as required. Further, it is clear that G and $\overline{G}$ are non-bipartite. This proves our theorem. □

4.2 Extendability of complementary prism of extendable graphs

In this section, we establish the extendability of the complementary prism $G\overline{G}$ of G where G and $\overline{G}$ are l_1-extendable and l_2-extendable non-bipartite graphs for $l_1 \geq 2$ and $l_2 \geq 2$, respectively. We begin with some lemmas.

Lemma 4.7. *Let G be a k-extendable graph for some integer $k \geq 2$ and let $S \subseteq V(G)$ be a cutset of G. If $G[S]$ contains $t \leq k-1$ independent edges, then $|S| \geq k+t+1$.*

Proof. Let $S' = S - V(F)$ where F is a matching of size t in $G[S]$. By Observation 2.10, $G' = G - V(F)$ is $(k-t)$-extendable. Observe that $k - t \geq 1$. By Theorem 2.2(b), G' is $(k-t+1)$-connected. Since S' is a cutset of G', $|S'| \geq k-t+1$ and thus $|S| \geq 2t + k - t + 1 = k + t + 1$ as required. This proves our lemma. □

Recall that, $\lfloor a \rfloor_e$ is denoted a greatest even integer less than or equal to a. Similarly, a greatest odd integer less than or equal to a may be denoted by $\lfloor a \rfloor_o$. Clearly, $\lfloor a \rfloor_o = 2\lfloor (a-1)/2 \rfloor + 1$.

Lemma 4.8. *Let G be a k-extendable non-bipartite graph for $k \geq 2$. Further, let $M \subseteq E(G)$ be a matching of size m and let $S = \phi$ or $S \subseteq V(G) - V(M)$ be an independent set such that $k - m - |S| = t \geq 0$ for some integer t. Then*

(a) If $|S|$ is even, then $G - (V(M) \cup S)$ is t-extendable. Further $G - (V(M) \cup S)$ is $\lfloor t \rfloor_e$-factor-critical. Consequently, there is a perfect matching in $G - (V(M) \cup S)$.

(b) If $|S|$ is odd and $t \geq 1$, then $G - (V(M) \cup S)$ is $\lfloor t \rfloor_o$-factor-critical. Thus $G - (V(M) \cup S)$ is 1-factor-critical.

(c) If $|S|$ is odd, $t = 0$ and there is a vertex $v \in V(G) - (V(M) \cup S)$ such that $vs \in E(G)$ for some $s \in S$, then $G - (V(M) \cup S \cup \{v\})$ contains a perfect matching.

Proof. We first suppose $m = k$. So $S = \phi$ and thus $G - (V(M) \cup S) = G - V(M)$ contains a perfect matching by Theorem 2.2(a) and it is 0-factor-critical as required. We now suppose that $m \leq k-1$. By Corollary 2.13, $G - V(M)$ is $(k-m)$-extendable non-bipartite. Since $k - m = |S| + t$, $G - V(M)$ is $(|S| + t)$-extendable non-bipartite.

(a) $|S|$ is even. By Theorem 2.2(a), $G - V(M)$ is $(|S| + \lfloor t \rfloor_e)$-extendable and thus it is $(|S| + \lfloor t \rfloor_e)$-factor-critical by Theorem 2.8. Hence, by Observation 2.11, $G - (V(M) \cup S)$ is $\lfloor t \rfloor_e$-factor-critical as required. It then follows by Theorem 2.2(a) that $G - (V(M) \cup S)$ contains a perfect matching. This proves (a).

(b) $|S|$ is odd and $t \geq 1$. By Theorem 2.2(a), $G - V(M)$ is $(|S| + \lfloor t \rfloor_o)$-extendable and thus it is $(|S| + \lfloor t \rfloor_o)$-factor-critical by Theorem 2.8. By Observation 2.11, $G - (V(M) \cup S)$ is $\lfloor t \rfloor_o$-factor-critical. Since $t \geq 1$, $\lfloor t \rfloor_o \geq 1$. Further, by Theorem 2.7, $G - (V(M) \cup S)$ is 1-factor-critical as required. This proves (b).

(c) Let $M' = M \cup \{vs\}$ and $S' = S - \{s\}$. Hence, our result follows from (a). This completes the proof of our lemma. □

Lemma 4.9. *Let G be a k-extendable graph for some integer k and let $S \subseteq V(G)$ be a cutset of G. If $G[S]$ contains t independent edges for $t \leq k$, then $c_o(G-S) \leq |S| - 2t$. Further, if $1 \leq t \leq k-1$ and $c_o(G-S) = |S| - 2t$ then $G-S$ contains no even components.*

Proof. Let F be a matching of size t in $G[S]$. Since G is a k-extendable graph, $G - V(F)$ contains a perfect matching by Theorem 2.2(a). By Theorem 2.1, $c_o(G-S) = c_o((G-V(F)) - (S-V(F))) \leq |S - V(F)| = |S| - 2t$, as required. We now suppose that $1 \leq t \leq k-1$ and $c_o(G-S) = |S| - 2t$. Let D be an even component of $G-S$. By Lemma 4.7 and the fact that $t \leq k-1 < k+1$, $V(F)$ is not a cutset of G. Then there is an edge $e = sd$ joining a vertex s in $S - V(F)$ and a vertex d in D. Since G is k-extendable and $F \cup \{e\}$ is a matching of size $t+1 \leq k$, it follows that there is a perfect matching in $G' = G - (V(F) \cup \{s,d\})$. Let $S' = S - (V(F) \cup \{s\})$. Clearly, $c_o(G'-S') = c_o(G-S) + 1 = |S| - 2t + 1$. Since G' contains a perfect matching, by Theorem 2.1, $|S| - 2t + 1 \leq c_o(G'-S') \leq |S| - (|V(F)| + 1)| = |S| - 2t - 1$, a contradiction. Hence, there is no even component in $G-S$. This proves our lemma. □

Lemma 4.10. *Let G be an l-extendable graph and let M be a matching of size $l+t$ where $t \geq 1$. Then there is a maximum matching in $G - V(M)$ saturates all except at most $2t$ non-adjacent vertices in $G - V(M)$.*

Proof. Let $T \subseteq M$ where $|T| = t$. Thus $M - T$ is a matching of size l in G. So there is a perfect matching F in $G - V(M-T)$. Clearly, $|V(F) \cap V(T)| = 2t$. Let $F_1 = \{xy \in F | \{x,y\} \cap V(M) = \phi\}$ and $F_2 = \{xy \in F | x \in V(M) \text{ and } y \notin V(M)\}$. Further, let F_2' be a maximum matching in $G[V(F_2) - V(M)]$. Then, $F_1 \cup F_2'$ is a matching in $G - V(M)$ saturates all except at most $2t$ non-adjacent vertices as required. □

By similar arguments as in the proof of Lemma 4.10, the next lemma follows.

Lemma 4.11. *Let G be a k-factor-critical graph and let $T \subseteq V(G)$ where $|T| = k+t$. Then there is a maximum matching in $G - V(T)$ saturates all except at most t non-adjacent vertices in $G - V(T)$.* □

Lemma 4.12. *Let G be an 1-extendable graph of order $p \geq 6$ and let v be a vertex of degree 2 in G. Then there are perfect matchings M_1, M_2 in G such that v is a vertex of C_{2n} in $M_1 \triangle M_2$ where $n \geq 3$. Further, there is a vertex $x \in V(C_{2n})$ where C_{2n} is a subgraph of $M_1 \triangle M_2$ such that $vx \notin E(G)$ and $G - \{v,x\}$ contains a perfect matching.*

Proof. Let $\{u_1, u_2\} = N_G(v)$. We first suppose $N_G(u_1) \cap N_G(u_2) = \{v\}$. Let M_1 be a perfect matching in G containing vu_1 and M_2 a perfect matching in G containing vu_2. Clearly, $\{vu_1, u_2u_3\} \subseteq M_1$ and $\{vu_2, u_1u_4\} \subseteq M_2$ for some $u_3, u_4 \in V(G)$. Since $\{v\} = N_G(u_1) \cap N_G(u_2), u_3 \neq u_4$. Hence, $u_3u_2vu_1u_4$ is a path of length 4 containing v. It must be contained in an even cycle of order at least 6 in $M_1 \triangle M_2$ as required.

So we now suppose that $N_G(u_1) \cap N_G(u_2) \neq \{v\}$. Then there is a vertex $v \neq u_3 \in N_G(u_1) \cap N_G(u_2)$. Since G is 2-connected by Theorem 2.2(b) and G is of order at least 6, it follows that u_3 is not a cut vertex. Then there is a vertex $u_4 \in N_G(u_1) \cup N_G(u_2)$ where $u_4 \neq u_3$. Without loss of generality, suppose $u_4 \in N_G(u_1)$. Let M_1 be a perfect matching in G containing u_1u_4 and M_2 be a perfect matching in G containing u_2u_3. It is easy to see that $\{u_1u_4, vu_2\} \subseteq M_1$ and $\{u_2u_3, vu_1\} \subseteq M_2$. Hence, $u_4u_1vu_2u_3$ is a path of length 4 containing v. It must be contained in an even cycle of order at least 6 in $M_1 \triangle M_2$ as required.

Further, let $x \in V(C_{2n})$ be such that the distance between v and x along the cycle C_{2n} is 3. Clearly, $xv \notin E(G)$ and it is easy to see that $G - \{v, x\}$ contains a perfect matching. This completes the proof of our lemma. □

Lemma 4.13. *For a graph G, let $A \subseteq V(G)$ and $\overline{B} \subseteq V(\overline{G})$. Suppose $c_o(G-A) = |A| - t_1$ and $c_o(\overline{G} - \overline{B}) = |\overline{B}| - t_2$, for some non-negative integers t_1, t_2. Then $c_o(G\overline{G} - (A \cup \overline{B})) \leq |A| + |\overline{B}| - (t_1 + t_2)$. Further, if $A \cup B \neq V(G)$ and $G - A$ and $\overline{G} - \overline{B}$ contain no even components, then $c_o(G\overline{G} - (A \cup \overline{B})) \leq |A| + |\overline{B}| - (t_1 + t_2) - 2$.*

Proof. It is easy to see that $c_o(G\overline{G} - (A \cup \overline{B})) \leq |A| + |\overline{B}| - (t_1 + t_2)$. We now suppose that $A \cup B \neq V(G)$ and $G - A$, $\overline{G} - \overline{B}$ contain no even components. Let $x \in V(G) - (A \cup B)$. Then x is in an odd component of $G - A$, say C. Clearly, $\bar{x} \notin \overline{A} \cup \overline{B}$ and thus $\bar{x}$ is in an odd component of $\overline{G} - \overline{B}$, say D. Hence, $G\overline{G}[V(C) \cup V(D)]$ forms an even component in $G\overline{G} - (A \cup \overline{B})$. Therefore $c_o(G\overline{G} - (A \cup \overline{B})) \leq |A| + |\overline{B}| - (t_1 + t_2) - 2$ as required. This proves our lemma. □

Lemma 4.14. *Let G and $\overline{G}$ be l_1-extendable and l_2-extendable graphs, respectively where l_1 and l_2 are positive integers. Further, let M be a matching of size $l+1$ in $G\overline{G}$ where $l = min\{l_1, l_2\}$. If either $M = \{x_i\bar{x}_i | x_i \in V(G) \text{ for } 1 \leq i \leq l+1\}$ or $M \subseteq (E(G) \cup E(\overline{G}))$, then $G\overline{G}$ has a perfect matching containing M.*

Proof. Clearly, if $M = \{x_i\bar{x}_i | x_i \in V(G) \text{ for } 1 \leq i \leq l+1\}$, then $\{v\bar{v} | v \in V(G)\}$ is a perfect matching in $G\overline{G}$ containing M as required. So we now suppose that $M \subseteq (E(G) \cup E(\overline{G}))$. Put $M_G = M \cap E(G)$ and $M_{\overline{G}} = M \cap E(\overline{G})$. If $1 \leq |M_G| \leq l$ and $1 \leq |M_{\overline{G}}| \leq l$, then it is easy to see that $M = M_G \cup M_{\overline{G}}$ can be extended to a perfect matching in $G\overline{G}$, by Theorem 2.2(a), since G is l_1-extendable and $\overline{G}$ is l_2-extendable. Hence, we suppose without loss of generality that $|M_G| = l+1$. Suppose there is no perfect matching in G containing M_G. By Lemma 4.10, there is a maximum matching F_1 in $G - V(M)$ saturates all except two non-adjacent vertices, say x and y. So $\bar{x}\bar{y} \in E(\overline{G})$. Since $\overline{G}$ is l_2-extendable where $l_2 \geq 1$ and by Theorem 2.2(a), it follows that there is a perfect matching F_2 in $\overline{G}$ containing $\bar{x}\bar{y}$. Hence, $M \cup F_1 \cup (F_2 - \{\bar{x}\bar{y}\}) \cup \{x\bar{x}, y\bar{y}\}$ is a perfect matching in $G\overline{G}$ containing M as required. This completes the proof of our lemma. □

We are now ready to prove our main result. We begin with the extendability of $G\overline{G}$ where G is l_1-extendable and $\overline{G}$ is l_2-extendable for $l_1 \geq 4$ and $l_2 \geq 4$.

Theorem 4.15. *For positive integers $l_1 \geq 4$, $l_2 \geq 4$, let G and $\overline{G}$ be l_1-extendable and l_2-extendable non-bipartite graphs, respectively. Then $G\overline{G}$ is $(l+1)$-extendable, where $l = min\{l_1, l_2\}$.*

Proof. Let $M \subseteq E(G\overline{G})$ be a matching of size $l+1$ in $G\overline{G}$. Put $M_G = M \cap E(G)$, $M_{\overline{G}} = M \cap E(\overline{G})$ and $M_{G\overline{G}} = M - (M_G \cup M_{\overline{G}})$. Note that $M_{G\overline{G}} = \{x\bar{x} | x \in V(G)\}$. If $M_{G\overline{G}} = M$ or $M_{G\overline{G}} = \phi$, then, by Lemma 4.14, there is a perfect matching in $G\overline{G}$ containing M as required. We now suppose that $M_{G\overline{G}} \neq M$ and $M_{G\overline{G}} \neq \phi$. Without loss of generality, we may suppose that $|M_G| \geq |M_{\overline{G}}|$. Hence, $M_G \neq \phi$.

Put $S = V(G) \cap V(M_{G\overline{G}})$. Let N_S be a maximum matching in $G[S]$. Put $I_S = S - V(N_S)$. Clearly, I_S is an independent set. Similarly, let $N_{\overline{S}}$ be a maximum matching in $\overline{G}[\overline{S}]$ and put $I_{\overline{S}} = \overline{S} - V(N_{\overline{S}})$. For simplicity, we denote the cardinalities of each set by its small letter, i.e., $m_G = |M_G|$, $m_{\overline{G}} = |M_{\overline{G}}|$, $m_{G\overline{G}} = |M_{G\overline{G}}|$, $s = |S|$, $i_S = |I_S|$, etc.

Clearly, $1 \leq m_G \leq l$, $s = \bar{s}$, $n_S + i_S \geq 1$ and $n_{\overline{S}} + i_{\overline{S}} \geq 1$ since $s = \bar{s} = m_{G\overline{G}} \geq 1$. Therefore,

$$l + 1 = m_G + m_{\overline{G}} + m_{G\overline{G}} \tag{4.1}$$

$$l + 1 = m_G + m_{\overline{G}} + s \tag{4.2}$$

$$l + 1 = m_G + m_{\overline{G}} + 2n_S + i_S \tag{4.3}$$

$$l + 1 = m_G + m_{\overline{G}} + 2n_{\overline{S}} + i_{\overline{S}}. \tag{4.4}$$

Consequently, $m_G + n_S = l + 1 - (m_{\overline{G}} + n_S + i_S) \leq l$ since $n_S + i_S \geq 1$ and $m_{\overline{G}} + n_{\overline{S}} = l + 1 - (m_G + n_{\overline{S}} + i_{\overline{S}}) \leq l$ since $n_{\overline{S}} + i_{\overline{S}} \geq 1$. Further, $s \equiv i_S \pmod 2$ and $\bar{s} \equiv i_{\overline{S}} \pmod 2$ because $s = 2n_S + i_S$ and $s = \bar{s} = 2n_{\overline{S}} + i_{\overline{S}}$.

We first suppose that $i_S = 0$. Since $m_G + n_S \leq l$, by Theorem 2.2(a), there is a perfect matching in $G - (V(M_G) \cup N_S)$, say F_G. Now consider $\overline{G}$. By Equation 4.4, $l - (m_{\overline{G}} + n_{\overline{S}} + i_{\overline{S}}) = m_G + n_{\overline{S}} - 1 \geq 0$ since $m_G \geq 1$. By Lemma 4.8(a), there is a perfect matching in $\overline{G} - (V(M_{\overline{G}} \cup N_{\overline{S}}) \cup I_{\overline{S}})$, say $F_{\overline{G}}$. Hence, $M \cup F_G \cup F_{\overline{G}}$ is a perfect matching in $G\overline{G}$ containing M as required.

So we now suppose that $i_S \geq 1$. We distinguish 2 cases according to parity of s.

Case 1 : s is even. So $i_S \geq 2$ and $i_{\overline{S}} \geq 0$ are also even. We distinguish 2 subcases according to $m_{\overline{G}} + n_S$.

Subcase 1.1 : $m_{\overline{G}} + n_S \geq 1$. So, by Equation 4.3, $l - (m_G + n_S + i_S) = m_{\overline{G}} + n_S - 1 \geq 0$. By Lemma 4.8(a), there is a perfect matching in $G - (V(M_G \cup N_S) \cup I_S)$, say F_G. Further, by Equation 4.4 and the fact that $m_G \geq 1$, $l - (m_{\overline{G}} + n_{\overline{S}} + i_{\overline{S}}) = m_G + n_{\overline{S}} - 1 \geq 0$. So, by Lemma 4.8(a), there is a perfect matching in $\overline{G} - (V(M_{\overline{G}} \cup N_{\overline{S}}) \cup I_{\overline{S}})$, say $F_{\overline{G}}$. Hence, $M \cup F_G \cup F_{\overline{G}}$ is a perfect matching in $G\overline{G}$ containing M as required.

Subcase 1.2 : $m_{\overline{G}} = n_S = 0$. We first show that $n_{\overline{S}} \leq \frac{l}{2}$. Since $n_S = 0$, $G[S]$ is independent and thus $\overline{G}[\overline{S}]$ is a complete graph. Because s is even, $n_{\overline{S}} = \frac{1}{2}\bar{s} = \frac{1}{2}s$. So, by Equation 4.2 and the fact that $m_G \geq 1$, $n_{\overline{S}} = \frac{1}{2}s = \frac{1}{2}(l + 1 - m_G - m_{\overline{G}}) = \frac{1}{2}(l + 1 - m_G) \leq \frac{l}{2}$ as required. By Equation 4.3, $l - m_G = m_{\overline{G}} + 2n_S + i_S - 1 = i_S - 1$ and $\lfloor i_S - 1 \rfloor_e = i_S - 2 \geq 0$ since $i_S = s$ is even. It follows by Lemma 4.8(a) that $G' = G - V(M_G)$ is $(i_S - 2)$-factor-critical. By Lemma 4.11, there is a maximum matching F_G in $G' - I_S$ saturates all except at most 2 vertices in $G' - I_S$.

We next consider $\overline{G}$. By Equation 4.4 and the fact that $m_G \geq 1$, $l-(m_{\overline{G}}+n_{\overline{S}}+i_{\overline{S}}) = m_G + n_{\overline{S}} - 1 \geq n_{\overline{S}} \geq 0$. By Lemma 4.8(a), there is a perfect matching in $\overline{G}-(V(M_{\overline{G}} \cup N_{\overline{S}}) \cup I_{\overline{S}})$, say $F_{\overline{G}}$. Clearly, if F_G is a perfect matching in $G' - I_S$, then $M \cup F_G \cup F_{\overline{G}}$ is a perfect matching in $G\overline{G}$ as required.

We now suppose that F_G is not a perfect matching. Let $x, y \in V(G') - I_S$ where x and y are unsaturated by F_G. Clearly, $xy \notin E(G)$. So $\bar{x}\bar{y} \in E(\overline{G})$. Because $n_{\overline{S}} \leq \frac{l}{2}$ and $l \geq 4$, it follows that $m_{\overline{G}} + n_{\overline{S}} + 1 = n_{\overline{S}} + 1 \leq \frac{l}{2} + 1 \leq \frac{l}{2} + (\frac{l}{2} - 1) \leq l - 1$. By Theorem 2.2(a), there is a perfect matching in $\overline{G} - V(M_{\overline{G}} \cup N_{\overline{S}} \cup \{\bar{x}\bar{y}\})$, say $F'_{\overline{G}}$. Hence, $M \cup F_G \cup (F'_{\overline{G}} - \{\bar{x}\bar{y}\}) \cup \{x\bar{x}, y\bar{y}\}$ is a perfect matching in $G\overline{G}$ containing M as required. This proves Case 1.

Case 2 : s is odd. So i_S and $i_{\overline{S}}$ are also odd. We distinguish 3 subcases according to $m_{\overline{G}} + n_S$.

Subcase 2.1 : $m_{\overline{G}} = n_S = 0$. By Equation 4.3, $l - (m_G + (i_S - 1)) = m_{\overline{G}} + 2n_S = 0$. Let $i \in I_S$, by Lemma 4.8(a), there is a perfect matching in $G - (V(M_G) \cup (I_S - \{i\}))$, say F_G. Let $iv \in F_G$. We now consider $\overline{G}$. Since $n_S = 0$, $G[S]$ is independent and thus $\overline{G}[S]$ is a complete graph of odd order s. Therefore, $n_{\overline{S}} = \frac{1}{2}(s-1)$ and $i_{\overline{S}} = 1$. By Equation 4.2, $l = m_G + s - 1$. So $l - (m_{\overline{G}} + n_{\overline{S}} + i_{\overline{S}}) = l - (n_{\overline{S}} + i_{\overline{S}}) = m_G + s - 1 - (\frac{1}{2}(s-1)+1) = m_G + \frac{1}{2}(s-3)$.

We next show that $m_G + \frac{1}{2}(s-3) \geq 1$. Suppose to the contrary that $m_G + \frac{1}{2}(s-3) = 0$. Since $m_G \geq 1$ and s is a positive odd integer, it follows that $m_G = 1$ and $s = 1$. By Equation 4.2, $l + 1 = m_G + m_{\overline{G}} + s = 1 + 0 + 1 = 2$. Thus $l = 1$, contradicting the fact that $l \geq 4$. Hence, $m_G + \frac{1}{2}(s-3) \geq 1$ as required.

Therefore, $l - (m_{\overline{G}} + n_{\overline{S}} + i_{\overline{S}}) = m_G + \frac{1}{2}(s-3) \geq 1$. By Lemma 4.8(b), $\overline{G} - (V(M_{\overline{G}} \cup N_{\overline{S}}) \cup I_{\overline{S}})$ is 1-factor-critical. Recall that $iv \in F_G$. Clearly, $\bar{v} \notin V(M_{\overline{G}})$ since $m_{\overline{G}} = 0$. So there is a perfect matching in $\overline{G} - ((V(M_{\overline{G}} \cup N_{\overline{S}}) \cup I_{\overline{S}}) \cup \{\bar{v}\})$, say $F_{\overline{G}}$. Hence, $M \cup (F_G - \{iv\}) \cup F_{\overline{G}} \cup \{v\bar{v}\}$ is a perfect matching in $G\overline{G}$ containing M as required. This proves Subcase 2.1.

Subcase 2.2 : $m_{\overline{G}} + n_S \geq 2$. By Equation 4.3, $l - (m_G + n_S + i_S) = m_{\overline{G}} + n_S - 1 \geq 1$. By Lemma 4.8(b), $G - (V(M_G \cup N_S) \cup I_S)$ is 1-factor-critical.

We now consider $\overline{G}$. By Equation 4.4, $l - (m_{\overline{G}} + n_{\overline{S}} + i_{\overline{S}}) = m_G + n_{\overline{S}} - 1$.

We first suppose that $l - (m_{\overline{G}} + n_{\overline{S}} + i_{\overline{S}}) = m_G + n_{\overline{S}} - 1 \geq 1$. By Lemma 4.8(b), $\overline{G} - (V(M_{\overline{G}} \cup N_{\overline{S}}) \cup I_{\overline{S}})$ is 1-factor-critical. Let $x \in V(G)$ such that $x, \bar{x} \notin V(M)$. Clearly, x exists because $|V(M_G \cup M_{\overline{G}}) \cup S| \leq 2l + 1$ and G and $\overline{G}$ are of order at least $2l+2$. Since $G - (V(M_G \cup N_S) \cup I_S)$ and $\overline{G} - (V(M_{\overline{G}} \cup N_{\overline{S}}) \cup I_{\overline{S}})$ are 1-factor-critical, it follows that there is a perfect matching in $G - (V(M_G \cup N_S) \cup I_S \cup \{x\})$, say F_G, and there is a perfect matching in $\overline{G} - (V(M_{\overline{G}} \cup N_{\overline{S}}) \cup I_{\overline{S}} \cup \{\bar{x}\})$, say $F_{\overline{G}}$. Hence, $M \cup F_G \cup F_{\overline{G}} \cup \{x\bar{x}\}$ is a perfect matching in $G\overline{G}$ containing M as required.

So we next suppose that $l - (m_{\overline{G}} + n_{\overline{S}} + i_{\overline{S}}) = m_G + n_{\overline{S}} - 1 = 0$. It follows that $n_{\overline{S}} = 0$ and $m_G = 1$ since $m_G \geq 1$. Thus $i_{\overline{S}} = \bar{s}$ and $m_{\overline{G}} \leq m_G = 1$. Put $M_G = \{xy\}$. Since $\overline{G}$ is l_2-extendable, for $l_2 \geq 4$, by Theorem 2.2(b), $\overline{G}$ is 5-connected. So $\{\bar{x}, \bar{y}\} \cup V(M_{\overline{G}})$ is not a cutset of $\overline{G}$ since $|\{\bar{x}, \bar{y}\} \cup V(M_{\overline{G}})| \leq 4$. Hence, there is an edge joining a vertex in $V(\overline{G}) - (\{\bar{x}, \bar{y}\} \cup V(M_{\overline{G}}))$, say $\bar{u}$, and a vertex in $\overline{S}$, say $\bar{w}$. Because $l - (m_{\overline{G}} + n_{\overline{S}} + i_{\overline{S}}) = 0$ and $i_{\overline{S}} = \bar{s}$, it follows that $l - (m_{\overline{G}} + n_{\overline{S}} + 1 + (i_{\overline{S}} - 1)) = l - (m_{\overline{G}} + n_{\overline{S}} + 1 + (\bar{s} - 1)) = 0$. By Lemma

4.8(a), there is a perfect matching in $\overline{G} - (V(M_{\overline{G}} \cup N_{\overline{S}} \cup \{\bar{u}\bar{w}\}) \cup (\overline{S} - \{\bar{w}\}))$, say $F_{\overline{G}}$. Since $G - (V(M_G \cup N_S) \cup I_S)$ is 1-factor-critical and $u \notin V(M_G)$, it follows that there is a perfect matching in $G - (V(M_G \cup N_S) \cup I_S \cup \{u\})$, say F_G. Hence, $M \cup F_G \cup F_{\overline{G}} \cup \{u\bar{u}\}$ is a perfect matching in $G\overline{G}$ containing M as required. This proves Subcase 2.2.

Subcase 2.3 : $m_{\overline{G}} + n_S = 1$. By Equation 4.3 and the fact that i_S is odd, $m_G + n_S = l + 1 - (m_{\overline{G}} + n_S + i_S) \leq l - 1$. We distinguish 2 subcases according to $m_{\overline{G}}$ and n_S.

Subcase 2.3.1 : $m_{\overline{G}} = 0$ and $n_S = 1$. Observe that $G[V(M_G \cup N_S)]$ contains $m_G + n_S \leq l - 1$ independent edges and $|V(M_G \cup N_S)| = 2(m_G + n_S) = (m_G + n_S) + m_G + n_S \leq l - 1 + (m_G + n_S)$. It follows by Lemma 4.7 that $V(M_G \cup N_S)$ is not a cutset of G. Then there are a vertex $u \in V(G) - (V(M_G) \cup S)$ and a vertex $z \in I_S$ such that $uz \in E(G)$. Since $l - ((m_G + n_S + 1) + (i_S - 1)) = l - (m_G + n_S + i_S) = m_{\overline{G}} + n_S - 1 = 0$, by Lemma 4.8(a), there is a perfect matching in $G - (V(M_G \cup N_S \cup \{uz\}) \cup (I_S - \{z\}))$, say F_G. We now consider $\overline{G}$. We next show that $m_G + n_{\overline{S}} \geq 2$. Suppose to the contrary that $m_G + n_{\overline{S}} = 1$. Since $m_G \geq 1$, $n_{\overline{S}} = 0$ and $m_G = 1$. By Equation 4.3, $l+1 = m_G + m_{\overline{G}} + 2n_S + i_S = 3 + i_S$. So $i_S = l - 2 \geq 4 - 2 = 2$. It follows that $\overline{G}[\overline{S}]$ contains K_2 as an induced subgraph. Thus $n_{\overline{S}} \geq 1$, contradicting the fact that $n_{\overline{S}} = 0$. Hence, $m_G + n_{\overline{S}} \geq 2$.

By Equation 4.4, $l - (m_{\overline{G}} + n_{\overline{S}} + i_{\overline{S}}) = m_G + n_{\overline{S}} - 1 \geq 1$. By Lemma 4.8(b), $\overline{G} - (V(M_{\overline{G}} \cup N_{\overline{S}}) \cup I_{\overline{S}})$ is 1-factor-critical. Recall that $m_{\overline{G}} = 0$. So $\bar{u} \notin V(M_{\overline{G}})$ Therefore, there is a perfect matching in $\overline{G} - (V(M_{\overline{G}} \cup N_{\overline{S}}) \cup I_{\overline{S}} \cup \{\bar{u}\})$, say $F_{\overline{G}}$. Hence, $M \cup F_G \cup F_{\overline{G}} \cup \{u\bar{u}\}$ is a perfect matching in $G\overline{G}$ containing M as required. This completes the proof of Subcase 2.3.1.

Subcase 2.3.2 : $m_{\overline{G}} = 1$ and $n_S = 0$. Put $m_{\overline{G}} = \{\bar{x}_1\bar{x}_2\}$. Note that $m_G + n_{\overline{S}} \geq 2$ since $|m| = l + 1 \geq 5$ and $n_S = 0$. If there is a vertex $u \in V(G) - (V(M_G) \cup S \cup \{x_1, x_2\})$ such that $uz \in E(G)$ for some $z \in S$, then by applying similar argument as in the proof of Subcase 2.3.1, there is a perfect matching in $G\overline{G}$ containing M as required. So we now suppose that there is no vertex $u \in V(G) - (V(M_G) \cup S \cup \{x_1, x_2\})$ such that $uz \in E(G)$ for some $z \in S$. Thus $V(M_G) \cup \{x_1, x_2\}$ is a cutset of G and $\{x_1, x_2\}$ is a cutset of $G - V(M_G)$. We next show that $s = 1$. Suppose to the contrary that $s \geq 3$. By Equation 4.2, $m_G = l + 1 - m_{\overline{G}} - s = l - s \leq l - 3$. By Observation 2.10, $G - V(M_G)$ is $(l - m_G)$-extendable. Because $l - m_G \geq 3$, by Theorem 2.2(b), $G - V(M_G)$ is 4-connected, contradicting the fact that $\{x_1, x_2\}$ is a cutset of $G - V(M_G)$. Hence, $s = 1$. Put $S = \{z\}$. Therefore, $zu \notin E(G)$ for $u \in V(G) - (V(M_G) \cup S \cup \{x_1, x_2\})$. So $N_G(z) \subseteq V(M_G) \cup \{x_1, x_2\}$

By Equation 4.2, $m_G = l + 1 - m_{\overline{G}} - s = l - 1$. By Observation 2.10, $G' = G - V(M_G)$ is 1-extendable. By Theorem 2.2(b), G' is 2-connected. Therefore, $N_{G'}(z) = \{x_1, x_2\}$ and $deg_{G'}(z) = 2$. By Lemma 4.12, there is a vertex $u \in V(G')$ such that $uz \notin E(G')$ and $G' - \{u, z\}$ contains a perfect matching, say F_G. We now consider $\overline{G}$. Since $l \geq 4$, $m_{\overline{G}} = 1$ and $\bar{s} = s = 1$, it follows that $l - (m_{\overline{G}} + \bar{s}) = l - 2 \geq 2$. By Lemma 4.8(b), $\overline{G}' = \overline{G} - V(M_{\overline{G}} \cup \overline{S})$ is 1-factor-critical. Then there is a perfect matching in $\overline{G}' - \{\bar{u}\}$, say $F_{\overline{G}}$. Hence, $M \cup F_G \cup F_{\overline{G}} \cup \{u\bar{u}\}$ is a perfect matching in $G\overline{G}$ containing M as required. This completes the proof of Subcase

2.3.2. and thus completes the proof of our theorem. □

We now turn our attention to studying the extendability of $G\overline{G}$ when G or $\overline{G}$ is l-extendable for $1 \leq l \leq 3$.

We first provide an example of a graph G suchthat both G and $\overline{G}$ are 1-extendable but $G\overline{G}$ is not 2-extendable. Let G be a graph where $V(G) = \{u_i|$ for $1 \leq i \leq 4\} \cup \{v_i|$ for $1 \leq i \leq 6\}$ and $E(G) = \{u_iu_{i+1}|$ for $1 \leq i \leq 3\} \cup \{v_iv_{i+1}|$ for $1 \leq i \leq 6$ where the subscript is read modulo $6\} \cup \{u_1v_i|$ for $1 \leq i \leq 6\} \cup \{u_3v_1, u_4v_2, u_4v_3\}$. Observe that $G[\{u_i|$ for $1 \leq i \leq 4\}]$ and $G[\{v_i|$ for $1 \leq i \leq 6\}]$ are a path of order 4 and a cycle of order 6, respectively. It is routine to verify that G and $\overline{G}$ are 1-extendable. But $G\overline{G}$ is not 2-extendable since $\{\bar{u}_2\bar{u}_4, u_3\bar{u}_3\}$ cannot be extended to a perfect matching in $G\overline{G}$.

We now scope our attention to extendability of $G\overline{G}$ where G is l_1-extendable and $\overline{G}$ is l_2-extendable for $l_1 \geq 2$ and $l_2 \geq 2$. We first consider the case where $l_1 = 2$ and $l_2 \geq 2$. We begin with the following lemma. Recall that if $\phi \neq \{x_1, \ldots, x_t\} \subseteq V(G)$, then $\{\bar{x}_1, \ldots, \bar{x}_t\} \subseteq V(\overline{G})$ is denoted by $\overline{X}$ and vice versa.

Lemma 4.16. *Let G and $\overline{G}$ be 2-extendable non-bipartite graphs of order $p \geq 10$ and let $M = \{x_1x_2, \bar{y}_1\bar{y}_2, z\bar{z}\}$ be a matching of size 3 in $G\overline{G}$, where $\{x_1, x_2, z\} \subseteq V(G)$ and $\{\bar{y}_1, \bar{y}_2, \bar{z}\} \subseteq V(\overline{G})$. Then there is a perfect matching in $G\overline{G}$ containing M.*

Proof. Suppose to the contrary that there is no perfect matching in $G\overline{G}$ containing M. By Theorem 2.1, there is a cutset $T \subseteq V(G\overline{G}) - V(M)$ such that $c_o(G\overline{G} - (V(M) \cup T)) > |T|$. By parity, $c_o(G\overline{G} - (V(M) \cup T)) \geq |T| + 2$. Put $S = T \cup V(M)$. So $c_o(G\overline{G} - S) \geq |S| - 4$. Put $A = S \cap V(G)$, $\overline{B} = S \cap V(\overline{G})$ and $C = V(G) - (A \cup B)$. Observe that $|A| \geq 3$ and $|\overline{B}| \geq 3$.

By Theorem 2.8, G and $\overline{G}$ are bicritical. Thus, by Theorem 2.6, $c_o(G - A) \leq |A| - 2$ and $c_o(\overline{G} - \overline{B}) \leq |\overline{B}| - 2$. We first show that $c_o(G - A) = |A| - 2$ and $c_o(\overline{G} - \overline{B}) = |\overline{B}| - 2$. Suppose to the contrary that $c_o(G - A) < |A| - 2$. By parity, $c_o(G - A) \leq |A| - 4$. It then follows by Lemma 4.13 that $c_o(G\overline{G} - S) \leq c_o(G - A) + c_o(\overline{G} - \overline{B}) \leq |A| + |\overline{B}| - 6$, contradicting the fact that $c_o(G\overline{G} - S) \geq |S| - 4$. Hence, $c_o(G - A) = |A| - 2$. Similarly, $c_o(\overline{G} - \overline{B}) = |\overline{B}| - 2$.

Since G and $\overline{G}$ are 2-extendable, by Theorem 2.5(b), $G[A]$ and $\overline{G}[\overline{B}]$ contain at most one independent edge. Because $\{x_1, x_2, z\} \subseteq A$ and $\{\bar{y}_1, \bar{y}_2, \bar{z}\} \subseteq \overline{B}$, $G[A]$ and $\overline{G}[\overline{B}]$ contain exactly 1 independent edge. By Lemma 4.9, $G - A$ and $\overline{G} - \overline{B}$ contain no even components. If $A \cup B \neq V(G)$, then, by Lemma 4.13, $c_o(G\overline{G} - S) = c_o(G\overline{G} - (A \cup \overline{B})) \leq |A| + |\overline{B}| - 6 = |S| - 6$, again a contradiction. Hence, $A \cup B = V(G)$. Observe that if $c_o(G - A) \geq 4$, $G[B] = G - A$ contains at least 4 independence vertices and thus $\overline{G}[\overline{B}]$ contains a matching of size at least two, a contradiction. Hence, $c_o(G - A) \leq 3$. Similarly, $c_o(\overline{G} - \overline{B}) \leq 3$ and each component of $\overline{G} - \overline{B}$ is singleton otherwise $G[A] = G - B$ contains at least 2 independent edges, a contradiction. Therefore, $c_o(G[B - A]) = c_o(G - A) \leq 3$ and $\overline{G}[\overline{A} - \overline{B}] = c_o(\overline{G} - \overline{B}) \leq 3$. Since $c_o(G - A) = |A| - 2$ and $c_o(\overline{G} - \overline{B}) = |\overline{B}| - 2$, it follows that $|A| = 2 + c_o(G - A) \leq 5$ and $|B| = |\overline{B}| = 2 + c_o(\overline{G} - \overline{B}) \leq 5$. Because $z \in A \cap B$, $|A \cup B| = |A| + |B| - |A \cap B| \leq 5 + 5 - 1 \leq 9$, contradicting the fact that $|V(G)| = p \geq 10$. This completes the proof of our lemma. □

The next theorem shows that if G is a 2-extendable non-bipartite graph and $\overline{G}$ is a l-extendable non-bipartite graph of order $p \geq 10$ and $l \geq 2$, then $G\overline{G}$ is 3-extendable.

Theorem 4.17. *Let G be a 2-extendable non-bipartite graph of order $p \geq 10$. If $\overline{G}$ is l-extendable non-bipartite for some positive integer $l \geq 2$, then $G\overline{G}$ is 3-extendable.*

Proof. By Theorem 2.2(b), $\overline{G}$ is 2-extendable non-bipartite graph. Let M be a matching of size 3 in $G\overline{G}$. Put $M_G = M \cap E(G)$, $M_{\overline{G}} = M \cap E(\overline{G})$ and $M_{G\overline{G}} = M - (M_G \cup M_{\overline{G}})$. Further, put $m_G = |M_G|, m_{\overline{G}} = |M_{\overline{G}}|$ and $m_{G\overline{G}} = |M_{G\overline{G}}|$. If $m_{G\overline{G}} = 0$ or $m_{G\overline{G}} = 3$, then, by Lemma 4.14, there is a perfect matching in $G\overline{G}$ containing M as required. So we now consider $1 \leq m_{G\overline{G}} \leq 2$. We distinguish 2 cases according to $m_{G\overline{G}}$.

Case 1: $m_{G\overline{G}} = 1$. If $m_G = m_{\overline{G}} = 1$, then, by Lemma 4.16, there is a perfect matching in $G\overline{G}$ containing M as required. So we suppose without loss of generality that $m_G = 2$, $m_{\overline{G}} = 0$. By applying similar arguments as in the proof of Subcase 2.1 in Theorem 4.15, there is a perfect matching in $G\overline{G}$ containing M as required.

Case 2: $m_{G\overline{G}} = 2$. By applying similar arguments as in the proof of Case 1 in Theorem 4.15, there is a perfect matching in $G\overline{G}$ containing M as required. This completes the proof of our theorem. □

We point out here that the bound on the order of graphs in Theorem 4.17 is best possible and the hypothesis that G and $\overline{G}$ are non-bipartite is essential. Let G be a 3-regular bipartite graph of order 8 with bipartition (X, Y) where $X = \{x_i | 1 \leq i \leq 4\}$ and $Y = \{y_i | 1 \leq i \leq 4\}$ and $E(G) = \{x_i y_j | 1 \leq i \neq j \leq 4\}$. It is not difficult to show that $\overline{G} \cong K_4 \times K_2$ and both G and $\overline{G}$ are 2-extendable. However, $G\overline{G}$ is not 3-extendable since $\{x_1\bar{x}_1, x_2y_1, \bar{y}_2\bar{y}_3\}$ cannot be extended to a perfect matching in $G\overline{G}$.

We finally turn our attention to 3-extendable graphs.

Lemma 4.18. *Suppose G and $\overline{G}$ are 3-extendable non-bipartite graphs of order $p \geq 8$. Let $\{x, y, z_1, z_2, z_3\} \subseteq V(G)$ and $\{\bar{z}_1, \bar{z}_2, \bar{z}_3\} \subseteq V(\overline{G})$ such that $G[\{z_1, z_2, z_3\}] \cong K_3$. Further, let $M = \{xy, z_1\bar{z}_1, z_2\bar{z}_2, z_3\bar{z}_3\}$ be a matching of size 4 in $G\overline{G}$. Then there is a perfect matching in $G\overline{G}$ containing M.*

Proof. Suppose there is no perfect matching in $G\overline{G} - V(M)$. Then by Theorem 2.1, there is a cutset $T \subseteq V(G\overline{G}) - V(M)$ such that $c_o(G\overline{G} - (T \cup V(M))) > |T|$. By parity, $c_o(G\overline{G} - (T \cup V(M))) \geq |T| + 2$. Put $S = T \cup V(M)$. So $c_o(G\overline{G} - S) \geq |S| - 6$. Since $G\overline{G}$ contains a perfect matching, by Theorem 2.1, $c_o(G\overline{G} - S) \leq |S|$. Thus $|S| - 6 \leq c_o(G\overline{G} - S) \leq |S|$. Put $A = S \cap V(G)$, $\overline{B} = S \cap V(\overline{G})$ and $C = V(G) - (A \cup B)$. Clearly, $\{z_1, z_2, z_3\} \subseteq A \cap B$. By Lemma 3.2(b), $c_o(G\overline{G} - S) \leq |S| - 6$. So $c_o(G\overline{G} - S) = |S| - 6$.

Since $xy, z_1z_2 \in E(G)$, by Lemma 4.9, $c_o(G - A) \leq |A| - 4$. On the other hand, since $\overline{G}$ is 3-extendable non-bipartite graph, by Theorems 2.2(a) and 2.8, $\overline{G}$ is bicritical. Therefore, by Theorem 2.6, $c_o(\overline{G} - \overline{B}) \leq |\overline{B}| - 2$. We first show that $c_o(G - A) = |A| - 4$ and $c_o(\overline{G} - \overline{B}) = |\overline{B}| - 2$. Suppose to the contrary that $c_o(G - A) \neq |A| - 4$. By parity, $c_o(G - A) \leq |A| - 6$. By Lemma 4.13(a), $c_o(G\overline{G} - S) = c_o(G\overline{G} - (A \cup \overline{B})) \leq |A| - 6 + |\overline{B}| - 2 = |S| - 8$, a contradiction. Hence, $c_o(G - A) = |A| - 4$. By similar argument, $c_o(\overline{G} - \overline{B}) = |\overline{B}| - 2$. By Lemma 4.9, $G - A$ contains no even components. We next show that $\overline{G} - \overline{B}$ contains no even components. Suppose this is not the case. Then $\overline{G} - \overline{B}$ contains an even component, say $\overline{D}$. Let $\bar{b}\bar{d} \in E(\overline{G})$ such that $\bar{b} \in \overline{B}$ and $\bar{d} \in V(\overline{D})$. By Corollary 2.13, $\overline{G}' = \overline{G} - \{\bar{b}, \bar{d}\}$ is 2-extendable non-bipartite. By Theorem 2.8, $\overline{G}'$ is bicritical. Since $c_o(\overline{G} - (\overline{B} \cup \{\bar{d}\})) = |\overline{B}| - 1$, $c_o(\overline{G}' - (\overline{B} - \{\bar{b}\})) = |\overline{B} - \{\bar{b}\}|$, contradicting Theorem 2.6. Hence, $\overline{G} - \overline{B}$ contains no even components.

If $A \cup B \neq V(G)$, then by Lemma 4.13, $c_o(G\overline{G} - S) = c_o(G\overline{G} - (A \cup \overline{B})) \leq c_o(G - A) + c_o(\overline{G} - \overline{B}) - 2 = |A| + |\overline{B}| - 8 = |S| - 8$, a contradiction. So $A \cup B = V(G)$.

Note that $G[A - B]$ contains the edge xy. We first show that $G[A - B]$ contains exactly one independent edge. Suppose $G[A - B]$ contains 2 independent edges. Since $z_1 z_2 \in E(G[A \cap B])$, there are at least 3 independent edges in $G[A]$. Therefore, by Lemma 4.9, $c_o(G - A) \leq |A| - 6$, contradicting the fact that $c_o(G - A) = |A| - 4$. Hence, $G[A - B]$ contains exactly one independent edge. We next show that $\overline{G}[\overline{B}]$ contains no edges. Suppose to the contrary that $\overline{B}$ contains an edge $\bar{u}_1 \bar{u}_2$. By Corollary 2.13, $\overline{G} - \{\bar{u}_1, \bar{u}_2\}$ is 2-extendable non-bipartite graph. By Theorem 2.8, $\overline{G} - \{\bar{u}_1, \bar{u}_2\}$ is bicritical. Then, by Theorem 2.6, $c_o(\overline{G} - \overline{B}) = c_o((\overline{G} - \{\bar{u}_1, \bar{u}_2\}) - (\overline{B} - \{\bar{u}_1, \bar{u}_2\})) \leq |\overline{B} - \{\bar{u}_1, \bar{u}_2\}| - 2 = |\overline{B}| - 4$, contradicting the fact that $c_o(\overline{G} - \overline{B}) = |\overline{B}| - 2$. Hence, $\overline{G}[\overline{B}]$ contains no edges and $\overline{G}[\overline{B}]$ is independent. So $G[B]$ and $B - A$ are clique and thus $c_o(G[B - A]) \leq 1$.

Therefore, $|A| - 4 = c_o(G - A) = c_o(G[B - A]) \leq 1$. So $|A| \leq 5$. If $\overline{G}[\overline{A} - \overline{B}]$ contains at least 4 components, then $G[A - B]$ contains at least two independent edges. But this contradicts the fact that $G[A - B]$ contains exactly one independent edges. Hence, $\overline{G}[\overline{A} - \overline{B}]$ contains at most 3 components. Therefore, $c_o(\overline{G}[\overline{A} - \overline{B}]) = c_o(\overline{G} - \overline{B}) = |\overline{B}| - 2 \leq 3$. Hence, $|B| = |\overline{B}| \leq 5$. It follows that $|V(G)| = |A \cup B| = |A| + |B| - |A \cap B| \leq 5 + 5 - 3 = 7$, a contradiction. This proves our lemma. □

Lemma 4.19. *Suppose G and $\overline{G}$ are 3-extendable non-bipartite graphs of order $p \geq 8$. Let $\{x, y, z_1, z_2, z_3\} \subseteq V(G)$ and $\{\bar{z}_1, \bar{z}_2, \bar{z}_3\} \subseteq V(\overline{G})$ such that $G[\{z_1, z_2, z_3\}] \ncong K_3$. Further, let $M = \{xy, z_1\bar{z}_1, z_2\bar{z}_2, z_3\bar{z}_3\}$ be a matching of size 4 in $G\overline{G}$. Then there is a perfect matching in $G\overline{G}$ containing M.*

Proof. Suppose $M = \{xy, z_1\bar{z}_1, z_2\bar{z}_2, z_3\bar{z}_3\}$ where $x, y \in V(G)$. Since $G[\{z_1, z_2, z_3\}] \ncong K_3$, we may suppose that $z_1 z_2 \notin E(G)$. Since $xy \in E(G)$, by Lemma 4.8(a), there is a perfect matching in $G - \{x, y, z_1, z_2\}$, say F_G. Let $z_3 w \in F_G$. Again, because $\bar{z}_1\bar{z}_2 \in E(\overline{G})$, by Lemma 4.8(a), there is a perfect matching in $\overline{G} - \{\bar{z}_1, \bar{z}_2, \bar{w}, \bar{z}_3\}$, say $F_{\overline{G}}$. Thus $M \cup (F_G - \{z_3 w\}) \cup F_{\overline{G}} \cup \{w\bar{w}\}$ is a perfect matching in $G\overline{G}$ containing M as required. This completes the proof of our lemma. □

Theorem 4.20. *Let G be a 3-extendable non-bipartite graph of order $p \geq 8$. If $\overline{G}$ is l-extendable non-bipartite for some positive integer $l \geq 3$, then $G\overline{G}$ is 4-extendable.*

Proof. By Theorem 2.2(b), $\overline{G}$ is 3-extendable non-bipartite graph. Let M be a matching of size 4 in $G\overline{G}$. Put $M_G = M \cap E(G)$, $M_{\overline{G}} = M \cap E(\overline{G})$ and $M_{G\overline{G}} = M - (M_G \cup M_{\overline{G}})$. Without loss of generallity, suppose $|M_G| \geq |M_{\overline{G}}|$. If $M_{G\overline{G}} = \phi$ or $M_{G\overline{G}} = M$, then, by Lemma 4.14, there is a perfect matching in $G\overline{G}$ containing M as required. So we now suppose that $M_{G\overline{G}} \neq \phi$ and $M_{G\overline{G}} \neq M$. Therefore, $1 \leq |M_{G\overline{G}}| \leq 3$. We distinguish 3 cases according to $|M_{G\overline{G}}|$.

Case 1: $|M_{G\overline{G}}| = 1$. By applying similar arguments as in the proof of Subcase 2.1 (if $|M_{\overline{G}}| = 0$) or Subcase 2.3 (if $|M_{\overline{G}}| = 1$) in Theorem 4.15, there is a perfect matching in $G\overline{G}$ containing M as required.

Case 2: $|M_{G\overline{G}}| = 2$. By applying similar arguments as in the proof of Case 1 in Theorem 4.15, there is a perfect matching in $G\overline{G}$ containing M as required.

Case 3: $|M_{G\overline{G}}| = 3$. Then, $|M_G| = 1$ and $|M_{\overline{G}}| = 0$. So, by Lemmas 4.18 and 4.19, there is a perfect matching in $G\overline{G}$ containing M as required.

This completes the proof of our theorem. □

The next Theorem follows by Theorems 4.15, 4.17 and 4.20.

Theorem 4.21. *For positive integers $l_1 \geq 2$, $l_2 \geq 2$, let G and $\overline{G}$ be l_1-extendable and l_2-extendable non-bipartite graphs, respectively. Then $G\overline{G}$ is $(l+1)$-extendable, where $l = min\{l_1, l_2\}$.*

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