4-TOTAL DOMINATION GAME CRITICAL GRAPHS

By

MISS Karnchana CHAROENSITTHICHAI

A Thesis Submitted in Partial Fulfillment of the Requirements

for Doctor of Philosophy MATHEMATICS

Department of MATHEMATICS

Silpakorn University

Academic Year 2022

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MISS KARNCHANA CHAROENSITTHICHAI : 4-TOTAL DOMINATION GAME CRITICAL GRAPHS. THESIS ADVISOR : ASSISTANT PROFESSOR CHALERMPONG WORAWANNOTAI, Ph.D.

The total domination game is played on a simple graph G with no isolated vertices by two players, Dominator and Staller, who alternate choosing a vertex in G. Each chosen vertex totally dominates its neighbors. In this game, each chosen vertex must totally dominates at least one new vertex not totally dominated before. The game ends when all vertices in G are totally dominated. Dominator's goal is to finish the game as soon as possible, and Staller's goal is to prolong it as much as possible. The game total domination number is the number of chosen vertices when both players play optimally, denoted by $\gamma_{tg}(G)$ when Dominator starts the game and denoted by $\gamma'_{tg}(G)$ when Staller starts the game. If a vertex v in G is declared to be already totally dominated, then we denote this graph by $G|v$. A total domination game critical graph is a graph G for which $\gamma_{tg}(G|v) < \gamma_{tg}(G)$ holds for every vertex v in G. Additionally, if $\gamma_{tg}(G) = k$, then G is called k-γ_{tg}-critical. In this thesis, we characterize some 4-γ_{tg}-critical graphs.

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MISS Karnchana CHEROENSITTHICHAI

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Introduction

A graph $G = (V(G), E(G))$ consists of a set $V(G)$ of vertices and a set $E(G)$ of edges, where each edge is associated with an unordered pair of elements of $V(G)$. Two vertices are *adjacent* if there is an edge associated with them; they are also the *end vertices* of the edge, and the edge is said to be *incident* to each of its end vertices. A *loop* is an edge connecting a vertex to itself. *Multiple edges* are two or more edges that are incident to the same two vertices. A graph without loops or multiple edges is called a *simple graph*. In this thesis, we only consider simple graphs.

Given a simple graph G and a subset S of some vertices in G, we say that S is a *dominating set* of G if every vertex in G is in S or is adjacent to some vertex in S. In this case, a vertex in G is *dominated* by some vertex in S. The *domination number* of G is the minimum size of a dominating set of G, denoted by $\gamma(G)$. Domination in graphs is one of the most studied topics in graph theory because many problems in real life can be modeled by domination in graphs. For example, domination can be used in the transportation route planning, the security system design, and the wireless network design. For an example of wireless network design in a building, we want to install multiple access points where the internet signal covers all areas in the building and use as few access points as possible to save costs and reduce maintenance. Then domination can be applied to solve this kind of resource allocation problem. To solve the problem, we can model the building using graph theory by representing each area (e.g. room) by a vertex and two vertices are adjacent if the signal from an access point in one area can cover the other. Then the minimum number of access points needed in this building is equal to the domination number of the graph and the access point's locations are determined by a minimum dominating set of the graph.

A fundamental problem in domination is to find the domination number of a graph but there is no known algorithm that can efficiently find the domination number of every graph. However, researchers have found the formulas of many families of graphs and produced many bounds of domination numbers. For more details about domination, we refer the readers to the books by Haynes, Hedetniemi, and Slater [14, 15] and the book by Haynes, Hedetniemi, and Henning [13]. In addition to finding the domination numbers of graphs, the study of domination also focuses on the problem of finding a minimum dominating set of the graph which leads to finding an effective way to solve the corresponding problems in real life. There is no known algorithm that can efficiently find a minimum dominating set of every graph. Thus, another fundamental problem in domination is the process of creating a small dominating set of a graph. In sense of a game, domination is a solo game that is played on a graph by a single player who tries to add vertices, one at a time, into a set until the set becomes a dominating set of the graph and he wishes to make the set as small as possible. A simple natural strategy for this player may be to add a vertex with the greatest degree into the set until the set becomes a dominating set. There are also other properties of graphs that can take into consideration when creating a dominating set. However, these strategies do not guarantee that the size of the dominating set will be smallest.

In 2010, Brešar, Klavžar, and Rall [5] developed domination into a competitive game among two players. Both players alternate adding a vertex into the same set until the set becomes a dominating set of the graph. One player, *Dominator*, is the original player who wants to finish the game by making the dominating set as small as possible, but the other player, *Staller*, has the opposite objective. Choosing a vertex will make all vertices in its closed neighborhood dominated. A vertex that can be chosen must dominate at least one new vertex. The game ends when all vertices are dominated. The format of playing of both players is a process of creating a dominating set of the graph so this is a game without winner or loser but the players want to play optimally according to their objectives. Thus, the resulting dominating set of the graph will depend on the strategy of both players. We call this game *domination game*. In case both players play optimally, the size of the dominating set of chosen vertices is called the *game domination number*. We denoted by $\gamma_g(G)$ when Dominator starts the game and denoted by $\gamma_g'(G)$ when Staller starts the game.

Given a graph G and a subset S of vertices, a *partially dominated graph* $G|S$ is G with a declaration that every vertex in S is already dominated, that is they do not need to be dominated during the game. The resulting game domination number is denoted by $\gamma_g(G|S)$ or $\gamma_g'(G|S)$ depending on who starts the game. In other words, the vertices in S are *predominated*.

Although the domination game was introduced not long ago, the topic has been widely studied, and recently Brešar, Henning, Klavžar and Rall [4] published a book that summarizes the results. Some fundamental results of the domination game include the following results. The relation between domination and domination game is shown in [5]; that is $\gamma(G) \leq \gamma_g(G) \leq 2\gamma(G) - 1$ hold for every graph G. The difference between two types of domination game of a graph is at most 1 [6]. A key lemma called *Continuation Principle* [21] is used to give a short proof of this result. It states the following. For a graph G and $A, B \subseteq V(G)$ with $B \subseteq A$, we have $\gamma_g(G|A) \leq \gamma_g(G|B)$ and $\gamma'_g(G|A) \leq \gamma'_g(G|B)$. Moreover, many variations of the game are derived and studied [1, 2, 5, 8, 16]. In this thesis, we study the total domination game which is a combination of two variations, domination game and total domination.

Let G be a simple graph with no isolated vertex. Given a subset S of vertices in G, we say that S is a *total dominating set* or S *totally dominates* G if every vertex in G is adjacent to some vertex in S. In other words, a vertex in S *totally dominates* its neighbors, but not itself. The *total domination number* of G is the minimum size of a total dominating set of G, denoted by $\gamma_t(G)$. We call a minimum total dominating set a γ_t-set of G. Since each vertex cannot totally dominate itself, graphs that contain an isolated vertex are not considered.

An application of total domination is the security system design. We want to set up a team of security guards in a village where all areas are covered and a guard is covered by at least one other guard. We first divide the village into smaller areas by considering all roads within the village. We will divide the entire road into sections. Each security guard is responsible for the safety of all members and all houses that are in the section or in the adjacent section, but we require one other security guard to look after him. We can model the area of the village to the graph by representing each section of the road by a vertex and two vertices are adjacent if the two sections are adjacent. Selecting one vertex in the graph means determining the main location of one security guard. The design is still based on comprehensive safety and the least cost. We solve the problem on the model graph by using total domination. We can conclude that the minimum number of security guards is equal to the total domination number, and the main locations of all security guards are determined by a minimum total domination set of the graph. An example of finding the total domination number on the graph is shown in Example 1.1.

Example 1.1. Let G be the graph in Figure 1.1. Observe that there is no subset

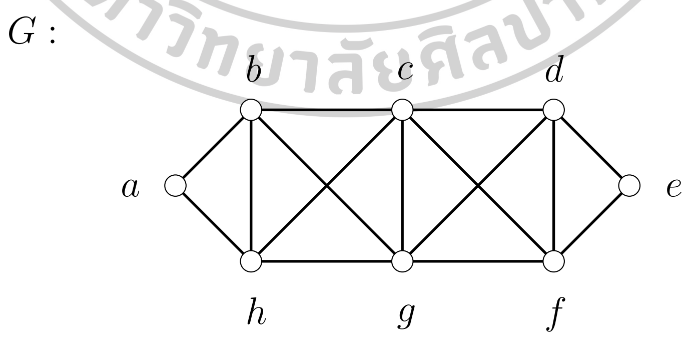

Figure 1.1: Graph G

S of two vertices in G such that S totally dominates G. Thus, $\gamma_t(G) \geq 3$. Since each vertex in $\{a, e, f, g, h\}$ is adjacent to at least one vertex in $\{b, c, d\}$ and b, d are adjacent to c, we get that $\{b, c, d\}$ is a total dominating set of G. Thus, $\gamma_t(G) \leq 3$. Hence, $\gamma_t(G) = 3$.

The *total domination game* is a game that is played on a graph G by two players, *Dominator* and *Staller*, who alternate taking turns choosing a vertex in G. Each chosen vertex must totally dominate at least one new vertex not totally dominated before. Such vertices are called *legal moves*; otherwise they are *illegal moves*. The game ends when the set of chosen vertices is a total dominating set. Dominator's goal is to finish the game as soon as possible, and Staller's goal is to prolong it as much as possible. The *game total domination number* is the size of the total dominating set of chosen vertices when both players play optimally, denoted by $\gamma_{tg}(G)$ when Dominator starts the game and denoted by $\gamma'_{tg}(G)$ when Staller starts the game.

This game was introduced by Henning, Klavžar and Rall [16] in 2015. They showed that many techniques in domination game can be adapted to use on total domination game and obtain many analogous results. It was proved that the game total domination number of a graph can be bounded in terms of the total domination number. The difference between the two types of game total domination numbers of a graph is at most 1. This was proved by using the key lemma called *Total Continuation Principle*. Henning, Klavžar and Rall [17] proved that if G is a graph on n vertices in which every component contains at least three vertices, then $\gamma_{tg}(G) \leq 4n/5$ and $\gamma'_{tg}(G) \leq (4n+2)/5$. Dorbec, Henning and Renault [11] determined the game total domination number for paths and cycles. Henning and Rall [19] showed that if G is a forest with no isolated vertex, then $\gamma_{tg}(G) \leq \gamma'_{tg}(G)$ and the trees with equal total domination and game total domination number are characterized.

A *partially total dominated graph* is the graph with a declaration that some vertices are already totally dominated, that is they do not need to be totally dominated during the game. A vertex that is already totally dominated can be a legal move if it has some totally undominated neighbors. Given a graph G and a subset S of vertices of G, let $G|S$ be the partially total dominated graph that every vertex in S has already been totally dominated. If $S = \{v\}$ for some vertex v in G, we write $G|v$ instead of $G|\{v\}$. The game total domination number of $G|S$ is the number of optimal moves remaining on $G|S$. We use $\gamma_{tg}(G|S)$ (or $\gamma'_{tg}(G|S)$) when Dominator (or Staller) starts the game on $G|S$. Observe that $\gamma_{tg}(G|S) \leq \gamma_{tg}(G)$ holds for any subset S of vertices in a graph G. When S contains only one vertex, Iršič [20] showed that $\gamma_{tg}(G|v) \geq \gamma_{tg}(G) - 2$ holds for every vertex v in G.

A critical graph with respect to a graph parameter and a graph operation is a graph whose value of parameter changes if we make any change to the graph using the specified operation, such as vertex removal, edge addition, etc. Criticality can be used to prove many important results in graph theory such as Ore's theorem on the necessary condition of Hamiltonian graphs. The knowledge of structures of critical graphs can lead to a general theorem describing general graph structures.

A critical graph with respect to the domination in [23] was presented as follows: a graph is *domination critical* if adding any edge will decrease its domination number. In [7], the domination critical graph was presented with the operation of removing a vertex. A graph is *vertex domination-critical* if removing any vertex will decrease its domination number.

Critical graphs with respect to the domination and (total) domination game were studied in several different operations such as vertex or edge removal in [3, 10], which has the following results. The game domination number can be increased or decreased by at most 2 when removing a vertex or an edge, and the game total domination number can be decreased by at most 2 when removing a vertex. By the Continuation Principle, $\gamma_g(G) \geq \gamma_g(G|v)$ for every vertex v in a graph G. In 2015, Bujtás, Klavžar, and Košmrlj [9] introduced the definition of domination game critical graphs with respect to predomination as follows. For a graph G, we say that G is *γ_g-critical* if $\gamma_g(G) > \gamma_g(G|v)$ for all $v \in V(G)$. If G is a γ_g-critical graph and $\gamma_g(G) = k$, then G is called *k-γ_g-critical*. The authors of [9] proved that if v is a vertex of a graph G, then $\gamma_g(G|v) \geq \gamma_g(G) - 2$. Moreover, they also characterized k-γ_g-critical graph with $k = 2, 3$. A graph G is 2-γ_g-critical if and only if $\gamma_g(G) = 2$ and every pair of vertices of G forms a dominating set. For a graph G with maximum degree at most $n-3$, G is 3-γ_g-critical if and only if there is no pair of vertices in G that have the same closed neighborhood, and for any $v \in V(G)$ there exists a vertex u of degree $n-3$ that is not adjacent to v.

The concept of domination game critical graphs does not make sense for Staller-start game. Suppose we define γ_g'-critical graphs analogously, that is, a graph G is γ_g'-critical if $\gamma_g'(G) > \gamma_g'(G|v)$ for all $v \in V(G)$. Let G be a graph. If there is an optimal first move u of Staller with degree at least one, then Staller can start on u in $G|u$ and we get that $\gamma'_g(G|u) \geq 1 + \gamma_g(G|N[u]) = \gamma'_g(G)$. This implies that graphs with no isolated vertex are not γ'_g-critical.

The total domination critical graphs were introduced by Henning, Klavžar, and Rall [18] in 2018 as follows. A graph G is *total domination game critical* or shortly γ_{tg}-*critical*, if $\gamma_{tg}(G) > \gamma_{tg}(G|v)$ for every vertex v in G. If G is γ_{tg}-critical and $\gamma_{tg}(G) = k$, then we say that G is k-γ_{tg}-*critical*. The authors characterized γ_{tg}-critical graphs for cycles and paths and they also characterized k-γ_{tg}-critical graphs when $k = 2, 3$. The results are as follows. A graph G is 2-γ_{tg}-critical if and only if every pair of vertices in G is adjacent. A graph G of order n is 3-γ_{tg}-critical if and only if there is no pair of vertices in G that share common neighborhood, there is no vertex in G adjacent to every other vertex of G, and for every vertex v of G of degree at most $n-3$, there exists a vertex u of degree $n-2$ that is not adjacent to v.

In this thesis, we study the characterization of 4-γ_{tg}-critical graphs. We proceed as follows. In the next chapter, we recall some definitions, notion and some useful results. In chapter 3, we characterize disconnected 4-γ_{tg}-critical graphs and show some properties of connected 4-γ_{tg}-critical graphs. In chapter 4, we characterize connected 4-γ_{tg}-critical graphs with diameter 4. In chapter 5, we characterize some connected 4-γ_{tg}-critical graphs with diameter 3. In the last chapter, we summarize all the results and state the remaining problem.

Chapter 2

Preliminaries

In this chapter, we recall some basic definitions and notation in graph theory and some useful results in total domination game. For graph theory in general, we follow the book [24] by West.

Definition 2.1. A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph G is *connected* if there is a path from any vertex to any other vertex; otherwise, G is *disconnected*.

Definition 2.2. An *isomorphism* from a simple graph G to a simple graph H is a bijection $f : V(G) \to V(H)$ such that for each $u, v \in V(G)$, $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. If there is an isomorphism from G to H, then we say that G is *isomorphic* to H, written $G \cong H$.

Definition 2.3. Let G be a graph and u, v be two vertices in G. The *distance* from u to v, written $d_G(u, v)$ or simply $d(u, v)$, is the length of a shortest path from u to v. The *diameter of* G, written $diam(G)$, is $\max_{u,v \in V(G)} d(u, v)$. The *eccentricity* of a vertex u, written $ecc(u)$, is $\max_{v \in V(G)} d(u, v)$.

Definition 2.4. Let G and H be two disjoint graphs. The disjoint union of graphs G and H is denoted by $G + H$.

In this study, it is useful to draw a graph by arranging the vertices according to their distances to a fixed vertex. For a vertex x in a graph G and $i \geq 0$, we let $\Gamma_i(x)$ be the subgraph of G induced by all vertices which have distance i to x in G. That is G is the disjoin union of subgraphs $\Gamma_i(x)$ where $i \in \{0, 1, 2, ..., ecc(x)\}$ together with some edges joining vertices in $\Gamma_i(x)$ and $\Gamma_{i+1}(x)$ for $i \in \{0, 1, 2, ..., ecc(x) - 1\}$.

Definition 2.5. Let G be a graph. The *order* of G is the number of vertices of G. The *degree* of the vertex v in G, written $deg(v)$, is the number of edges incident to v. If $deg(v) = 0$, then v is said to be an *isolated vertex*. If $deg(v) = 1$, then v is said to be a *pendent* or *leaf*.

Definition 2.6. A *path* P_n is a graph of order n whose vertices can be listed in the order $v_1, v_2, ..., v_n$ such that v_i and v_{i+1} are adjacent where $i = 1, 2, ..., n-1$ and no other pair of vertices are adjacent. A *cycle* C_n with $n \geq 3$ is a graph obtained from P_n by adding an edge connecting the two leaves.

Definition 2.7. An *independent set* in a graph is a set of pairwise nonadjacent vertices. A graph G is said to be a *bipartite graph* if $V(G)$ is the union of two disjoint (possibly empty) independent sets X and Y. Then we say that X, Y are *partite sets* of G and (X, Y) is a *bipartition* of G.

Definition 2.8. A *complete graph* K_n is a graph of order n which every pair of distinct vertices are adjacent. A *complete bipartite graph* is a bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. We denote a complete bipartite graph with two partite sets of sizes m and n by $K_{m,n}$.

Definition 2.9. A *matching* in a graph is a set of non-loop edges with no common end vertices. A *perfect matching* in a graph G is a matching M such that every vertex in G is incident to exactly one edge in M.

Definition 2.10. Let G and H be two disjoint graphs. The *join* of disjoint graphs G and H, written $G \vee H$, is the graph obtained from the disjoint union of graphs G and H by adding the edges uv for all $u \in V(G)$ and $v \in V(H)$.

Definition 2.11. The *chain* $G_0 \vee G_1 \vee G_2 \vee ... \vee G_d$ of disjoint graphs $G_0, G_1, G_2, ..., G_d$ is the graph obtained from the disjoint union $G_0 + G_1 + G_2 + ... + G_d$ by adding the edges uv for all $i \in \{0, 1, 2, ..., d-1\}$, $u \in V(G_i)$ and $v \in V(G_{i+1})$.

Definition 2.12. The *open neighborhood* $N_G(v)$ of v in a graph G is the set of vertices adjacent to v; that is $N_G(v) = \{u \in V(G) | uv \in E(G)\}$. The *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. We simply write $N(v)$ and $N[v]$ if the graph is understood. If $N[v] = V(G)$, then v is said to be a *universal vertex* in G.

Definition 2.13. Let G be a graph. Two vertices u and v of G are *twins* in G if their closed neighborhoods are the same, that is $N[u] = N[v]$. G is said to be *twin-free* if it does not contain any twins. Two vertices u and v of G are *open twins* in G if their open neighborhoods are the same, that is $N(u) = N(v)$. G is said to be *open twin-free* if it does not contain any open twins.

Next, we recall some important results on total domination game. By the definition of total domination game, we have the following lemma.

Lemma 2.14. *Let G be a graph. Then the following statements hold.*

(i) *For Dominator-start game, if Dominator has a strategy that can end the game within k moves, then* $\gamma_{tg}(G) \leq k$.

(ii) *For Staller-start game, if Dominator has a strategy that can end the game within k moves, then* $\gamma'_{tg}(G) \leq k$.

(iii) *For Dominator-start game, if Staller has a strategy that can end the game with at least k moves, then* $\gamma_{tg}(G) \geq k$.

(iv) *For Staller-start game, if Staller has a strategy that can end the game with at least k moves, then* $\gamma'_{tg}(G) \geq k$.

By Lemma 2.14, we can prove a bound for game total domination number by devising an appropriate strategy for a player.

Example 2.15. Let G be the graph in Figure 1.1. For Dominator-start game, if Dominator starts on c, then only three vertices a, c and e are not totally dominated by c. Thus, at most 3 more moves are played to totally dominated them. By this strategy of Dominator, the game will end within 4 moves so by Lemma 2.14(i), $\gamma_{tg}(G) \leq 4$. Next, we show that Staller has a strategy to end the game using at least 4 moves as follows. By symmetry, Dominator has 3 ways to start the game.

Case 1: Dominator starts on a. Then Staller responds by playing on e. Now, a and e are not totally dominated yet. Since a and e cannot be totally dominated at the same time, at least 2 more moves are played to totally dominate them. In this case, at least 4 moves are played in this game.

Case 2: Dominator starts on b. Then Staller responds by playing on e and only two vertices b and e are not totally dominated. Since they cannot be totally dominated at the same time, at least 2 more moves are played to totally dominate them. In this case, at least 4 moves are played in this game.

Case 3: Dominator starts on c. Then Staller responds by playing on g and only two vertices a and e are not totally dominated. Since they cannot be totally dominated at the same time, at least 2 more moves are played to totally dominate them. In this case, at least 4 moves are played in this game.

By this strategy of Staller, at least 4 moves are played in this game. By Lemma 2.14(iii), $\gamma_{tg}(G) \geq 4$. Hence, we conclude that $\gamma_{tg}(G) = 4$.

In addition, we can use the comparison of some vertices or use the value of the total domination number to bound the game total domination number of a graph, see in the following theorem.

Theorem 2.16. [16] *Let G be a graph with no isolated vertex. Then $\gamma_t(G) \leq \gamma_{tg}(G) \leq 2\gamma_t(G) - 1$.*

Theorem 2.17. [16]*(Total Continuation Principle) Let G be a graph and let A and B be subsets of $V(G)$. If $B \subseteq A$, then $\gamma_{tg}(G|A) \leq \gamma_{tg}(G|B)$ and $\gamma'_{tg}(G|A) \leq \gamma'_{tg}(G|B)$.*

The Total Continuation Principle implies that whenever two vertices u and v are legal moves and $N(u) \subseteq N(v)$, Dominator prefers to play on v over u but Staller prefers to play on u over v.

The next result shows that predominating an open twin does not change the game total domination number of a graph.

Lemma 2.18. [18] *If u and v are open twins in a graph G, then* $\gamma_{tg}(G) = \gamma_{tg}(G|u) = \gamma_{tg}(G|v)$.

As an immediate consequence of Lemma 2.18, we have the following result.

Corollary 2.19. [18] *Every γ_{tg}-critical graph is open twin-free.*

By Total Continuation Principle, we have a fundamental property of γ_{tg}-critical graphs.

Lemma 2.20. [18] *If G is a γ_{tg}-critical graph, then a neighbor of a vertex v is not an optimal first move of Dominator in the Dominator-start game on $G|v$.*

For an example of a 4-γ_{tg}-critical graph, we consider the graph G in Figure 1.1. From Example 2.15, we know that $\gamma_{tg}(G) = 4$. Next, we show that for each $v \in V(G)$, Dominator has a strategy to end the game in $G|v$ within 3 moves. By symmetry, we can assume that $v \in \{a, b, c\}$. In $G|a$, Dominator starts on c. Then only two vertices c and e are not totally dominated. Thus, at most 2 more moves are played to totally dominate the graph. In $G|b$, Dominator starts on b. Then only three vertices d, e and f are not totally dominated, and Staller is forced to totally dominate two new vertices in his move so this game ends within 3 moves. In $G|c$, Dominator starts on c. Then only two vertices a and e are not totally dominated. Thus, at most 2 more moves are played to totally dominate the graph. By this strategy of Dominator, we get that at most 3 moves are played in $G|v$ for any $v \in V(G)$. By Lemma 2.14(i), we have $\gamma_{tg}(G|v) \leq 3$ for any $v \in V(G)$. Therefore, G is 4-γ_{tg}-critical.

Total domination game critical graphs were characterized for the cycles and paths. We will see that C_6 is the only 4-γ_{tg}-critical cycle and there are no 4-γ_{tg}-critical path.

Theorem 2.21. [11] *For $n \geq 3$,*

$$\gamma_{tg}(C_n) = \begin{cases} \left\lfloor \frac{2n+1}{3} \right\rfloor - 1; & n \equiv 4 \pmod 6 \\ \left\lfloor \frac{2n+1}{3} \right\rfloor; & \text{otherwise.} \end{cases}$$

Theorem 2.22. [18] *For $n \geq 3$, the cycle C_n is γ_{tg}-critical if and only if $n \pmod 6 \in \{0, 1, 3\}$.*

By Theorem 2.21, $\gamma_{tg}(C_n) = 4$ if and only if $n = 6$. Thus, by Theorem 2.22, C_6 is the only cycle that is 4-γ_{tg}-critical.

Theorem 2.23. [11] *For $n \geq 2$,*

$$\gamma_{tg}(P_n) = \begin{cases} \left\lfloor \frac{2n}{3} \right\rfloor; & n \equiv 5 \pmod 6 \\ \left\lceil \frac{2n}{3} \right\rceil; & \text{otherwise.} \end{cases}$$

Theorem 2.24. [18] *For $n \geq 2$, the path P_n is γ_{tg}-critical if and only if $n \pmod 6 \in \{2, 4\}$.*

By Theorem 2.23, $\gamma_{tg}(P_n) = 4$ if and only if $n = 6$ so by Theorem 2.24, P_6 is not a 4-γ_{tg}-critical graph. Thus, P_n is not 4-γ_{tg}-critical for any n.

Chapter 3

Some properties of 4-γ_{tg}-critical graphs

In this chapter, we present some general properties of 4-γ_{tg}-critical graphs and characterize disconnected 4-γ_{tg}-critical graphs. First, we show that Staller must end the game on a $2k$-γ_{tg}-critical graph by totally dominating one new vertex.

Lemma 3.1. *Let G be a $2k$-γ_{tg}-critical graph. If both players play optimally in G, then the last move of the game always totally dominates exactly one new vertex.*

Proof. Suppose that the last move of the game totally dominates two new vertices x and y. It implies that $N(x) = N(y)$; otherwise Staller can choose not to end the game in the $2k$-th move. This contradicts with Corollary 2.19 where G is open twin-free. Hence, exactly one new vertex is totally dominated in the last move of the game. □

To study 4-γ_{tg}-critical graphs, we sometimes need to consider several partially total dominated graphs of the form $G|v$ simultaneously. Thus, we will use the following notation to specify moves (not necessary optimal). Let d_i be the i-th move of Dominator's in a graph G and let s_i be the i-th move of Staller's in G. For a vertex v in G, we let d_i^v be the i-th move of Dominator's in $G|v$ and let s_i^v be i-th move of Staller's in $G|v$.

If there is a vertex v in a graph G such that $\gamma_{tg}(G|v) = 1$, then v is a universal vertex in G. In this case, $\gamma_{tg}(G) = 2$, and thus, G is not 4-γ_{tg}-critical.

Lemma 3.2. *Let G be a graph. If there exists a vertex v in G such that $\gamma_{tg}(G|v) = 2$, then $\gamma_{tg}(G) \leq 3$.*

Proof. Let v be a vertex in G with $\gamma_{tg}(G|v) = 2$ and let d_1^v be an optimal first move for Dominator in $G|v$. If $d_1^v = v$, then every legal move of Staller's is played in $N(v)$. Thus, $\{u, v\}$ is a total dominating set of G for every $u \in N(v)$ such that $N(u)\backslash N[v] \neq \varnothing$. We consider a total dominating set $\{u, v\}$ for some $u \in N(v)$ such that $N(u)\backslash N[v] \neq \varnothing$. Then the game in G can be ended within 3 moves if Dominator starts on v and then plays u. Therefore, $\gamma_{tg}(G) \leq 3$. If $d_1^v \in N(v)$, then $\gamma_{tg}(G) = \gamma_{tg}(G|v) = 2$. Assume that $d(d_1^v, v) \geq 2$. If $d(d_1^v, v) > 2$, then v is a legal move of Staller's in $G|v$. Since d_1^v is not totally dominated by v and Staller can play v, we have $\gamma_{tg}(G|v) \geq 3$. It is a contradiction. Thus, $d(d_1^v, v) = 2$ so at least one vertex u in $N(v)$ is totally dominated by d_1^v. Since Staller can play u and $\gamma_{tg}(G|v) = 2$, we get that $\{d_1^v, u\}$ is a total dominating set of G. Therefore, $\gamma_{tg}(G) \leq 3$. □

From the above results, we have the following conclusion.

Corollary 3.3. *Let G be a 4-γ_{tg}-critical graph. Then $\gamma_{tg}(G|v) = 3$ for all $v \in V(G)$.*

By Theorem 2.16, we have the following lemma.

Lemma 3.4. *Let G be a 4-γ_{tg}-critical graph. Then $\gamma_t(G) \in \{3, 4\}$.*

Next, we characterize disconnected 4-γ_{tg}-critical graphs.

Theorem 3.5. *Let G be a disconnected graph. Then G is 4-γ_{tg}-critical if and only if $G \cong K_m + K_n$ for some $m, n \geq 2$.*

Proof. Clearly, $K_m + K_n$ is 4-γ_{tg}-critical for all $m, n \geq 2$. Assume that G is 4-γ_{tg}-critical. Since we require at least two vertices to totally dominate a component, we get that G has exactly 2 components H_1 and H_2. Let $v \in V(H_1)$. Since G is 4-γ_{tg}-critical, we have $\gamma_{tg}(H_1|v) = 1$. Then v is a universal vertex in H_1. Since v is an arbitrary vertex in H_1, we have $H_1 \cong K_m$ for some $m \geq 2$. Similarly, $H_2 \cong K_n$ for some $n \geq 2$. Hence, $G \cong K_m + K_n$. □

For the rest of the thesis, we only consider connected graphs. We determine the possible value of the diameters of connected 4-γ_{tg}-critical graphs.

Lemma 3.6. *Let G be a connected 4-γ_{tg}-critical graph. Then $2 \leq diam(G) \leq 4$.*

Proof. If $diam(G) = 1$, then G is a complete graph which has $\gamma_{tg}(G) = 2$. This contradicts with G being 4-γ_{tg}-critical. Thus, $diam(G) \geq 2$. Suppose that $diam(G) \geq 5$. Let $x \in V(G)$ with $ecc_G(x) \geq 5$. Let v be a vertex in $\Gamma_2(x)$. We show that Staller has a strategy to end the game in $G|v$ by using at least four moves as follows. If Dominator starts in $\{x\} \cup \Gamma_1(x) \cup \Gamma_2(x)$, then Staller plays in $\Gamma_5(x)$. If Dominator starts elsewhere, then Staller plays x. By this strategy, at least four moves are needed to end the game. Therefore, $\gamma_{tg}(G|v) \geq 4$. This contradicts with Corollary 3.3. Therefore, $diam(G) \leq 4$. □

Examples of 4-γ_{tg}-critical graphs with diameters 2, 3 and 4 are shown in Figure 3.1.

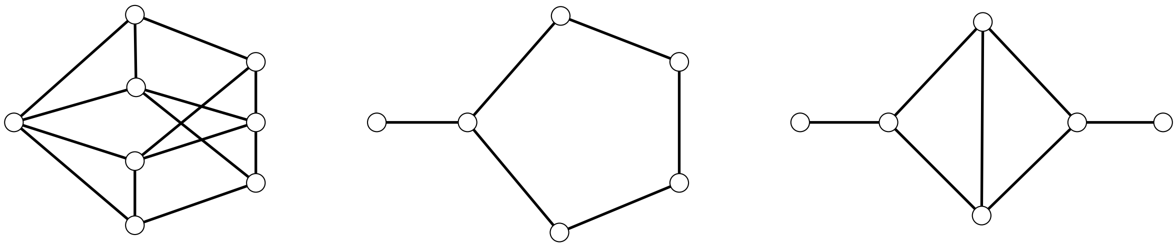

Figure 3.1: Examples of 4-γ_{tg}-critical graphs with diameters 2,3 and 4, respectively.

Chapter 4

4-γ_{tg}-critical graphs with diameter 4

In this chapter, we characterize 4-γ_{tg}-critical graphs with diameter 4.

Theorem 4.1. *Let G be a connected graph with diameter 4. Then G is 4-γ_{tg}-critical if and only if $G \cong K_1 \vee K_{n_1} \vee K_{n_2} \vee K_{n_3} \vee K_1$ for some positive integers n_1, n_2, n_3 where $n_2 \geq 2$.*

Proof. First, we suppose that $G \cong K_1 \vee K_{n_1} \vee K_{n_2} \vee K_{n_3} \vee K_1$ for some positive integers n_1, n_2, n_3 where $n_2 \geq 2$. Then there exists a vertex x in G such that $ecc(x) = 4, \Gamma_1(x) \cong K_{n_1}, \Gamma_2(x) \cong K_{n_2}, \Gamma_3(x) \cong K_{n_3}, \Gamma_4(x) \cong K_1$ and each vertex in $\Gamma_i(x)$ is adjacent to every vertex in $\Gamma_{i+1}(x)$ for $i \in \{0, 1, 2, 3\}$. We show that $\gamma_{tg}(G) = 4$. If Dominator starts in G by playing in $\Gamma_2(x)$, then exactly three vertices are not yet totally dominated. Thus, at most four moves are played in G. It implies that $\gamma_{tg}(G) \leq 4$. On the other hand, we show that Staller has a strategy that can end the game in G using at least four moves as follows. If Dominator plays d_1 in $\{x\} \cup \Gamma_4(x)$, then Staller totally dominates d_1. If Dominator plays d_1 in $\Gamma_1(x) \cup \Gamma_3(x)$, then Staller responds in $\{x\} \cup \Gamma_4(x)$ to totally dominate d_1. In these two cases, either $\Gamma_3(x) \cup \Gamma_4(x)$ or $\{x\} \cup \Gamma_1(x)$ is totally undominated and at least two more moves are played to end the game. If Dominator plays d_1 in $\Gamma_2(x)$, then Staller responds in $\Gamma_2(x)$ to totally dominate d_1. It implies that x and

$\Gamma_4(x)$ are totally undominated. Since x and $\Gamma_4(x)$ cannot be totally dominated simultaneously, at least two more moves are played to end the game. Thus, at least four moves are played in G. Hence, $\gamma_{tg}(G) \geq 4$. Consequently, $\gamma_{tg}(G) = 4$.

Next, we show that $\gamma_{tg}(G|v) \leq 3$ for every $v \in V(G)$. Let $v \in V(G)$. If v is in $\{x\} \cup \Gamma_4(x)$, then Dominator starts in $\Gamma_2(x)$. It implies that exactly two vertices are not yet totally dominated. Otherwise, Dominator starts on v. In either case, after Staller's first move, exactly one vertex is not yet totally dominated. By this strategy of Dominator's, the number of moves in $G|v$ is at most three. Thus, $\gamma_{tg}(G|v) \leq 3$, implying that G is a 4-γ_{tg}-critical graph.

To prove the other direction, we assume that G is a 4-γ_{tg}-critical graph. Let x be a vertex in G such that $ecc(x) = 4$. We show that $\Gamma_1(x) \cong K_{n_1}, \Gamma_2(x) \cong K_{n_2}, \Gamma_3(x) \cong K_{n_3}, \Gamma_4(x) \cong K_1$ and each vertex in $\Gamma_i(x)$ is adjacent to every vertex in $\Gamma_{i+1}(x)$ for $i \in \{0, 1, 2, 3\}$.

Claim 1. $\Gamma_4(x) \cong K_1$.

Proof. Let v be a vertex in $\Gamma_2(x)$. We consider an optimal first move d_1^v for Dominator in $G|v$. If d_1^v is in $\Gamma_1(x) \cup \{x\}$, then Staller can play in $\Gamma_4(x)$. If d_1^v is in $\Gamma_3(x) \cup \Gamma_4(x)$, then Staller can play x. From the two cases, Dominator cannot end this game in the third move. Since $\gamma_{tg}(G|v) = 3$, we get that d_1^v is in $\Gamma_2(x)$. Then the remaining two moves must be played in $\Gamma_1(x)$ and $\Gamma_3(x)$ to totally dominate x and vertices in $\Gamma_4(x)$, respectively. Therefore, Staller cannot play a vertex in $\{x\} \cup \Gamma_2(x) \cup \Gamma_4(x)$, which implies that the following properties hold

- d_1^v totally dominates all vertices in $\Gamma_1(x) \cup \Gamma_3(x)$.

- There is no edge in $\Gamma_4(x)$.

- Each legal move s_1^v in $\Gamma_3(x)$ must totally dominate all vertices in $\Gamma_4(x)$.

Since G is open twin-free, we have $|V(\Gamma_4(x))| = 1$. ■

Next, we let w be the vertex in $\Gamma_4(x)$.

Claim 2. Each vertex in $\Gamma_3(x)$ is adjacent to w.

Proof. Suppose that there is a vertex v in $\Gamma_3(x)$ such that v is not adjacent to w. We show that Staller has a strategy that can end the game in $G|v$ using at least four moves as follows. If d_1^v is in $\Gamma_1(x) \cup \{x\}$, then Staller plays w so at least two more moves are played to totally dominate d_1^v and w. If $d_1^v = w$, then Staller plays x. If d_1^v is in $\Gamma_2(x)$ and v is a legal move, then Staller plays v. If d_1^v is in $\Gamma_2(x)$ and v is an illegal move, then Staller plays in $\Gamma_2(x)$ to totally dominate d_1^v. In these three cases at least two more moves are played to totally dominate x and w. If d_1^v is in $\Gamma_3(x)$ and w is totally dominated, then Staller plays w. If d_1^v is in $\Gamma_3(x)$ but w is not totally dominated, then Staller plays in $\Gamma_3(x)$ to totally dominate w. In these two cases at least two moves are played to totally dominate x and $\Gamma_1(x)$. From all cases, the number of moves in $G|v$ is at least four. It implies that $\gamma_{tg}(G|v) \geq 4$, a contradiction. Therefore $N(w) = V(\Gamma_3(x))$ and each vertex in $\Gamma_3(x)$ is adjacent to w. ■

Claim 3. Each vertex in $\Gamma_2(x)$ is adjacent to every vertex in $\Gamma_1(x) \cup \Gamma_3(x)$.

Proof. We show that each vertex in $\Gamma_1(x)$ is adjacent to every vertex in $\Gamma_2(x)$. Suppose that there are two vertices v, y such that v is in $\Gamma_1(x)$, y is in $\Gamma_2(x)$ and v

is not adjacent to y. Then we have $|\Gamma_1(x)| > 1$ since y is adjacent to some vertex in $\Gamma_1(x)$. We show that Staller has a strategy that can end the game in $G|v$ using at least four moves as follows. We consider an optimal first move d_1^v of Dominator's in $G|v$. If $d_1^v = v$, then Staller plays in $\Gamma_1(x)$ to totally dominate y. If d_1^v is in $\Gamma_1(x)\backslash\{v\}$, then Staller plays x. In these two cases at least two more moves are played to totally dominate $\Gamma_3(x)$ and w. If d_1^v is in $\Gamma_3(x)$, then Staller plays x so at least two more moves are played to totally dominate $\Gamma_3(x)$ and x. If $d_1^v = w$, then Staller plays x so at least two more moves are played to totally dominate x and w. It remains to consider the case that d_1^v is in $\Gamma_2(x)$. Suppose that $\Gamma_2(x)$ does not contain any edges. Since $\gamma_{tg}(G|v) = 3$, d_1^v must totally dominate all vertices in $(\Gamma_1(x) \cup \Gamma_3(x))\backslash\{v\}$. It implies that for each $a \in V(\Gamma_1(x))$, there is $b \in V(\Gamma_3(x))$ such that $N(a) \cup N(b)$ contains all vertices in $\Gamma_2(x)$. Also for each $b \in V(\Gamma_3(x))$, there is $a \in V(\Gamma_1(x))$ such that $N(a) \cup N(b)$ contains all vertices in $\Gamma_2(x)$. We now consider an optimal first move d_1^y of Dominator's in $G|y$. If d_1^y is not in $\Gamma_2(x)$, then Staller can totally dominate d_1^y without playing in $\Gamma_2(x)$ so at least two more moves are played to totally dominate all remaining vertices. If d_1^y is in $\Gamma_2(x)$ but d_1^y does not totally dominate some vertex in $\Gamma_1(x) \cup \Gamma_3(x)$, then Staller can totally dominate that vertex by playing in $\{x\} \cup \Gamma_4(x)$ so at least two more moves are played to totally dominate x and $\Gamma_4(x)$. In these two cases, at least four moves are played in this game. Since $\gamma_{tg}(G|y) = 3$, the move d_1^y must be played in $\Gamma_2(x)$ and it totally dominates all of $\Gamma_1(x) \cup \Gamma_3(x)$. Now in a game on G, assume that Dominator starts at $d_1 = d_1^y$. Since $\Gamma_2(x)$ does not contain any edges, Staller is forced to play in $\Gamma_1(x)$ or $\Gamma_3(x)$. By the above results, Dominator can end the game on G in his second move. It implies that $\gamma_{tg}(G) \leq 3$, a contradiction. Therefore, $\Gamma_2(x)$ contains some edge. It implies that $|\Gamma_2(x)| \geq 2$. Since d_1^v is in $\Gamma_2(x)$ in $G|v$, Staller can respond by playing in $\Gamma_2(x)$. So at least two more moves are played to totally dominate x and w. From all cases, we get that $\gamma_{tg}(G|v) \geq 4$. This contradicts with Corollary 3.3. Hence, each vertex in $\Gamma_2(x)$ is adjacent to every vertex in $\Gamma_1(x)$. Similarly, each vertex in $\Gamma_2(x)$ is adjacent to every vertex in $\Gamma_3(x)$. ■

Claim 4. $\Gamma_2(x)$ contains at least one edge.

Proof. Suppose that $V(\Gamma_2(x)) = \{v\}$. If Dominator starts at $d_1 = v$ in G, then only three vertices x, v and w are not totally dominated by d_1. By Claim 3, Staller cannot totally dominate only v. It implies that Dominator can end this game in his second move. Thus, $\gamma_{tg}(G) \leq 3$, a contradiction. Therefore $|\Gamma_2(x)| \geq 2$. Since G is open twin-free and by Claim 3, we get that $\Gamma_2(x)$ contains some edge. ■

Claim 5. $\Gamma_1(x)$ and $\Gamma_3(x)$ are complete graphs.

Proof. Suppose there are nonadjacent vertices u and v in $\Gamma_1(x)$. We show that Staller has a strategy that can end the game in $G|v$ using at least four moves as follows. If $d_1^v = w$, then Staller plays x. If d_1^v is in $\Gamma_2(x)$, then by Claim 4, Staller can play in $\Gamma_2(x)$. In these two cases at least two more moves are played to totally dominate x and w. If d_1^v is in $\Gamma_3(x)$, then Staller plays w. Since v cannot totally dominate u, at least two more moves are played to totally dominate $\{x\} \cup \Gamma_1(x) \setminus \{v\}$. If d_1^v is in $\Gamma_1(x)$, then by assumption, Staller can play x. So at least two more moves are played to totally dominate $\Gamma_3(x)$ and w. By this strategy, the number of moves in $G|v$ is at least four. It implies that $\gamma_{tg}(G|v) \geq 4$, which is a contradiction. Therefore, $\Gamma_1(x)$ is a complete graph. Similarly, $\Gamma_3(x)$ is a complete graph. ■

Claim 6. $\Gamma_2(x)$ is a complete graph (of order at least 2).

Proof. Let v be an arbitrary vertex in $\Gamma_2(x)$. From the proof of Claim 1, we have an optimal first move d_1^v of Dominator's in $G|v$ is in $\Gamma_2(x)$. By Claim 3, d_1^v totally dominates all vertices in $\Gamma_1(x) \cup \Gamma_3(x)$. We show that v is adjacent to some vertex in $\Gamma_2(x)$. If v is isolated in $\Gamma_2(x)$, then we can assume that $d_1^v \neq v$ and d_1^v is incident to an edge in $\Gamma_2(x)$ by using Total Cotinuation Principle and Claim 4. In this case, Staller can play in $\Gamma_2(x)$ to totally dominate d_1^v. It implies that at least two more moves are played to totally dominate x and w. Thus, $\gamma_{tg}(G|v) \geq 4$, a contradiction. Therefore there is a vertex v' in $\Gamma_2(x)$ that is adjacent to v.

Suppose there is a vertex u in $\Gamma_2(x)$ such that u and v are not adjacent. We consider an optimal first move d_1^u of Dominator's in $G|u$. If $d_1^u = u$, then Staller plays v' to totally dominate v. Otherwise, Staller plays in $\Gamma_2(x)$ to totally dominate d_1^u. So at least two more moves are played to totally dominate x and w. It implies that $\gamma_{tg}(G|v) \geq 4$, which is a contradiction. Hence, $\Gamma_2(x)$ is a complete graph. ■

Thus, $G \cong K_1 \vee K_{n_1} \vee K_{n_2} \vee K_{n_3} \vee K_1$ for some positive integers n_1, n_2, n_3 where $n_2 \geq 2$. □

Chapter 5

Some 4-γ_{tg}-critical graphs with diameter 3

In this chapter, we characterize some 4-γ_{tg}-critical graphs with diameter 3. By Lemma 3.4, a 4-γ_{tg}-critical graph has total domination number 3 or 4.

Lemma 5.1. *Let G be a connected 4-γ_{tg}-critical graph. If $\gamma_t(G) = 4$, then $diam(G) = 3$. Furthermore, if $diam(G) \in \{2, 4\}$, then $\gamma_t(G) = 3$.*

Proof. Let $v \in V(G)$ with $ecc(v) = diam(G)$. Suppose $\gamma_t(G) = 4$. Then we have $ecc(v) \geq 2$. By Corollary 3.4, $\gamma_{tg}(G|v) = 3$. Assume that d_1^v and d_2^v are the optimal moves for Dominator in $G|v$. Since G is γ_{tg}-critical, by Lemma 2.20 d_1^v is not played in $N(v)$. If either s_1^v or d_2^v is in $N(v)$, then $\{d_1^v, s_1^v, d_2^v\}$ is a total dominating set of G. It is a contradiction with $\gamma_t(G) = 4$. Thus, Staller cannot play in $N(v)$. It means that d_1^v totally dominates all vertices in $\Gamma_2(v)$. Since d_1^v is not in $\Gamma_1(v)$, we get that d_1^v is in $\Gamma_3(v)$. Hence, $diam(G) \geq 3$. If $diam(G) = 4$, then by Theorem 4.1, we have $\gamma_t(G) = 3$. Thus, $diam(G) = 3$. By Lemma 3.4 and Lemma 3.6, the second statement of the lemma is equivalent to the first one. □

5.1 4-γ_{tg}-critical graph G with $diam(G) = 3$ and $\gamma_t(G) = 4$

Theorem 5.2. *Let G be a graph with $diam(G) = 3$ and $\gamma_t(G) = 4$. Then G is 4-γ_{tg}-critical if and only if G is obtained from the complete bipartite graph $K_{n,n}$*

for some $n \geq 3$ by removing a perfect matching.

Proof. First, we assume that G is 4-γ_{tg}-critical. Let $x \in V(G)$ with $ecc(x) = 3$. We consider that d_1^x is an optimal move of Dominator's in $G|x$. As in the proof of Lemma 5.1 (with $v = x$), the following results hold in $G|x$.

- d_1^x is in $\Gamma_3(x)$.
- d_1^x totally dominates all vertices in $\Gamma_2(x)$.
- Staller cannot play in $N(x) = V(\Gamma_1(x))$.

Consequently, $\Gamma_1(x)$ does not contain any edges. Let $d_1^x = y$. Then $V(\Gamma_2(x)) \subseteq N(y)$, implying that $d(y,z) = 1$ or 2 for any vertex z in $\Gamma_3(x)$ and $z \neq y$. Next, we consider an optimal move d_1^y of Dominator's in $G|y$. Since $d(x,y) = 3 = diam(G)$, we get that $ecc(y) = 3$. Similarly with d_1^x in $G|x$, we have d_1^y totally dominates all vertices in $\Gamma_2(y)$ in $G|y$. Since $\Gamma_1(x)$ does not contain any edges and $ecc(y) = 3$, each vertex in $\Gamma_1(x)$ is adjacent to some vertex in $\Gamma_2(x)$. Since y is adjacent to every vertex in $\Gamma_2(x)$, we get that d_1^y is not in $\Gamma_2(x)$ and $V(\Gamma_1(x)) \subseteq V(\Gamma_2(y))$. Similar to d_1^x, the move d_1^y totally dominates all vertices in $\Gamma_2(y)$, and hence $\Gamma_1(x)$. If there is a vertex z in $\Gamma_3(x)$ such that $d(y,z) = 2$, then d_1^y must totally dominate z so d_1^y is in $\Gamma_2(x)$ to totally dominate $\Gamma_1(x) \cup \{z\}$ and d_1^y is adjacent to y, a contradiction. We conclude that $N[y] = V(\Gamma_2(x) \cup \Gamma_3(x))$ so $\Gamma_2(y) = \Gamma_1(x)$ and $\Gamma_3(y) = \{x\}$. Since d_1^y is in $\Gamma_3(y)$, we have $d_1^y = x$. If $\Gamma_3(x)$ contains at least two vertices, then Staller can respond in $\Gamma_3(x) \backslash \{y\}$. It implies that at least two more moves are played to totally dominate x and some vertices in $\Gamma_3(x)$ so $\gamma_{tg}(G|y) \geq 4$, a contradiction. Therefore $V(\Gamma_3(x)) = \{y\}$, and thus $\Gamma_1(y) = \Gamma_2(x)$. Similar to $\Gamma_1(x)$, the subgraph $\Gamma_1(y)$ does not contain any edges. Since $\gamma_{tg}(G) = 4$, we get that $|V(\Gamma_1(x))| > 1$ and $|V(\Gamma_2(x))| > 1$. So far, each of $\Gamma_1(x)$ and $\Gamma_2(x)$ contains at least two vertices but has no edges, and $|\Gamma_3(x)| = 1$.

Since $\gamma_t(G) = 4$ and each of $\Gamma_1(x)$ and $\Gamma_2(x)$ has no edges, there are two vertices u, v such that $u \in V(\Gamma_1(x)), v \in V(\Gamma_2(x))$ and $d(u, v) = 3$. Then $ecc(u) = 3$. From the above argument, we have $V(\Gamma_3(u)) = \{v\}$. In $G|u$, we have $d_1^u = v$. Since $\Gamma_1(x)$ has no edges, $V(\Gamma_1(x))\backslash\{u\} \subseteq V(\Gamma_2(u))$, which implies that d_1^u totally dominates all vertices in $\Gamma_1(x)$ except u. It means that v is adjacent to every vertex in $\Gamma_1(x)$ except u. Similarly, u is adjacent to every vertex in $\Gamma_2(x)$ except v. Therefore, $|V(\Gamma_1(x))| = |V(\Gamma_2(x))|$. Let $n = |V(\Gamma_1(x))| + 1$. Hence, G is obtained from $K_{n,n}$ for some $n \geq 3$ by removing a perfect matching.

Conversely, we let $n \geq 3$ and a graph G be the graph obtained from $K_{n,n}$ by removing a perfect matching. Clearly $diam(G) = 3$ and $\gamma_t(G) = \gamma_{tg}(G) = 4$. Let (X, Y) be a bipartition of G. We show that $\gamma_{tg}(G|v) \leq 3$ for any $v \in V(G)$. Let $v \in V(G)$. Without loss of generality, we assume that $v \in X$. Then there exists $w \in Y$ such that v is not adjacent to w. We show that Dominator has a strategy to end the game in $G|v$ within 3 moves as follows. Dominator starts on w. After this move, all vertices in X are totally dominated. Since there is no edge in Y, Staller is forced to play in X to totally dominate $|Y| - 1$ vertices in Y. Then the number of moves in $G|v$ is at most three, implying that $\gamma_{tg}(G|v) \leq 3$. Hence, G is a 4-γ_{tg}-critical graph. $\square$

5.2 4-γ_{tg}-critical graph G with $diam(G) = 3$ and $\gamma_t(G) = 3$

For a graph G with $diam(G) = 3$ and $\gamma_t(G) = 3$, we characterize 4-γ_{tg}-critical graph G when there is a vertex v in G such that some optimal first move of Dominator's in $G|v$ is not in any γ_t-set of G.

Lemma 5.3. *Let G be a graph with $diam(G) = 3$ and $\gamma_t(G) = 3$. Assume that there are vertices $v, x \in V(G)$ such that x is an optimal first move of Dominator's in $G|v$ and x is not in any γ_t-set of G. If G is 4-γ_{tg}-critical, then the following conditions hold.*

(i) $V(\Gamma_3(x)) = \{v\}$ *and* $N(v) = V(\Gamma_2(x))$.

(ii) $\Gamma_1(x)$ *contains at least one edge and* $\Gamma_2(x)$ *does not contain any edges.*

(iii) Each vertex in $\Gamma_1(x)$ *is adjacent to every vertex in* $\Gamma_2(x)$ *except one vertex.*

(iv) Each vertex in $\Gamma_2(x)$ *is not adjacent to at least one vertex in* $\Gamma_1(x)$.

(v) $|V(\Gamma_1(x))| \geq |V(\Gamma_2(x))| \geq 2$.

Proof. Let $v, x \in V(G)$ such that x is an optimal first move of Dominator's in $G|v$ and x is not in any γ_t-set of G. We show that G satisfies the conditions (i) to (v).

Claim 7. For each $u \in V(G)$, there is no total dominating set of size 3 of $G|u$ that contains both x and a neighbor of u (distinct from x).

Proof. Clear by assumption. ■

Claim 8. $V(\Gamma_3(x)) = \{v\}$ and each vertex in $N(v)$ is not adjacent to any vertex in $\Gamma_2(x)$.

Proof. Recall that x is an optimal first move of Dominator's. If $v = x$ or v is in $\Gamma_2(x)$, then Staller can play s_1^v on a vertex in $\Gamma_1(x)$ that is adjacent to v. Since $\gamma_{tg}(G|v) = 3$, Dominator can end the game in $G|v$ with d_2^v. It implies that $\{x, s_1^v, d_2^v\}$ is a total dominating set of $G|v$ that contains $s_1^v \in N(v)$, a contradiction with Claim 7. Therefore, $v \neq x$ and v is not in $\Gamma_2(x)$. Since $x \notin N(v)$ and $diam(G) = 3$, we have v is in $\Gamma_3(x)$.

If $\{v\} \subsetneqq \Gamma_3(x)$, then Staller can respond with a vertex s_1^v in $\Gamma_3(x)\backslash\{v\}$ and make $\gamma_{tg}(G|v) \geq 4$, a contradiction. Thus, $V(\Gamma_3(x)) = \{v\}$. Since $\gamma_{tg}(G|v) = 3$, $d_1^v = x$ and by Claim7, we get that every vertex in $N(v)$ is not a legal response for Staller in $G|v$. So every vertex in $N(v)$ is not adjacent to any vertices in $\Gamma_2(x)$. ■

Claim 9. There is no vertex in $\Gamma_1(x)$ that is adjacent to every vertex in $\Gamma_2(x)$. Furthermore, $|V(\Gamma_1(x))| \geq 2$ and $|V(\Gamma_2(x))| \geq 2$.

Proof. Suppose that there is a vertex u in $\Gamma_1(x)$ that is adjacent to every vertex in $\Gamma_2(x)$. By Claim 8, $V(\Gamma_3(x)) = \{v\}$, and we get that $\{x, u, w\}$ is a γ_t-set of G where $w \in N(v)$, a contradiction with the assumption. So there is no vertex in $\Gamma_1(x)$ that is adjacent to every vertex in $\Gamma_2(x)$. It implies that $|V(\Gamma_1(x))| \geq 2$ and $|V(\Gamma_2(x))| \geq 2$. ■

Claim 10. $V(\Gamma_2(x)) = N(v)$. Moreover, there is no vertex in $\Gamma_2(x)$ that is adjacent to every vertex in $\Gamma_1(x)$, and $\Gamma_2(x)$ does not contain edges.

Proof. Suppose that there is a vertex in $\Gamma_2(x)$ that is not adjacent to v. By Claim 8, each vertex in $N(v)$ is not adjacent to any vertices in $\Gamma_2(x)$. By Total

Continuation Principle, there is no vertex in $N(v)$ that is adjacent to every vertex in $\Gamma_1(x)$; otherwise this vertex is a neighbor of v that is no worse than the optimal first move x in $G|v$.

In $G|x$, we consider an optimal first move d_1^x of Dominator's. If $d_1^x = x$, then Staller can play s_1^x in $\Gamma_1(x)$. Since $\gamma_{tg}(G|x) = 3$, Dominator can end the game in $G|v$ with d_2^x so $\{x, s_1^x, d_2^x\}$ is a total dominating set of $G|x$ that contains $s_1^x \in N(x)$. It contradicts with Claim 7. Therefore, d_1^x is in $\{v\} \cup \Gamma_2(x)$.

Suppose that Staller cannot respond with x. Then d_1^x is in $\Gamma_2(x)$ and totally dominates all vertices in $\Gamma_1(x)$. Since there is no vertex in $N(v)$ that is adjacent to every vertex in $\Gamma_1(x)$, we have $d_1^x \notin N(v)$. Since each vertex in $N(v)$ is not adjacent to any vertices in $\Gamma_2(x)$, Staller can play v. It implies that at least two more moves are played to totally dominate d_1^x and v, which contradicts with $\gamma_{tg}(G|x) = 3$. Thus, Staller can always play s_1^x on x.

If d_1^x is in $N(v)$, then Staller plays x. Since $\gamma_{tg}(G|x) = 3$ and by Claim 8, Dominator can end the game by playing in $\Gamma_1(x)$ to totally dominate all vertices in $\Gamma_2(x)$. This contradicts with Claim 9. Thus, $d_1^x \notin N(v)$. So Staller plays x and there is a vertex u in $\Gamma_2(x)$ such that $u \notin N(v)$ and u has not been totally dominated. It implies that at least two more moves are played to totally dominate u and v. Thus, $\gamma_{tg}(G|x) \geq 4$, a contradiction. We conclude that v is adjacent to every vertex in $\Gamma_2(x)$. It implies that there is no vertex in $\Gamma_2(x)$ that is adjacent to every vertex in $\Gamma_1(x)$. By Claim 8, we get that $\Gamma_2(x)$ does not contain edges. ■

Claim 11. $\Gamma_1(x)$ contains at least one edge.

Proof. Suppose that $\Gamma_1(x)$ does not contain any edges. Since each γ_t-set of G has size 3, its induced subgraph is isomorphic to the path P_3 or the cycle C_3. Since $\Gamma_2(x)$ does not contain edges and x is not in any γ_t-set of G, each γ_t-set of G must contain one vertex in $\Gamma_1(x)$ that is adjacent to every vertex in $\Gamma_2(x)$ or contain one vertex in $\Gamma_2(x)$ that is adjacent to every vertex in $\Gamma_1(x)$. It contradicts with Claim 9 or Claim 10. Thus, $\Gamma_1(x)$ contains at least one edge. ■

Claim 12. Each vertex in $\Gamma_1(x)$ is adjacent to every vertex in $\Gamma_2(x)$ except one vertex.

Proof. Suppose that there is a vertex u in $\Gamma_1(x)$ that is not adjacent to at least two vertices y and z in $\Gamma_2(x)$. In $G|u$, we consider an optimal first move d_1^u of Dominator's. Since $|V(\Gamma_1(x))| \geq 2$, we have $d_1^u \neq v$; otherwise $\gamma_{tg}(G|u) \geq 4$.

If d_1^u is in $\Gamma_1(x)$ but $d_1^u \neq u$, then Staller responds with x. By Claim 9, there is no vertex in $\Gamma_1(x)$ that is adjacent to every vertex in $\Gamma_2(x)$ so at least two more moves are played to totally dominate v and its neighbor(s). Thus, $\gamma_{tg}(G|u) \geq 4$, a contradiction.

If $d_1^u = u$, then v, y, z are not totally dominated by u. Since $\Gamma_2(x)$ does not contain any edges, v and one of $\{y, z\}$ cannot be totally dominated simultaneously. Since $\gamma_{tg}(G|u) = 3$, Staller cannot totally dominate exactly one of $\{y, z\}$. It implies that $N(y) = N(z)$ so y and z are open twins. This contradicts with Corollary 2.19. Thus, d_1^u is not in $\Gamma_1(x)$. Therefore, d_1^u is in $\Gamma_2(x)$.

If Staller can respond with $s_1^u = x$, then an optimal move d_2^u is in $\Gamma_1(x)$. Since $\gamma_{tg}(G|u) = 3$, we have $\{d_1^u, x, d_2^u\}$ is a total dominating set of $G|u$ that contains $x \in N(u)$. It contradicts with Claim 7. So Staller cannot play x. It means that d_1^u totally dominates every vertex in $\Gamma_1(x)$ except u. Without loss of generality, we can assume that $d_1^u = y$. Then $V(\Gamma_1(x))\backslash\{u\} \subseteq N(y)$.

In $G|y$, we consider an optimal first move d_1^y for Dominator. If $d_1^y = x$, then Staller responds with v. So at least two more moves are played to totally dominate v and x, a contradiction. If $d_1^y = u$, then Staller plays x. So at least two more moves are played to totally dominate v and z, a contradiction. If d_1^y is in $\Gamma_2(x)$, then Staller plays x. Since $\gamma_{tg}(G|y) = 3$, an optimal move d_2^y is in $\Gamma_1(x)$. It means that there is a vertex w in $\Gamma_1(x)$ such that $N(w)$ contains all vertices in $\Gamma_2(x)$ except y. Since u is not adjacent to z, we have $u \neq w$. Since $V(\Gamma_1(x))\backslash\{u\} \subseteq N(y)$, we get that $w \in N(y)$. It is a contradiction. Thus, each vertex in $\Gamma_1(x)$ is adjacent to every vertex in $\Gamma_2(x)$ except one vertex. ■

Claim 13. $|V(\Gamma_1(x))| \geq |V(\Gamma_2(x))|$

Proof. Suppose that $|V(\Gamma_1(x))| < |V(\Gamma_2(x))|$. By Claim 12, each vertex in $\Gamma_1(x)$ is not adjacent one vertex in $\Gamma_2(x)$. It implies that at least one vertex in $\Gamma_2(x)$ must be adjacent to every vertex in $\Gamma_1(x)$, a contradiction with Claim 10. Hence, $|V(\Gamma_1(x))| \geq |V(\Gamma_2(x))|$. ■

Thus, the conditions (i) to (v) hold. □

Theorem 5.4. *Let G be a graph with $diam(G) = 3$ and $\gamma_t(G) = 3$. Assume that there are $v, x \in V(G)$ such that x is an optimal first move of Dominator's in $G|v$ and x is not in any γ_t-set of G. If $|V(\Gamma_1(x))| = |V(\Gamma_2(x))|$, then G is 4-γ_{tg}-critical if and only if G is obtained from $K_{n,n}$ for some $n \geq 3$ by removing a perfect* *matching and adding at least one edge to one of the partite sets and the resulting set contains at least one isolated vertex.*

Proof. Assume that $|V(\Gamma_1(x))| = |V(\Gamma_2(x))| = k$ and G is 4-γ_{tg}-critical. Let $V(\Gamma_1(x)) = \{y_1, y_2, ..., y_k\}$ and $V(\Gamma_2(x)) = \{z_1, z_2, ..., z_k\}$. Suppose that there is a vertex in $\Gamma_2(x)$ that is not adjacent to at least two vertices in $\Gamma_1(x)$. Without loss of generality, we can assume that y_1, y_2 are not adjacent to z_1. Since z_1 is adjacent to some vertex in $\Gamma_1(x)$, we have $k = |V(\Gamma_1(x))| \geq 3$. For each $3 \leq i \leq k$, by Lemma 5.3(iii), y_i is adjacent to every vertex in $\Gamma_2(x)$ except one vertex. Without loss of generality, for $i \geq 2$ we can assume that y_i is not adjacent to one of $\{z_1, z_2, ..., z_{i-1}\}$. It implies that z_k is adjacent to every vertex in $\Gamma_1(x)$, a contradiction with Lemma 5.3(iv). Therefore, each vertex in $\Gamma_2(x)$ is adjacent to every vertex in $\Gamma_1(x)$ except one vertex.

From Lemma 5.3 and the above result, we get that G is obtained from $K_{n,n}$ with bipartition $(\Gamma_1(x) \cup \{v\}, \Gamma_2(x) \cup \{x\})$ by removing a perfect matching in $(\Gamma_1(x), \Gamma_2(x))$ and adding at least one edge in $\Gamma_1(x)$. Note that v is an isolated vertex in the partite set $\Gamma_1(x) \cup \{v\}$.

To prove the other direction, let $n \geq 3$ and (Y, Z) be a bipartition of $K_{n,n}$. Assume that G is obtained from $K_{n,n}$ by removing a perfect matching, and adding at least one edge to Y so that vertex $y \in Y$ is isolated in $G[Y]$. Let z be the vertex in Z that is not adjacent to y. We show that G is 4-γ_{tg}-critical. Clearly $\gamma_{tg}(G) \leq 4$.

We show that Staller has a strategy to end this game in G using at least four moves as follows. We consider Dominator's first move in G. If Dominator starts on one of $\{y, z\}$, then Staller plays the other vertex in $\{y, z\}$. Since y and z cannot be totally dominated together, we get that at least two more moves are played to totally dominate y and z. If Dominator starts in $Z \backslash \{z\}$, then Staller plays z. So every vertex in Z is not totally dominated. Since there is no vertex that is adjacent to all of Z, we get that at least two more moves are played to totally dominate Z. If Dominator starts in $Y \backslash \{y\}$, then Staller plays z. So y and one vertex in Z are not totally dominated. Since y does not have neighbor in Y, we get that at least two more moves are played. From all cases, we get that at least four moves are played to finish the game in G. Thus, $\gamma_{tg}(G) \geq 4$. Consequently, $\gamma_{tg}(G) = 4$.

It remains to show that $\gamma_{tg}(G|u) \leq 3$ for any $u \in V(G)$. Let $u \in V(G)$. We show that Dominator has a strategy to end the game in $G|u$ within 3 moves as follows. We consider four possibilities.

- Case $u = y$. Then Dominator starts on z.
- Case $u \in Y \backslash \{y\}$. Then there is a vertex w in Z such that w is adjacent to every vertex in Y except u. Dominator starts on w.

From these two cases, after Dominator makes his first move, every vertex in Y is totally dominated and at most two more moves are played to totally dominate Z.

- Case $u = z$. Then Dominator starts on y.

- Case $u \in Z \backslash \{z\}$. Then there is a vertex w in Y such that w is adjacent to every vertex in Z except u. Dominator starts on w.

From these two cases, after Dominator makes his first move, every vertex in Z is totally dominated. By assumption on G, Dominator can end the game in his second move. By this strategy of Dominator's, the number of moves in $G|u$ is three. It implies that $\gamma_{tg}(G|u) \leq 3$. Hence G is 4-γ_{tg}-critical. This completes the proof for Theorem 5.4. □

Theorem 5.5. *Let G be a graph with $diam(G) = 3$ and $\gamma_t(G) = 3$. Assume that there are $v, x \in V(G)$ such that x is an optimal first move of Dominator's in $G|v$ and x is not in any γ_t-set of G. If $|V(\Gamma_1(x))| > |V(\Gamma_2(x))|$, then G is 4-γ_{tg}-critical if and only if the following conditions hold.*

(i) $V(\Gamma_3(x)) = \{v\}$ *and* $N(v) = V(\Gamma_2(x))$ *which contains at least two vertices.*

(ii) $\Gamma_1(x)$ *contains at least one edge and* $\Gamma_2(x)$ *does not contain any edges.*

(iii) Each vertex in $\Gamma_1(x)$ *is adjacent to every vertex in* $\Gamma_2(x)$ *except one vertex.*

(iv) Each vertex in $\Gamma_2(x)$ *is not adjacent to at least one vertex in* $\Gamma_1(x)$*.*

(v) If y is a vertex in $\Gamma_2(x)$, then y is adjacent to every vertex in $\Gamma_1(x)$ except one vertex or for every vertex in $\Gamma_1(x)$ that is not adjacent to y, its closed neighborhood contains every vertex in $\Gamma_1(x)$.

Proof. Assume that $|V(\Gamma_1(x))| > |V(\Gamma_2(x))|$. To prove the forward direction, we assume that G is 4-γ_{tg}-critical. By Lemma 5.3, conditions (i)-(iv) hold. It remains to show that condition (v) holds. Let y be a vertex in $\Gamma_2(x)$. By condition (iv), y is not adjacent to at least one vertex in $\Gamma_1(x)$, say z.

Suppose that $V(\Gamma_1(x)) \nsubseteq N[z]$. Then we show that y is adjacent to every vertex in $\Gamma_1(x)$ except z. We consider $G|z$. By Lemma 2.20, we have $d_1^z \neq x$. If $d_1^z = v$, then Staller responds with x so at least two more moves are played to totally dominate x and v, a contradiction. If d_1^z is in $\Gamma_1(x)$, then one vertex in $\Gamma_2(x)$ and at least one vertex in $\Gamma_1(x)$ are not totally dominated. Staller can respond with x so at least two more moves are played to totally dominate v and the undominated vertex in $\Gamma_2(x)$, a contradiction. Thus, d_1^z is in $\Gamma_2(x)$. By condition (iii) and Lemma 2.20, we get that $d_1^z = y$. Suppose $V(\Gamma_1(x)\backslash\{z\}) \nsubseteq N(y)$. Then Staller can respond with x. So x and every vertex in $\Gamma_2(x)$ are not totally dominated. Since $\gamma_{tg}(G|z) = 3$, Dominator can end this game by playing in $\Gamma_1(x)$. Thus, there is a vertex in $\Gamma_1(x)$ that is adjacent every vertex in $\Gamma_2(x)$. It contradicts with condition (iii). Hence y is adjacent to every vertex in $\Gamma_1(x)$ except z. Thus, condition (v) holds.

To prove the other direction, we assume that conditions (i) to (v) hold and show that G is 4-γ_{tg}-critical. First, we show that $\gamma_{tg}(G) = 4$. Clearly $\gamma_{tg}(G) \leq 4$. We show that Staller has a strategy to end this game in G using at least 4 moves as follows. We consider Dominator's first move in G. If Dominator starts on a vertex in $\{x, v\}$, then Staller plays the other vertex in $\{x, v\}$. Since x and v cannot be totally dominated by the same vertex, we get that at least two more moves are played to totally dominate x and v. If Dominator starts in $\Gamma_1(x)$, then Staller responds with x so one vertex in $\Gamma_2(x)$ and v are not totally dominated. Since

$\Gamma_2(x)$ does not contains any edges, we get that two more moves are played to totally dominate them. If Dominator starts in $\Gamma_2(x)$, then Staller responds with x so x and every vertex in $\Gamma_2(x)$ are not totally dominated. By (iii), at least two more moves are played to totally dominate x and $\Gamma_2(x)$. From all cases, we get that at least four moves are played to finish the game in G. Thus, $\gamma_{tg}(G) \geq 4$. Consequently, $\gamma_{tg}(G) = 4$.

It remains to show that $\gamma_{tg}(G|u) \leq 3$ for any $u \in V(G)$. Let $u \in V(G)$. We show that Dominator has a strategy to end the game in $G|u$ within 3 moves as follows. We consider four possibilities.

Case 1 : $u = v$. Then Dominator starts on x. If Staller totally dominates x, then Dominator can end this game by playing v. Otherwise, Staller plays v so Dominator can end this game by playing to totally dominate x.

Case 2 : u is in $\Gamma_2(x)$. By (iv), there is a vertex w in $\Gamma_1(x)$ that is not adjacent to u. Then Dominator starts on w and every vertex in $\Gamma_2(x)$ are totally dominated.

Subcase 2.1 For every vertex in $\Gamma_1(x)$ that is not adjacent to u, its closed neighborhood contains every vertex in $\Gamma_1(x)$. In particular, $V(\Gamma_1(x)) \subseteq N[w]$. Then after Dominator plays w, only two vertices w and v are not totally dominated so at most two more moves are played to totally dominate them.

Subcase 2.2 u is adjacent to every vertex in $\Gamma_1(x)$ except w. If Staller responds with x, then Dominator can end this game by playing u. If Staller responds with a vertex in $\Gamma_2(x)$, then Dominator can end this game by playing x.

Assume that Staller plays s_1^u in $\Gamma_1(x)$. If s_1^u totally dominates w, then Dominator can end this game by playing u. Otherwise, we can assume that s_1^u totally dominates $w' \neq w$. By (iii), there is a vertex u' in $\Gamma_2(x)$ that is not adjacent to w'. Then $u' \neq u$. We consider u' in $\Gamma_2(x)$.

Suppose that there are at least two vertices in $\Gamma_1(x)$ that are not adjacent to u'. By (v), $V(\Gamma_1(x)) \subseteq N[w']$ so w' is totally dominated by $d_1^u = w$. It implies that u' is adjacent to every totally undominated vertices in $\Gamma_1(x)$. Thus, Dominator can end this game by playing u'.

Case 3 : u is in $\Gamma_1(x)$. Then $d_1^u \notin \{v, x\}$; otherwise Staller can make the game last more than three moves. By (iii), u is not adjacent to exactly one vertex in $\Gamma_2(x)$, say w. By (v), we consider two possibilities.

Subcase 3.1 w is adjacent to every vertex in $\Gamma_1(x)$ except u.

Then Dominator starts on w. After Staller's response, at least one vertex in $\Gamma_2(x)$ is totally dominated. By (iii), Dominator can end this game by playing in $\Gamma_1(x)$.

Subcase 3.2 w is not adjacent to at least two vertices in $\Gamma_1(x)$.

Then by (v), $V(\Gamma_1(x)) \subseteq N[u]$. Dominator starts on u and only two vertices v, w are not totally dominated so at most two more moves are played to totally dominate them.

Case 4 : $u = x$. Then Dominator starts on v. If Staller responds in $\Gamma_2(x)$, then Dominator can end this game by playing x. If Staller responds with x, then only v is not totally dominated so Dominator can end this game in his next move. Assume that Staller plays s_1^u in $\Gamma_1(x)$. Then s_1^u totally dominates at least one vertex in $\Gamma_1(x)$, say y. By (iii), there is a vertex z in $\Gamma_2(x)$ that is not adjacent to y. If s_1^u is not adjacent to z, then by (v), $V(\Gamma_1(x)) \subseteq N[s_1^u]$. Thus, only two vertices s_1^u and v are not totally dominated so Dominator can totally dominate them in his next move.

By this strategy of Dominator's, at most 3 moves are played in $G|u$. It implies that $\gamma_{tg}(G|u) \leq 3$. Hence G is 4-γ_{tg}-critical. This completes the proof for Theorem 5.5. $\square$

In Figure 5.1, the graphs G_1 and G_2 are examples of 4-γ_{tg}-critical graphs that correspond to Theorem 5.5.

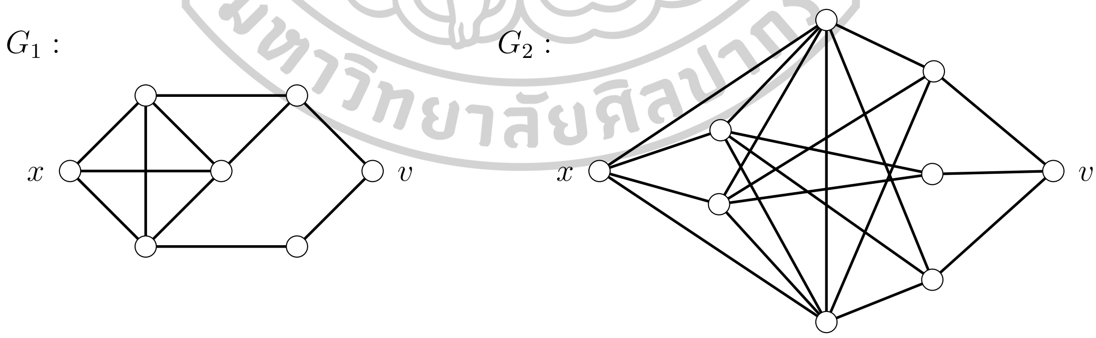

Figure 5.1: Examples of 4-γ_{tg}-critical graphs with diameter 3 and total domination numbers 3 that correspond to Theorem 5.5.

Chapter 6

Conclusions

In this chapter, we summarize all the results obtained in this study. We start by characterizing the disconnected 4-γ_{tg}-critical graphs. The disconnected 4-γ_{tg}-critical graph is the graph obtained from the union of two nontrivial complete graphs.

Theorem 3.5. *Let G be a disconnected graph. Then G is 4-γ_{tg}-critical if and only if $G \cong K_m + K_n$ for some $m, n \geq 2$.*

For a 4-γ_{tg}-critical connected graph G, we have $2 \leq diam(G) \leq 4$. Next, the 4-γ_{tg}-critical graph with diameter 4 is characterized.

Theorem 4.1. *Let G be a connected graph with $diam(G) = 4$. Then G is 4-γ_{tg}-critical if and only if $G \cong K_1 \vee K_{n_1} \vee K_{n_2} \vee K_{n_3} \vee K_1$ for some positive integers n_1, n_2, n_3 where $n_2 \geq 2$.*

Recall Lemma 3.4, if G is 4-γ_{tg}-critical, then $\gamma_t(G) \in \{3, 4\}$. The next result shows the characterization of 4-γ_{tg}-critical graph G with $diam(G) = 3$ and $\gamma_t(G) = 4$.

Theorem 5.2. *Let G be a graph with $diam(G) = 3$ and $\gamma_t(G) = 4$. Then G is 4-γ_{tg}-critical if and only if G is obtained from the complete bipartite graph $K_{n,n}$*

for some $n \geq 3$ by removing a perfect matching.

For a 4-γ_{tg}-critical graph G with $diam(G) = 3$ and $\gamma_t(G) = 3$, we characterize such graph when it has a vertex v in G such that some optimal first move of Dominator in $G|v$ is not in any γ_t-set of G.

Theorem 5.4. *Let G be a graph with $diam(G) = 3$ and $\gamma_t(G) = 3$. Assume that there are $v, x \in V(G)$ such that x is an optimal first move of Dominator's in $G|v$ and x is not in any γ_t-set of G. If $|V(\Gamma_1(x))| = |V(\Gamma_2(x))|$, then G is 4-γ_{tg}-critical if and only if G is obtained from $K_{n,n}$ for some $n \geq 3$ by removing a perfect matching and adding at least one edge to one of the partite sets and the resulting set contains at least one isolated vertex.*

Theorem 5.5. *Let G be a graph with $diam(G) = 3$ and $\gamma_t(G) = 3$. Assume that there are $v, x \in V(G)$ such that x is an optimal first move of Dominator's in $G|v$ and x is not in any γ_t-set of G. If $|V(\Gamma_1(x))| > |V(\Gamma_2(x))|$, then G is 4-γ_{tg}-critical if and only if the following conditions hold.*

(i) $V(\Gamma_3(x)) = \{v\}$ *and* $N(v) = V(\Gamma_2(x))$ *which contains at least two vertices.*

(ii) $\Gamma_1(x)$ *contains at least one edge and* $\Gamma_2(x)$ *does not contain any edges.*

(iii) Each vertex in $\Gamma_1(x)$ *is adjacent to every vertex in* $\Gamma_2(x)$ *except one vertex.*

(iv) Each vertex in $\Gamma_2(x)$ *is not adjacent to at least one vertex in* $\Gamma_1(x)$*.*

(v) If y is a vertex in $\Gamma_2(x)$, then y is adjacent to every vertex in $\Gamma_1(x)$ except one vertex or for every vertex in $\Gamma_1(x)$ that is not adjacent to y, its closed neighborhood contains every vertex in $\Gamma_1(x)$.

Therefore, we are left with one more case for 4-γ_{tg}-critical graphs with diameter 3, that is to characterize 4-γ_{tg}-critical graph G with $\gamma_t(G) = 3$, and for every vertex v in G, every optimal first move of Dominator's in $G|v$ is in some γ_t-set of G.

The graphs in Figure 6.1 are examples of 4-γ_{tg}-critical graphs that correspond to the conditions in this case.

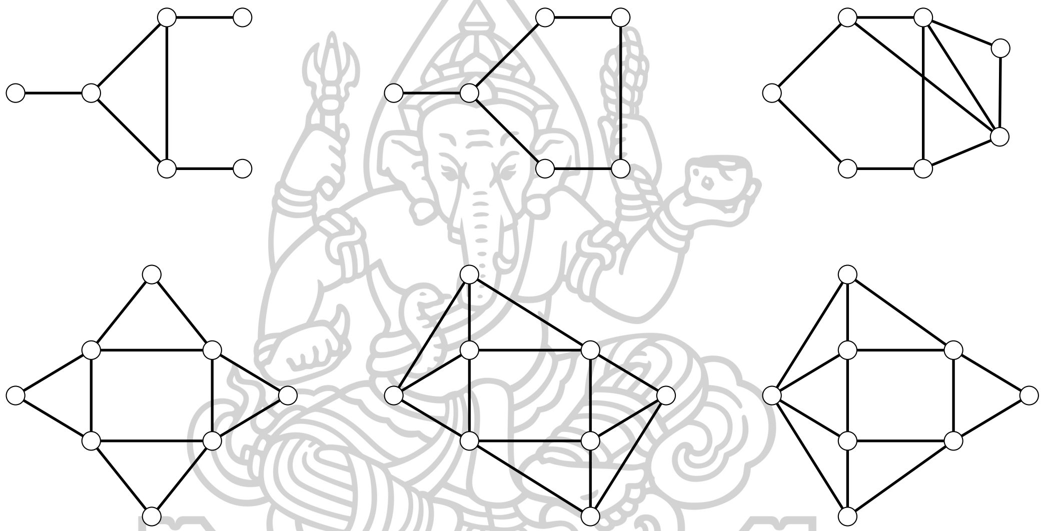

Figure 6.1: Examples of other 4-γ_{tg}-critical graphs with diameter 3.

For a 4-γ_{tg}-critical graph G with diameter 2, we show that for every vertex v in G, every optimal first move of Dominator's in $G|v$ is in some γ_t-set of G.

Proposition 6.1. *Let G be a 4-γ_{tg}-critical graph with diam$(G) = 2$. If v is a vertex in G, then every optimal first move of Dominator's in $G|v$ is in some γ_t-set of G.*

Proof. By Lemma 5.1, we have $\gamma_t(G) = 3$. Let $v \in V(G)$. In $G|v$, we consider that d_1^v is optimal. By Lemma 2.20, we have $d_1^v \notin N(v)$. Since $diam(G) = 2$ and $\gamma_{tg}(G|v) = 3$, we get that $ecc(v) = 2$. If $d_1^v = v$, then Staller can respond with s_1^v in $\Gamma_1(v)$ to totally dominate some vertices of $\Gamma_2(v)$. If d_1^v is in $\Gamma_2(v)$, then Staller can respond with s_1^v in $\Gamma_1(v)$ to totally dominate d_1^v. Note that $s_1^v \in N(v)$ and it cannot end the game; otherwise $\{d_1^v, s_1^v\}$ is a total dominating set of size 2. Therefore, $\{d_1^v, s_1^v, d_2^v\}$ is a γ_t-set of G. □

Figure 6.2 shows examples of 4-γ_{tg}-critical graphs with diameter 2.

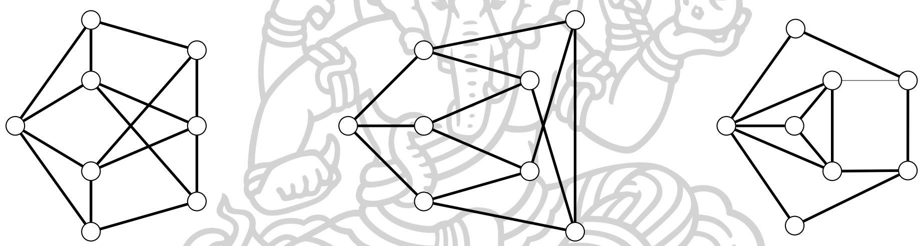

Figure 6.2: Examples of 4-γ_{tg}-critical graphs with diameter 2.

Lastly, we summarize the remaining problem in this topic.

Problem 6.2. Characterize 4-γ_{tg}-critical graph G where $\gamma_t(G) = 3, diam(G) \in \{2, 3\}$, and for every vertex v in G, every optimal first move of Dominator's in $G|v$ is in some γ_t-set of G.

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- Chalermpong Worawannotai, and Karnchana Charoensitthichai. "4-Total domination game critical graphs." **Discrete Mathematics, Algorithms and Applications** (accepted).

Conferences

- Presentation of the work titled "4-Total domination game critical graphs" at "The 10th Slovenian International Conference on Graph Theory" held in Kranjska Gora, Slovenia, from June 18-24, 2023.

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