EFFECT OF VERTEX-REMOVAL ON GAME TOTAL DOMINATION NUMBERS

By

MISS Karnchana CHAROENSITTHICHAI

A Thesis Submitted in Partial Fulfillment of the Requirements

for Master of Science (MATHEMATICS)

Department of MATHEMATICS

Graduate School, Silpakorn University

Academic Year 2018

ผลกระทบของการลบจุดต่อค่าโดมิเนชันเกมแบบรวม

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Title	Effect of vertex-removal on game total domination numbers
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MISS KARNCHANA CHAROENSITTHICHAI : EFFECT OF VERTEX-REMOVAL ON GAME TOTAL DOMINATION NUMBERS THESIS ADVISOR : ASSISTANT PROFESSOR CHALERMPONG WORAWANNOTAI, Ph.D.

The total domination game is played on a simple graph G with no isolated vertices by two players, named Dominator and Staller. They alternately select a vertex of G; each chosen vertex totally dominates its neighbors. In this game, each chosen vertex must totally dominates at least one new vertex not totally dominated before. The game ends when all vertices in G are totally dominated. Dominator's goal is to finish the game as soon as possible, and Staller's goal is to prolong it as much as possible. The game total domination number is the number of chosen vertices when both players play optimally, denoted by $\gamma_{tg}(G)$ when Dominator starts the game and denoted by $\gamma'_{tg}(G)$ when Staller starts the game.

In this thesis, we show that for any graph G and a vertex v such that $G - v$ has no isolated vertex, we have $\gamma_{tg}(G) - \gamma_{tg}(G - v) \leq 2$ and $\gamma'_{tg}(G) - \gamma'_{tg}(G - v) \leq 2$. Moreover, all such differences can be realized by some connected graphs.

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Karnchana CHEROENSITTHICHAI

TABLE OF CONTENTS

Page

LIST OF FIGURES

Chapter 1

Introduction

A subset S of vertices in a graph G is called a *dominating set* if every vertex not in S is adjacent to some vertex in S. The *domination number* of G is the minimum cardinality among dominating sets for G, denoted by $\gamma(G)$. Domination is one of the most popular topics in graph theory. So far there are already over a thousand papers on this topic. For more detail about domination we refer the readers to the books by Haynes, Hedetniemi, and Slater [5, 6].

Domination has many practical applications such as the transportation route planning, the security system design, and the wireless network installation. Allocation of utilities efficiently in such a way that everybody has access to utilities and the cost is minimum is an important objective to study in domination. There are many variations of domination. In this thesis, we study a combination of two variations, domination game and total domination, which is called the total domination game. First we recall the notion of domination game.

The domination game was introduced by Brešar, Klavžar, and Rall [2] in 2010, where the original idea was attributed to Henning in 2003. This game is played on a graph G by two players, *Dominator* and *Staller*, who alternate taking turns choosing a vertex from G. Playing a vertex will make all vertices in its closed neighborhood dominated. A vertex is *valid* to choose or *legal* if at least one additional vertex is dominated by playing that vertex. The game ends when the chosen vertices form a dominating set, i.e., all vertices are dominated. Dominator's goal is to finish the game as soon as possible, and Staller's goal is to prolong it as much as possible. Note that a domination game is a game without the winner or loser, but the players want to play optimally according to their purposes. The *game domination number* is the size of the dominating set of chosen vertices when both players play optimally, denoted by $\gamma_g(G)$ when Dominator starts the game and denoted by $\gamma_g'(G)$ when Staller starts the game.

Example 1.1. Let G be the graph in Figure 1.1. Then the set $\{c, d\}$ is a dominating set of G. Since there exists no vertex adjacent to all other vertices, we get that $\gamma(G) = 2$. It implies that both games use at least 2 moves. Now let's consider the Dominator-start game. Dominator can play on vertex c. Then e is the only undominated vertex and Staller is forced to dominate e in his turn. Therefore, $\gamma_g(G) \leq 2$. We can conclude that $\gamma_g(G) = 2$. Finally let's consider the Staller-start game. No matter how Staller starts the game, Dominator can end the game by playing an appropriate vertex from $\{c, d\}$. Thus $\gamma_g'(G) \leq 2$. We can conclude that $\gamma_g'(G) = 2$.

Now we present the total version of domination. A vertex u *totally dominates* another vertex v if they are adjacent. A set S of vertices of a graph G is a *total dominating set*, abbreviated TD-set, if every vertex of G is totally dominated by some vertex in S. Note that in total domination a vertex does not

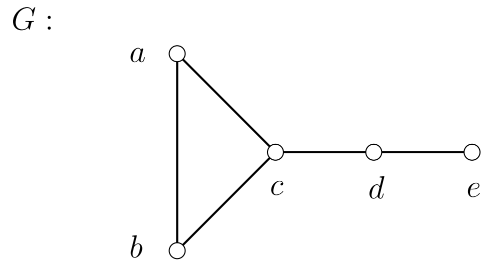

Figure 1.1: Graph G

dominate itself, so it is required that there is no isolated vertex in a graph. All graphs considered here have no isolated vertex. The *total domination number* of a graph G is the minimum cardinality of a total dominating set of G, denoted by $\gamma_t(G)$. For any graph G which has no isolated vertex, $\gamma_t(G)$ exists and $\gamma_t(G) \geq 2$.

Finally we present a total domination game. A *total domination game* was recently introduced in [7] as follows. Two players, named Dominator and Staller, alternate taking turns choosing a vertex from a graph G. Each chosen vertex must totally dominate at least one new vertex not totally dominated before. The game ends when the set of chosen vertices is a total dominating set of G. Dominator's goal is to finish the game as soon as possible, and Staller's goal is to prolong it as much as possible. The *game total domination number* is the size of the total dominating set of chosen vertices when both players play optimally, denoted by $\gamma_{tg}(G)$ when Dominator starts the game and denoted by $\gamma'_{tg}(G)$ when Staller starts the game.

There are many results about the effect of graph operations on domination game. In 2010, Brešar, Klavžar, and Rall [2] showed a lower bound on the game domination number of an arbitrary Cartesian product of two graphs. In 2014, Brešar, Dorbec, Klavžar, and Košmrlj [1] showed that removing an edge of a graph can change the game domination numbers by at most 2. They also showed that removing a vertex of a graph can decrease the game domination numbers by at most 2 or increase them by any amount. In 2015, Dorbec, Košmrlj, and Renault [4] showed how the game domination number of the union of two no-minus graphs corresponds to the game domination numbers of the initial graphs. In 2018, Onphaeng, Ruksasakcha and Worawannotai [10] showed the game domination numbers of a disjoint union of paths and cycles.

Most recently, Iršič [8] studied the effect of a vertex removal on a total domination game. In particular, she showed that removing a vertex of a graph can decrease the game total domination numbers by at most 4 or increase them by any amount. However, the results are not sharp. In this thesis, we also study the effect of a vertex removal on a total domination game and obtain sharp bounds (in fact, we were unaware of [8] until recently). In chapter 2, we recall some definitions and known results of game total domination numbers. In chapter 3, we show that removing a vertex of a graph can decrease the game total domination numbers by at most 2 or increase them by any amount. Finally, in chapter 4, we determine all pairs (a, b) such that there is a graph G and a vertex v with $(\gamma_{tg}(G), \gamma_{tg}(G-v)) = (a, b)$, and determine all pairs (c, d) such that there is a graph G and a vertex v with $(\gamma'_{tg}(G), \gamma'_{tg}(G-v)) = (c, d)$.

Chapter 2

Preliminaries

In this chapter, we recall some definitions and useful results.

First, we introduce some terminologies, backgrounds, and concepts of graph theory. A graph $G = (V(G), E(G))$ consists of a set $V(G)$ of vertices and a set $E(G)$ of edges where each edge is identified with an unordered pair of vertices (not necessary distinct vertices). Two vertices are *adjacent* if they are connected by an edge; they are also the *end vertices* of the edge, and the edge is said to be *incident* to each of its end vertices. *Multiple edges* are two or more edges that are incident to the same two vertices. A *loop* is an edge connecting a vertex to itself. A graph without loops or multiple edges is called a *simple graph*. From now on, we only consider simple graphs.

For any graph G, the number of vertices in G is called the *order* of G and it is denoted by $|V(G)|$ or $|G|$. The *open neighborhood* of a vertex v of G is the set of vertices adjacent to v, denoted by $N_G(v)$, and the *closed neighborhood* of a vertex v of G is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of v, denoted by $deg(v)$, is the number of edges incident with v, or equivalently, $deg(v) = |N_G(v)|$. A vertex of degree 1 is called a *pendant vertex*, and a vertex of degree 0 is called an *isolated* vertex.

Consider a graph G, a graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a subset S of $V(G)$, the graph $G-S$ is the graph obtained from G by removing all vertices in S and all edges incident with vertices in S. Observe that a graph $G-S$ is a subgraph of G for all $S \subseteq V(G)$. If $S = \{v\}$, we write $G-v$. We say that a subset S of $V(G)$ is an *independent set* if no two vertices in S are adjacent. We say G is *connected* if for any pair $u, v \in V(G)$, there exists a sequence $u_0e_1u_1e_2u_2...e_ku_k$ of distinct vertices and edges where $u = u_0, v = u_k$ and $e_i = u_{i-1}u_i$ for all $i = 1, 2, ..., k$. Otherwise, G is *disconnected*.

Example 2.1. Let T be the graph in Figure 2.1. We see that T is connected, $\{u, v\}$ is an independent set, and $T - \{u, v\}$ is a disconnected subgraph of T.

Figure 2.1: Graph T and graph $T - \{u, v\}$

Next, we introduce some families of graphs which are used in this work. A *path* P_n is a graph whose vertices can be listed in the order $v_1, v_2, ..., v_n$ such that v_i and v_{i+1} are adjacent where $i = 1, 2, ..., n-1$. A *cycle* C_n is a connected graph of order n such that every vertex has degree 2. A connected graph with no cycles is called a *tree*. A pendant vertex in a tree is called a *leaf*. Moreover, a disjoint union of trees is called a *forest*. In Figure 2.1, the graph T is a tree, the vertex v is a leaf and the graph $T - \{u, v\}$ is a forest.

A *complete graph* K_n is a graph of order n such that every two distinct vertices are adjacent. A graph G is *bipartite* if its vertex set can be partitioned into two disjoint sets X and Y such that each edge of G incident with one vertex in X and the other in Y. The sets X and Y are called the *partite sets* and the pair (X, Y) is called a *bipartition* of the bipartite graph. A bipartite graph G with bipartition (X, Y) is a *complete bipartite* graph if each vertex of X is adjacent to all the vertices of Y. In this case, G is denoted by $K_{m,n}$, where $|X| = m$ and $|Y| = n$. As an example, the complete graph K_4 and the complete bipartite graph $K_{2,3}$ are shown in Figure 2.2.

K_4 : $K_{2,3}$:

Figure 2.2: The complete graph K_4 and the complete bipartite graph $K_{2,3}$

The *corona* of two graphs G_1 and G_2 is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2. In particular, the corona $H \circ K_1$ is the graph obtained from H by joining each vertex in H to a new vertex. Figure 2.3 shows the corona of P_4 and K_1.

Finally, some significant definitions, theorems and lemmas for total domination games are shown below. In a total domination game, if Dominator starts

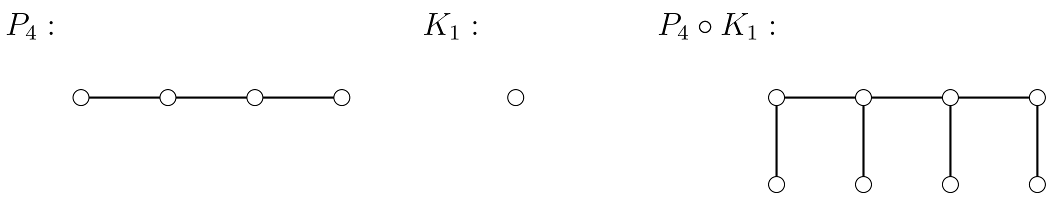

Figure 2.3: The corona $P_4 \circ K_1$

the game, this game is said to be a *Domination-start game*. Otherwise, it is said to be a *Staller-start game*. By the definition of game total domination numbers, we have the following lemma.

Lemma 2.2. *Let G be a graph. Then the following statements hold.*

(i) *For a Dominator-start game, if Dominator has a strategy that can end the game within k moves, then $\gamma_{tg}(G) \leq k$.*

(ii) *For a Staller-start game, if Dominator has a strategy that can end the game within k moves, then $\gamma'_{tg}(G) \leq k$.*

(iii) *For a Dominator-start game, if Staller has a strategy that can end the game with at least k moves, then $\gamma_{tg}(G) \geq k$.*

(iv) *For a Staller-start game, if Staller has a strategy that can end the game with at least k moves, then $\gamma'_{tg}(G) \geq k$.*

Lemma 2.2 is useful for determining game total domination numbers of graphs. We can get the upper bound by presenting an appropriate Dominator's strategy and get the lower bound by presenting an appropriate Staller's strategy.

Example 2.3. Recall the graph G in Example 1.1. We see that the set $\{c, d\}$ is a minimum total dominating set of G. It follows that $\gamma_t(G) = 2$. Now let's consider the Dominator-start game. Dominator can try c first and then d. This ensures that the game ends within 3 moves. Therefore $\gamma_{tg}(G) \leq 3$. Observe that Staller can always totally dominate at most one new vertex in his first turn and thus prolong the game to at least 3 turns. Therefore $\gamma_{tg}(G) \geq 3$. We can conclude that $\gamma_{tg}(G) = 3$. Finally let's consider the Staller-start game. Dominator can try playing vertices from $\{c, d\}$ and ensures that the game ends within 4 moves. Thus $\gamma'_{tg}(G) \leq 4$. Observe that Staller can totally dominate only one new vertex in each of his first two moves. So he can prolong the game to at least 4 turns and thus $\gamma'_{tg}(G) \geq 4$. We can conclude that $\gamma'_{tg}(G) = 4$.

If Dominator plays his moves in a graph G by playing vertices in a minimum TD-set of G, then he guarantees that the game will end within $2\gamma_t(G) - 1$ moves. By Lemma 2.2 (i), we have the following theorem.

Theorem 2.4. *If a graph G has at least two vertices, then $\gamma_t(G) \leq \gamma_{tg}(G) \leq 2\gamma_t(G) - 1$.*

For a graph G and a subset $S \subseteq V(G)$, we denote by $G|S$ the *partially totally dominated graph* G where the vertices of S are considered already totally dominated in the game. In particular, if $S = \{x\}$, we write $G|x$. We can find the game total domination numbers of $G|S$ by considering only the number of vertices chosen after S is totally dominated.

In a partially totally dominated graph G, A vertex v in G is *saturated* if all vertices in its closed neighborhood are totally dominated. The *residual graph* of a partially totally dominated graph G is the graph obtained from G by removing all saturated vertices and all edges joining totally dominated vertices. Note that a partially totally dominated graph and its residual graph have the same game total domination numbers. In particular, for any graph G and a vertex v of G, we have $\gamma_{tg}(G|N_G[v]) = \gamma_{tg}(G - v|N_G(v))$ and $\gamma'_{tg}(G|N_G[v]) = \gamma'_{tg}(G - v|N_G(v))$.

Example 2.5. Let G be the graph in Figure 2.4 and let $S = \{b, d, e, f\}$. Then $G|S$ is the partially totally dominated G with S totally dominated and vertex e is saturated. And $G|S - e$ is the residual graph of $G|S$.

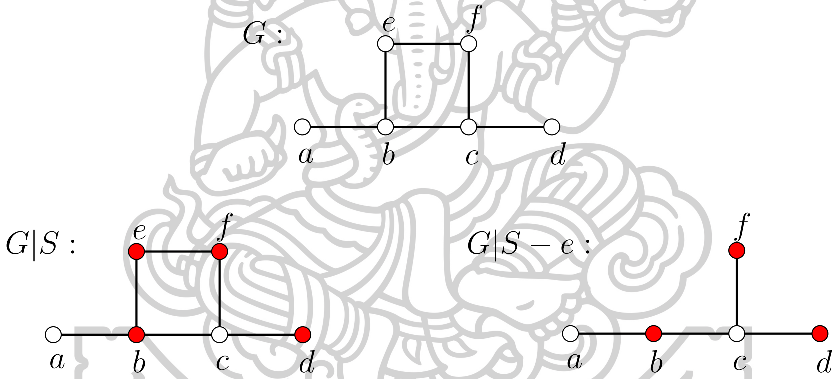

Figure 2.4: The graph G, partially totally dominated graph G, and residual graph of $G|S$

Theorem 2.6 ([7, Lemma 2.1 (Total Continuation Principle)]). *Let G be a graph and $A, B \subseteq V(G)$. If $B \subseteq A$, then $\gamma_{tg}(G|A) \le \gamma_{tg}(G|B)$ and $\gamma'_{tg}(G|A) \le \gamma'_{tg}(G|B)$.*

Total Continuation Principle is proved by Henning, Klavžar, and Rall [7]. It is useful for comparing certain choices of a move by a player. In Example 2.5, we see that $N_G(a) \subseteq N_G(e)$. By Total Continuation Principle, we have $\gamma_{tg}(G|N_G(e)) \leq \gamma_{tg}(G|N_G(a))$ and $\gamma'_{tg}(G|N_G(e)) \leq \gamma'_{tg}(G|N_G(a))$. So in Dominator-start game, starting on e is better than or as good as starting on a for Dominator. And in Staller-start game, starting on a is better than or as good as starting on e for Staller. However, this principle may not be able to guarantee that these vertices are optimal for the players. Henning, Klavžar, and Rall [7] also showed a relationship of the two types of game total domination numbers.

Theorem 2.7 ([7, Theorem 2.2])**.** *For any graph G, we have $|\gamma_{tg}(G) - \gamma'_{tg}(G)| \leq 1$.*

Moreover, we recall some useful results of the game total domination numbers of some families of graphs.

Proposition 2.8 ([7, Proposition 2.3])**.** *Let G_0 be a graph. For positive integer r, let G_i, $1 \leq i \leq r$, be graphs with $\gamma_{tg}(G_i) = \gamma'_{tg}(G_i) = 2$. Then $\gamma_{tg}(\cup_{i=0}^{r} G_i) = \gamma_{tg}(G_0) + 2r$ and $\gamma'_{tg}(\cup_{i=0}^{r} G_i) = \gamma'_{tg}(G_0) + 2r$.*

Consider a graph G with $\gamma_{tg}(G) = \gamma'_{tg}(G) = 2$, one of examples is a graph G such that there is a vertex v adjacent to every other vertex. In this case, we say that v is a *universal* vertex. For Dominator-start game, Dominator can start on a universal vertex v so only v is still undominated and $\gamma_{tg}(G) \leq 2$. For Staller-start game, Dominator can end the game by replying on v if Staller does not start on v. So $\gamma'_{tg}(G) \leq 2$. Since $\gamma_{tg}(G) \geq 2$ and $\gamma'_{tg}(G) \geq 2$, it implies that $\gamma_{tg}(G) = \gamma'_{tg}(G) = 2$. For Dominator-start game, we characterize graphs G with $\gamma_{tg}(G) = 2$ in Chapter 4.

In 2016, Dorbec and Henning [3] determined the game total domination numbers for cycles and paths as follows.

Theorem 2.9 ([3]). *For $n \geq 3$,*

$$\gamma_{tg}(C_n) = \begin{cases} \left\lfloor \frac{2n+1}{3} \right\rfloor - 1; & n \equiv 4 \mod 6 \\ \left\lfloor \frac{2n+1}{3} \right\rfloor; & \text{otherwise}, \end{cases}$$

$$\gamma'_{tg}(C_n) = \begin{cases} \left\lfloor \frac{2n}{3} \right\rfloor - 1; & n \equiv 2 \mod 6 \\ \left\lfloor \frac{2n}{3} \right\rfloor; & \text{otherwise}, \end{cases}$$

$$\gamma_{tg}(P_n) = \begin{cases} \left\lfloor \frac{2n}{3} \right\rfloor; & n \equiv 5 \mod 6 \\ \left\lceil \frac{2n}{3} \right\rceil; & \text{otherwise}, \end{cases}$$

$$\gamma'_{tg}(P_n) = \left\lceil \frac{2n}{3} \right\rceil.$$

Chapter 3

Vertex removal

In this chapter, we show that removing a vertex can increase the total game domination numbers by at most 2 or decrease them by any amount.

Theorem 3.1. *For any graph G and a vertex $v \in V(G)$ such that $G - v$ has no isolated vertex, we have $\gamma_{tg}(G) - \gamma_{tg}(G - v) \leq 2$.*

Proof. We show that Dominator has a strategy that can end the game in G within $\gamma_{tg}(G-v)+2$ moves. His strategy is to play the game on G as follows. Dominator imagines playing another game on $G - v$ by copying every move of Staller to this game and responds in $G - v$. Each response of Dominator in the imagined game is copied back to the real game in G. In G, Staller plays using an optimal strategy while Dominator is responding optimally in $G - v$. For any vertex u in $G - v$, we have $N_{G-v}[u] \subseteq N_G[u]$ so Dominator can always copy the move into G. Consider the following possibilities:

Case 1 all the moves are legal in both two games. By Lemma 2.2, the number of moves for $G - v$ is less than or equal to $\gamma_{tg}(G - v)$. In G, after $\gamma_{tg}(G - v)$ moves all vertices except maybe v are totally dominated. So in G at most $\gamma_{tg}(G - v) + 1$ moves are played.

Case 2 Staller played the k-th move in G but this move is not legal in $G - v$. Let

X be the set of totally dominated vertices after $k-1$ moves in $G-v$.

In $G-v$, we have

$$(k-1)+\gamma'_{tg}(G-v|X) \leq \gamma_{tg}(G-v). \tag{3.1}$$

Subcase 2.1 Staller played the k-th move such that only v was totally dominated among vertices that have not yet been totally dominated in G. Then the set of totally dominated vertices after $k-1$ moves are played in G is equal to X.

If $N_G(v) \subseteq X$, then we get

$$\begin{aligned}
\gamma_{tg}(G) &\leq k+\gamma_{tg}(G|X\cup\{v\}) \\
&= k+\gamma_{tg}(G-v|X) && \text{(since } v \text{ is saturated in } G|X\cup\{v\}) \\
&\leq k+\gamma'_{tg}(G-v|X)+1 && \text{(by Theorem 2.7)} \\
&\leq \gamma_{tg}(G-v)+2. && \text{(by (3.1))}
\end{aligned}$$

If there is a vertex in $N_G(v)$ which is not totally dominated after the k-th move, then Dominator can reply with the $(k+1)$-th move on v in G. Therefore

$$\begin{aligned}
\gamma_{tg}(G) &\leq k+1+\gamma'_{tg}(G|X\cup N_G[v]) \\
&= k+1+\gamma'_{tg}(G-v|X\cup N_G(v)) && \text{(since } v \text{ is saturated in } G|X\cup N_G[v]) \\
&\leq k+1+\gamma'_{tg}(G-v|X) && \text{(by Total Continuation Principle)} \\
&\leq \gamma_{tg}(G-v)+2. && \text{(by (3.1))}
\end{aligned}$$

Subcase 2.2 Staller played the vertex v in the k-th move.

If v was totally dominated, then we get

$$\begin{aligned}
\gamma_{tg}(G) &\leq k + \gamma_{tg}(G|X \cup N_G[v]) \\
&= k + \gamma_{tg}(G - v|X \cup N_G(v)) && \text{(since } v \text{ is saturated in } G|X \cup N_G[v]) \\
&\leq k + \gamma_{tg}(G - v|X) && \text{(by Total Continuation Principle)} \\
&\leq k + \gamma'_{tg}(G - v|X) + 1 && \text{(by Theorem 2.7)} \\
&\leq \gamma_{tg}(G - v) + 2. && \text{(by (3.1))}
\end{aligned}$$

If v is not totally dominated, then Dominator can reply with the $(k+1)$-th move on a vertex adjacent with v, say w, in G. Therefore

$$\begin{aligned}
\gamma_{tg}(G) &\leq k + 1 + \gamma'_{tg}(G|X \cup N_G[v] \cup N_G[w]) \\
&= k + 1 + \gamma'_{tg}(G - v|X \cup N_G(v) \cup N_G[w]) && \text{(since } v \text{ is saturated in} \\
& && G|X \cup N_G[v] \cup N_G[w]) \\
&\leq k + 1 + \gamma'_{tg}(G - v|X) && \text{(by Total Continuation Principle)} \\
&\leq \gamma_{tg}(G - v) + 2. && \text{(by (3.1))}
\end{aligned}$$

□

In the Staller-start game, we have a similar result and the proof uses a similar argument.

Theorem 3.2. *For any graph G and a vertex $v \in V(G)$ such that $G - v$ has no isolated vertex, we have $\gamma'_{tg}(G) - \gamma'_{tg}(G - v) \leq 2$.*

Proof. We show that Dominator has a strategy that can end the game in G within $\gamma'_{tg}(G - v) + 2$ moves. His strategy is to play the game on G as follows. Dominator imagines playing another game on $G-v$ by copying every move of Staller to this game and responds in $G-v$. Each response of Dominator's in the imagined game is copied back to the real game in G. In G, Staller plays using an optimal strategy while Dominator is responding optimally in $G-v$. For any vertex u in $G-v$, we have $N_{G-v}[u] \subseteq N_G[u]$ so Dominator can always copy the move into G. Consider the following possibilities:

Case 1 all the moves are legal in both two games. By Lemma 2.2, the number of moves for $G-v$ is less than or equal to $\gamma'_{tg}(G-v)$. In G, after $\gamma'_{tg}(G-v)$ moves all vertices except maybe v are totally dominated. So in G at most $\gamma'_{tg}(G-v)+1$ moves are played.

Case 2 Staller played the k-th move in G but this move is not legal in $G-v$. Let X be the set of totally dominated vertices after $k-1$ moves in $G-v$.

In $G-v$, we have

$$(k-1)+\gamma'_{tg}(G-v|X) \leq \gamma'_{tg}(G-v). \tag{3.2}$$

Subcase 2.1 Staller played the k-th move such that only v was totally dominated among vertices that have not yet been totally dominated in G. Then the set of totally dominated vertices after $k-1$ moves are played in G is equal to X.

If $N_G(v) \subseteq X$, then we get

$$\begin{aligned} \gamma'_{tg}(G) &\le k + \gamma_{tg}(G|X \cup \{v\}) \\ &= k + \gamma_{tg}(G - v|X) && \text{(since } v \text{ is saturated in } G|X \cup \{v\}) \\ &\le k + \gamma'_{tg}(G - v|X) + 1 && \text{(by Theorem 2.7)} \\ &\le \gamma'_{tg}(G - v) + 2. && \text{(by (3.2))} \end{aligned}$$

If there is a vertex in $N_G(v)$ which is not totally dominated after the k-th move, then Dominator can reply with the $(k+1)$-th move on v in G. Therefore

$$\begin{aligned} \gamma'_{tg}(G) &\le k + 1 + \gamma'_{tg}(G|X \cup N_G[v]) \\ &= k + 1 + \gamma'_{tg}(G - v|X \cup N_G(v)) && \text{(since } v \text{ is saturated in } G|X \cup N_G[v]) \\ &\le k + 1 + \gamma'_{tg}(G - v|X) && \text{(by Total Continuation Principle)} \\ &\le \gamma'_{tg}(G - v) + 2. && \text{(by (3.2))} \end{aligned}$$

Subcase 2.2 Staller played the vertex v in the k-th move.

If v was totally dominated, then we get

$$\begin{aligned} \gamma'_{tg}(G) &\le k + \gamma_{tg}(G|X \cup N_G[v]) \\ &= k + \gamma_{tg}(G - v|X \cup N_G(v)) && \text{(since } v \text{ is saturated in } G|X \cup N_G[v]) \\ &\le k + \gamma_{tg}(G - v|X) && \text{(by Total Continuation Principle)} \\ &\le k + \gamma'_{tg}(G - v|X) + 1 && \text{(by Theorem 2.7)} \\ &\le \gamma'_{tg}(G - v) + 2. && \text{(by (3.2))} \end{aligned}$$

If v is not totally dominated, then Dominator can reply with the $(k+1)$-th move on a vertex adjacent with v, say w, in G. Therefore

$$\begin{aligned}
\gamma'_{tg}(G) &\leq k+1+\gamma'_{tg}(G|X \cup N_G[v] \cup N_G[w]) \\
&= k+1+\gamma'_{tg}(G-v|X \cup N_G(v) \cup N_G[w]) && \text{(since } v \text{ is saturated in} \\
& && G|X \cup N_G[v] \cup N_G[w]) \\
&\leq k+1+\gamma'_{tg}(G-v|X) && \text{(by Total Continuation Principle)} \\
&\leq \gamma'_{tg}(G-v)+2. && \text{(by (3.2))}
\end{aligned}$$

□

Chapter 4

Realization

In this section, we show that all possibilities in Theorem 3.1 and Theorem 3.2 can be realized by infinite families of connected graphs. More precisely, we find all pairs (a, b) such that there is a graph G and a vertex v with $(\gamma_{tg}(G), \gamma_{tg}(G - v)) = (a, b)$, and all pairs (c, d) such that there is a graph G and a vertex v with $(\gamma'_{tg}(G), \gamma'_{tg}(G - v)) = (c, d)$. First we consider the Dominator-start game. We begin by characterizing graphs G with $\gamma_{tg}(G) = 2$.

Lemma 4.1. *For any graph G, we have $\gamma_{tg}(G) = 2$ if and only if G is a complete bipartite graph or is obtained from a complete bipartite graph with nonempty partite sets by adding edges to one of the partite sets.*

Proof. Let G be a graph. If G is a complete bipartite graph, then $\gamma_{tg}(G) = 2$. Assume that G is obtained from a complete bipartite graph with nonempty partite sets by adding edges to one of the partite sets. Let A be the partite set of G without edges and let B be the other partite set of G. If Dominator plays his first move on a vertex in A, then all vertices in B are totally dominated. Since A is an independent set, Staller is forced to play a vertex in B and then all vertices in A are totally dominated. Therefore $\gamma_{tg}(G) \le 2$. Since $\gamma_{tg}(G) \ge 2$, it implies that $\gamma_{tg}(G) = 2$.

Assume that $\gamma_{tg}(G) = 2$. We show that G is a complete bipartite graph or is obtained from a complete bipartite graph with nonempty partite sets by adding edges to one of the partite sets. Let x be an optimal first move of Dominator. Since $\gamma_{tg}(G) = 2$, each vertex in $N_G(x)$ is adjacent to all remaining undominated vertices. Suppose that there are two vertices y, z in $G \setminus N_G(x)$ which are adjacent. Then Staller can play y and y is still undominated. So $\gamma_{tg}(G) \geq 3$. It is a contradiction with $\gamma_{tg}(G) = 2$. Therefore there are no edges in $G \setminus N_G(x)$. If $N_G(x)$ has no edges, then G is the complete bipartite graph with bipartition $(N_G(x), V(G) \setminus N_G(x))$. Otherwise, G is obtained from a complete bipartite graph with nonempty partite sets by adding edges to one the partite sets. $\square$

4.1 $\gamma_{tg}(G) - \gamma_{tg}(G - v) = 2$

Proposition 4.2. *For any positive integer $l \geq 5$, there exists a graph G with a vertex v such that $\gamma_{tg}(G) = l$ and $\gamma_{tg}(G - v) = l - 2$. In particular, $\gamma_{tg}(G) - \gamma_{tg}(G - v) = 2$.*

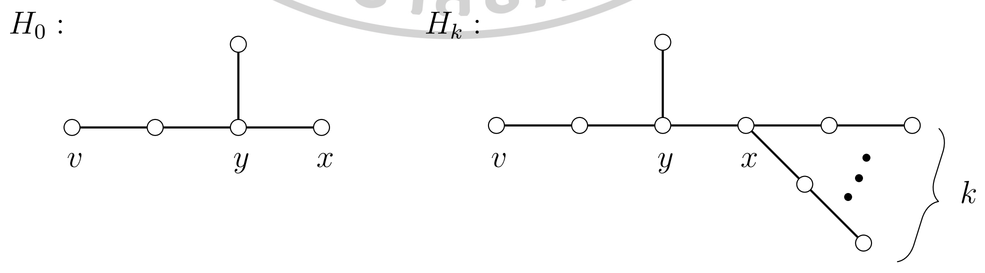

Figure 4.1: Graph H_0 and Graph H_k

Proof. Let H_0 and x be the graph and the vertex shown in Figure 4.1. For $k \geq 1$, the graph H_k is obtained from H_0 by identifying the left end vertices of k copies of P_3 with x, see Figure 4.1. Let $l = k + 4$ and $G = H_k$. We claim that $\gamma_{tg}(H_k) = k + 4 = l$ and $\gamma_{tg}(H_k - v) = k + 2 = l - 2$. By Theorem 3.1, it suffices to show that $\gamma_{tg}(H_k) \geq k + 4$ and $\gamma_{tg}(H_k - v) \leq k + 2$.

First, we show that Dominator has a strategy that can end the game in $H_k - v$ within $k + 2$ moves as follows. Dominator plays his first move on x. Then any first move of Staller totally dominates at least two vertices. After that, there are k undominated vertices any two of which have no common neighbors. Therefore the number of moves for this graph is at most $k + 2$. It implies that $\gamma_{tg}(H_k - v) \leq k + 2$.

Lastly, we show that Staller has a strategy that can end the game in H_k using at least $k + 4$ moves as follows. If Dominator starts on x, then Staller plays on v. Excluding x, there are $k + 2$ undominated vertices any two of which have no common neighbors. So at least $k + 4$ moves are played. If Dominator does not start on x, then Staller can force at least 3 moves to totally dominate H_0. Since at least $k + 1$ moves are played in k copies of P_3, this strategy can end the game using at least $k + 4$ moves. Therefore $\gamma_{tg}(H_k) \geq k + 4$. $\square$

Next, we show that there exists no graph G and a vertex v with $\gamma_{tg}(G) = 4$ and $\gamma_{tg}(G - v) = 2$. Indeed, we have the following:

Lemma 4.3. *Let G be a graph and let v be a vertex in G. If $\gamma_{tg}(G) = 4$, then $\gamma_{tg}(G - v) \geq 3$.*

Proof. Let G be a graph with $\gamma_{tg}(G) = 4$. Suppose that $\gamma_{tg}(G-v) = 2$. By Lemma 4.1, the graph $G - v$ is a complete bipartite graph or is obtained from a complete bipartite graph with nonempty partite sets by adding edges to one of the partite sets. Then there are adjacent vertices x and y in G such that x is adjacent to v and $N_G(x) \cup N_G(y) = V(G)$. Dominator can play x and then y (if the game has not ended) so the game ends within 3 moves. So $\gamma_{tg}(G) \leq 3$, a contradiction. $\square$

4.2 $\gamma_{tg}(G) - \gamma_{tg}(G-v) = 1$

For $k \geq 2$, let v be a pendant vertex on $K_k \circ K_1$, see Figure 4.2. We show that $\gamma_{tg}(K_k \circ K_1) = k+1$ and $\gamma_{tg}((K_k \circ K_1) - v) = k$. Let x be the vertex in $K_k \circ K_1$ that is adjacent to v.

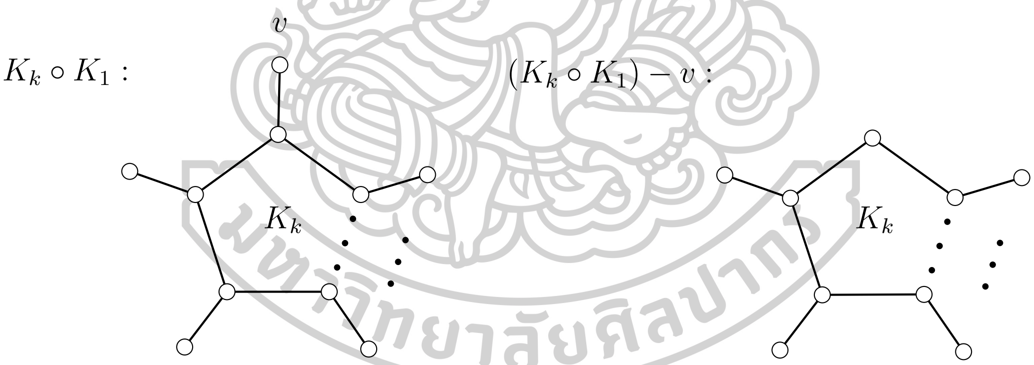

Figure 4.2: Graph $K_k \circ K_1$ and Graph $(K_k \circ K_1) - v$

In graph $K_k \circ K_1$, by Total Continuation Principle we can assume that Dominator plays his first move on x and then Staller plays his first move on v. So there are $k-1$ undominated vertices any two of which have no common neighbors. Hence $\gamma_{tg}(K_k \circ K_1) = k + 1$.

We show that Dominator has a strategy that can end the game in $(K_k \circ K_1) - v$ within k moves as follows. Dominator starts on x. Then Staller is forced to totally dominate two new vertices. After that there are $k-2$ undominated vertices left. Thus $\gamma_{tg}((K_k \circ K_1) - v) \leq 2 + (k-2) = k$.

We show that Staller has a strategy that can end the game in $(K_k \circ K_1) - v$ using at least k moves as follows. If Dominator starts on x, then the number of moves for $(K_k \circ K_1) - v$ is equal to k. Otherwise, Staller responds by playing a pendant vertex. Then there are at least $k-2$ undominated vertices any two of which have no common neighbors. Thus $\gamma_{tg}((K_k \circ K_1) - v) \geq 2 + (k-2) = k$. Hence $\gamma_{tg}((K_k \circ K_1) - v) = k$.

4.3 $\gamma_{tg}(G) - \gamma_{tg}(G - v) = 0$

Figure 4.3: Graph G and Graph $G - v$

For $k \geq 1$, let G be the graph obtained from k copies of P_3 by identifying the left end vertices with the vertex x, see Figure 4.3. We claim that $\gamma_{tg}(G) = \gamma_{tg}(G-v) = k+1$. Since $\gamma_t(G) = k+1$, we have $\gamma_{tg}(G) \geq k+1$. If Dominator plays first move on x, then Staller is forced to totally dominate x and one end vertex. So there are $k-1$ undominated vertices left. Therefore $\gamma_{tg}(G) \leq 2+(k-1) = k+1$. Hence $\gamma_{tg}(G) = k+1$.

In graph $G-v$, we show that Dominator has a strategy that can end the game in $G-v$ within $k+1$ moves as follows. Dominator starts on x. Then there are k undominated vertices left. Therefore $\gamma_{tg}(G-v) \leq k+1$.

If $k=1$, then $\gamma_{tg}(G-v) = \gamma_t(G) = 2$. Suppose $k \geq 2$. If Dominator starts on x, then Staller replies by playing the leaf adjacent to x. Otherwise, Staller replies by playing an end vertex. In both case, there are $k-1$ undominated vertices any two of which have no common neighbors. So at least $k+1$ moves are played in $G-v$. Therefore $\gamma_{tg}(G-v) \geq k+1$. Hence $\gamma_{tg}(G-v) = k+1$.

4.4 $\gamma_{tg}(G) - \gamma_{tg}(G-v) < 0$

Proposition 4.4. *For any positive integers $k > l \geq 1$, there exists a graph G with a vertex v such that $\gamma_{tg}(G) = k-l+1$ and $\gamma_{tg}(G-v) = k+1$. In particular, $\gamma_{tg}(G) - \gamma_{tg}(G-v) = -l$.*

Proof. For positive integers $k > l \geq 1$, let $G_{k,l}$ be the graph obtained from $K_k \circ K_1$ by adding one vertex v and joining v to all vertices of K_k and $l+1$ pendant vertices, see Figure 4.4. Note that $\gamma_{tg}(G_{k,l} - v) = \gamma_{tg}(K_k \circ K_1) = k+1$. We claim that $\gamma_{tg}(G_{k,l}) = k-l+1$. We show that Dominator has a strategy that can end the game in $G_{k,l}$ within $k-l+1$ moves as follows. Dominator plays first move on v. Then there are $k-l$ undominated vertices left. Therefore $\gamma_{tg}(G_{k,l}) \leq 1+(k-l) = k-l+1$.

We show that Staller has a strategy that can end the game in $G_{k,l}$ using at least $k-l+1$ moves as follows. By Total Continuation Principle, we may assume

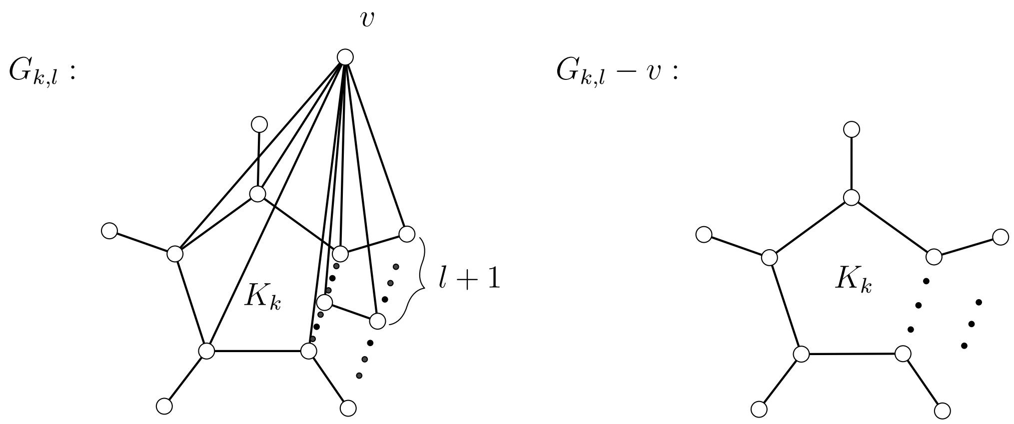

Figure 4.4: Graph $G_{k,l}$ and Graph $G_{k,l} - v$

that Dominator starts on a vertex in $K_k \cup \{v\}$. If Dominator starts on v, then Staller responds by playing a vertex of degree 2 (adjacent to v). Otherwise, Staller responds by playing the pendant vertex in $K_k \circ K_1$ that is adjacent to Dominator's move. After that there are $k-l-1$ undominated vertices any two of which have no common neighbors. Therefore $\gamma_{tg}(G_{k,l}) \geq k-l+1$. Hence $\gamma_{tg}(G_{k,l}) = k-l+1$. □

From the above discussions we have the following theorem.

Theorem 4.5. *For any positive integers $a \geq 2$ and $b \geq 2$ with $a - b \leq 2$, there is a graph G with a vertex v such that $(\gamma_{tg}(G), \gamma_{tg}(G - v)) = (a, b)$ except for $(a, b) = (4, 2)$.*

Next, we consider the Staller-start game.

4.5 $\gamma'_{tg}(G) - \gamma'_{tg}(G - v) = 2$

Proposition 4.6. *For any positive integer $l \geq 4$, there exists a graph G with a* *vertex* v *such that* $\gamma'_{tg}(G) = l$ *and* $\gamma'_{tg}(G-v) = l-2$. *In particular,* $\gamma'_{tg}(G)-\gamma'_{tg}(G-v) = 2$.

Proof. For $4 \leq l \leq 6$, we present some graphs G_l with $\gamma'_{tg}(G_l) = l$ and $\gamma'_{tg}(G_l-v) = l-2$ in Figure 4.5. For $l \geq 7$, recall the infinite family of graphs H_k in Figure

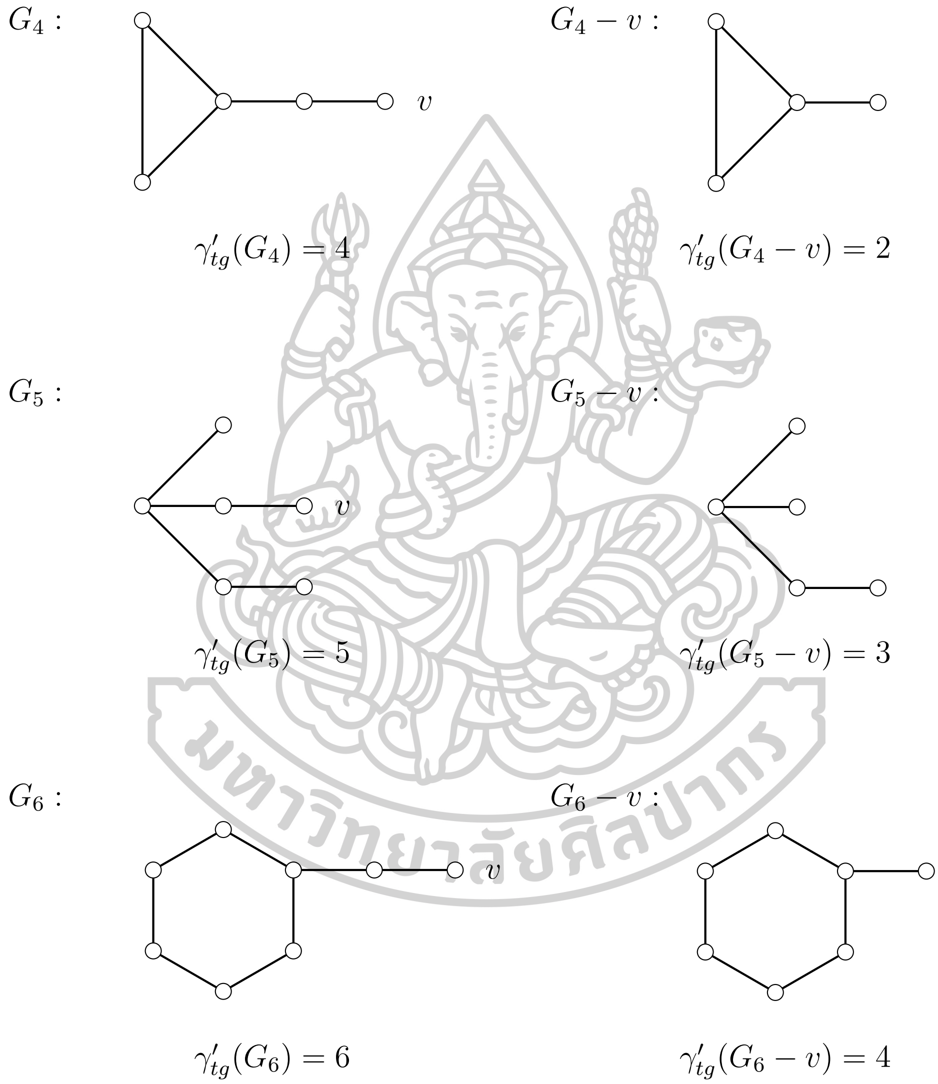

Figure 4.5: Some graphs G_l with $\gamma'_{tg}(G_l) = l$ and $\gamma'_{tg}(G_l - v) = l-2$ for $4 \leq l \leq 6$

4.1. Let $l = k+5$ and $G = H_k$. We claim that $\gamma'_{tg}(H_k) = k+5 = l$ and

$\gamma'_{tg}(H_k - v) = k+3 = l-2$. By Theorem 3.1, it suffices to show that $\gamma'_{tg}(H_k) \geq k+5$ and $\gamma'_{tg}(H_k - v) \leq k+3$.

First, we show that Dominator has a strategy that can end the game in $H_k - v$ within $k+3$ moves as follows. If Staller starts on x, then Dominator replies by playing on y. Otherwise, Dominator replies by playing on x. Then there are at most $k+3$ undominated vertices. Since two leaves are totally dominated together by playing y and x is not dominated alone, at most $k+1$ moves are played. So the number of moves for H_k is at most $k+3$. Therefore $\gamma'_{tg}(H_k - v) \leq k+3$.

Lastly, we show that Staller has a strategy that can end the game in H_k using at least $k+5$ moves as follows. Staller starts on an end vertex of a copy of P_3. If Dominator replies on x, then Staller plays on v. Excluding x, there are $k+2$ undominated vertices any two of which have no common neighbors. So at least $k+5$ moves are played. If Dominator does not reply on x, Staller can force at least 3 moves to totally dominate H_0 by not playing on x. Since at $k+1$ moves are played in k copies of P_3, this strategy can end the game using at least $k+5$ moves. Therefore $\gamma'_{tg}(H_k) \geq k+5$. □

4.6 $\gamma'_{tg}(G) - \gamma'_{tg}(G - v) = 1$

For $k \geq 2$, let v be a pendant vertex of $K_k \circ K_1$, see Figure 4.2. We show that $\gamma'_{tg}(K_k \circ K_1) = k+1$ and $\gamma'_{tg}((K_k \circ K_1) - v) = k$.

In graph $K_k \circ K_1$, by Total Continuation Principle we can assume that Staller plays his first move on v and then Dominator plays his first move on the vertex adjacent to v. There are $k-1$ undominated vertices any two of which have no common neighbors. Hence $\gamma'_{tg}(K_k \circ K_1) = k+1$.

We show that Dominator has a strategy that can end the game in $(K_k \circ K_1) - v$ within k moves as follows. After Staller plays his first move, Dominator replies on a vertex in K_k that totally dominates one pendant vertex and all vertices in K_k in first move. Then there are at most $k-2$ undominated vertices left. Thus $\gamma'_{tg}((K_k \circ K_1) - v) \leq 2 + (k-2) = k$.

We show that Staller has a strategy that can end the game in $(K_k \circ K_1) - v$ using at least k moves as follows. Staller starts on a pendant vertex. Then there are $k-1$ undominated vertices any two of which have no common neighbors. Thus $\gamma'_{tg}((K_k \circ K_1) - v) \geq 1 + (k-1) = k$. Hence $\gamma'_{tg}((K_k \circ K_1) - v) = k$.

4.7 $\gamma'_{tg}(G) - \gamma'_{tg}(G-v) = 0$

For $k \geq 2$, let x be a pendant vertex of $K_k \circ K_1$ and let v be the unique vertex adjacent to x. Take $G = (K_k \circ K_1) - x$. Then $\gamma'_{tg}(G) = k$ and $\gamma'_{tg}(G-v) = \gamma'_{tg}(K_k \circ K_1) = k$.

4.8 $\gamma'_{tg}(G) - \gamma'_{tg}(G-v) < 0$

Proposition 4.7. *For any positive integers k, l with $l+2 < k$, there exists a graph G with a vertex v such that $\gamma'_{tg}(G) = k-l+1$ and $\gamma'_{tg}(G-v) = k+1$. In particular, $\gamma'_{tg}(G) - \gamma'_{tg}(G-v) = -l$.*

Proof. For positive integers k, l with $l+2 < k$, let $G_{k,l}$ be the graph obtained from $K_k \circ K_1$ by adding one vertex v and joining v to all vertices of the subgraph K_k

and $l+2$ pendant vertices. Note that $\gamma'_{tg}(G_{k,l}-v)=\gamma'_{tg}(K_k\circ K_1)=k+1$. We claim that $\gamma'_{tg}(G_{k,l})=k-l+1$.

First, we show that Dominator has a strategy that can end the game in $G_{k,l}$ within $k-l+1$ moves as follows. If Staller starts on v, then there are $k-l-1$ undominated vertices left. Otherwise, Dominator responds by playing on v. After that there are at most $k-l-1$ undominated vertices left. Therefore $\gamma'_{tg}(G_{k,l})\leq k-l+1$.

Lastly, we show that Staller has a strategy that can end the game in $G_{k,l}$ using at least $k-l+1$ moves as follows. Staller starts on a pendant vertex and then plays on a vertex in K_k which is adjacent to a vertex of degree 2. After that there are $k-l-2$ undominated vertices any two of which have no common neighbors. So this strategy can end the game using at least $k-l+1$ moves. Therefore $\gamma'_{tg}(G_{k,l})\geq k-l+1$. Hence $\gamma'_{tg}(G_{k,l})=k-l+1$. $\square$

By Proposition 4.7, for any positives $c\geq 4$ and $d>c$ there exists a graph G with a vertex v such that $(\gamma'_{tg}(G),\gamma'_{tg}(G-v))=(c,d)$. It remains to consider the pairs (c,d) where $c\in\{2,3\}$ and $d>c$.

For $l\geq 1$, let H be an arbitrary graph with $\gamma'_{tg}(H)=l+2$. Let G be the graph obtained from H by adding one new vertex v and joining v to all vertices of H. We have $\gamma'_{tg}(G)=2$ and $\gamma'_{tg}(G-v)=\gamma'_{tg}(H)=l+2$.

For $l=1,2$, we present some graphs G_l with $\gamma'_{tg}(G_l)=3$ and $\gamma'_{tg}(G_l-v)=l+3$ in Figure 4.6. And for $l\geq 3$, we show that there exists no graph G with $\gamma'_{tg}(G)=3$ and $\gamma'_{tg}(G-v)=l+3$ in the following lemmas.

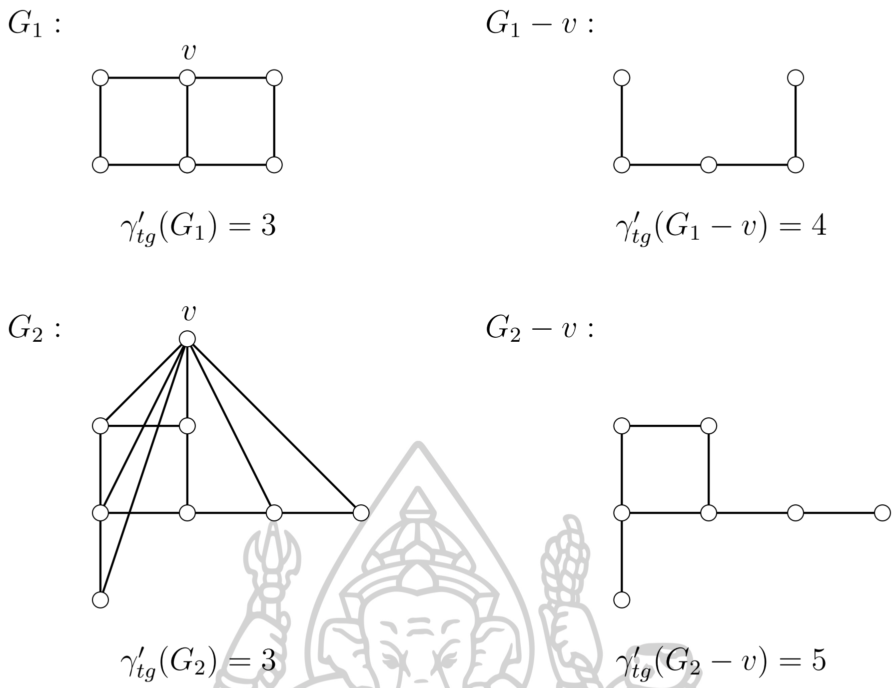

Figure 4.6: Graphs G_l with $\gamma'_{tg}(G_l) = 3$ and $\gamma'_{tg}(G_l - v) = l + 3$ for $l = 1, 2$

Lemma 4.8. *If G is a graph with $\gamma'_{tg}(G) = 3$, then Dominator's optimal first move is adjacent to Staller's optimal first move.*

Proof. Let G be a graph with $\gamma'_{tg}(G) = 3$. Let x and y be the optimal first moves for Staller and Dominator respectively. Suppose that y is not adjacent to x. Then there is a vertex z in $G \setminus N_G[x]$ such that z is totally dominated by y. Since $\gamma'_{tg}(G) = 3$, any legal third move will end the game. Therefore, each legal move is adjacent all undominated vertices. Now z is a legal move since it has an undominated neighbor y. Since x is undominated, the legal move z is adjacent to x, a contradiction. So y is adjacent to x. $\square$

Lemma 4.9. *Let G be a graph and let v be a vertex in G. If $\gamma'_{tg}(G) = 3$, then*

$\gamma'_{tg}(G - v) \leq 5$.

Proof. Observe that $\gamma_t(G) \in \{2, 3\}$. First we consider the case $\gamma_t(G) = \gamma'_{tg}(G) = 3$. It follows that any vertex in G is an optimal first move of Staller, and there is a vertex u of G not adjacent to v. If Staller starts on u, then by Lemma 4.8, we get that v is not an optimal first move of Dominator. Let x be an optimal first move of Dominator. Then $\{u, x, y\}$ is a minimum TD-set of G for any $y \in V(G)$ such that $N_G(y) \nsubseteq N_G(u) \cup N_G(x)$. In $G - v$, Dominator's strategy is playing the vertices in $\{u, x\}$. So $\gamma'_{tg}(G - v) \leq 5$.

Now we consider the case $\gamma_t(G) = 2$. Consider the following possibilities.

Case 1 There is a minimum TD-set T of G such that $v \notin T$. Then in $G - v$, Dominator's strategy is playing the vertices in T. So $\gamma'_{tg}(G - v) \leq 4$.

Case 2 v is in every minimum TD-set of G. Let S be a minimum TD-set of G. We show that there is an optimal first move u of Staller in G that is not adjacent to v. Since $\gamma'_{tg}(G) = 3$, there is a vertex u in $G \setminus S$ that is not adjacent to v. Since $\gamma_t(G) = 2$ and v is in every minimum TD-set of G, we get that u is not adjacent to some vertices in $N_G(v)$. If Staller starts on u, then Dominator cannot end the game in his first move. Since $\gamma'_{tg}(G) = 3$, it implies that u is an optimal first move of Staller. By Lemma 4.8, we get that v is not an optimal first move of Dominator. Let x be an optimal first move of Dominator. Then $\{u, x, y\}$ is a TD-set of G for any $y \in V(G)$ such that $N_G(y) \nsubseteq N_G(u) \cup N_G(x)$. In $G - v$, Dominator's strategy is playing the vertices in $\{u, x\}$. So $\gamma'_{tg}(G - v) \leq 5$. $\square$

From the above discussions we have the following theorem.

Theorem 4.10. *For any positive integers* $c \geq 2$*,* $d \geq 2$ *with* $c - d \leq 2$*, there is a graph* G *with a vertex* v *such that* $(\gamma'_{tg}(G), \gamma'_{tg}(G - v)) = (c, d)$ *except for* $(c, d) = (3, l)$ *for all* $l \geq 6$*.*

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DISSEMINATIONS

Publications

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