SOME NECESSARY CONDITIONS FOR GRAPHS WITH EXTREMAL CONNECTED 2-DOMINATING SETS

By
MR. Piyawat WONGTHONGCUE

A Thesis Submitted in Partial Fulfillment of the Requirements
for Master of Science MATHEMATICS
Department of MATHEMATICS
Silpakorn University
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Title	Some necessary conditions for graphs with extremal connected 2-dominating sets
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Field of Study	MATHEMATICS
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MR. PIYAWAT WONGTHONGCUE : SOME NECESSARY CONDITIONS FOR GRAPHS WITH EXTREMAL CONNECTED 2-DOMINATING SETS. THESIS ADVISOR : ASSISTANT PROFESSOR CHALERMPONG WORAWANNOTAI, Ph.D.

Let G be a graph with no multiple edges or loops. A set S of vertices of G is a dominating set if every vertex in $V(G) \setminus S$ is adjacent to at least one vertex of S. A connected k-dominating set of G is a subset S of vertices in $V(G)$ such that every vertex in $V(G) \setminus S$ has at least k neighbors in S, and the subgraph $G[S]$ is connected. The domination number of G is the number of vertices in a minimum dominating set of G, denoted by $\gamma(G)$. The connected k-domination number of G, denoted by $\gamma_k^c(G)$, is the minimum cardinality of a connected k-dominating set of G. For $k = 1$, we simply write $\gamma_c(G)$. For $k = 2$, it is known that the bounds $\gamma_2^c(G) \geqslant \gamma(G)+1$ and $\gamma_2^c(G) \geqslant \gamma_c(G)+1$ are sharp. In this thesis, we study one of the open problems posted by Volkmann in 2009. In particular, we study graphs with the smallest possible connected 2-domination numbers with respect to domination numbers and connected domination numbers. We provide a characterization of the connected graphs G with $\gamma(G) = 1$ and $\gamma_2^c(G) = 2$. Moreover, we present necessary conditions of the connected graphs G with $\gamma_2^c(G) = \gamma(G)+1$ and $\gamma_2^c(G) = \gamma_c(G)+1$, respectively, when $\gamma_2^c(G) \geqslant 3$. Lastly, we present a graph construction that takes

in any connected graph with k vertices and gives a graph G with $\gamma_2^c(G) = k$, $\gamma_c(G) = k - 1$ and $\gamma(G) \in \{k - 1, k - 2\}$.

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Piyawat Wongthongcue

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CHAPTER 1

INTRODUCTION

1.1 Introduction

Let $G = (V(G), E(G))$ be a graph with a vertex set $V(G)$ and an edge set $E(G)$, where each edge of $E(G)$ is associated with an unordered pair of vertices (not necessary distinct vertices) of $V(G)$. Two vertices are *adjacent* and are *neighbors* if there exists an edge associated with the two vertices; in this case, we say that the edge *joins* the two vertices and the vertices are its *end vertices*. An edge is called a *loop* if it joins a vertex to itself. *Multiple edges* are edges that join the same pair of vertices. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a subset S of $V(G)$, the *induced subgraph* $G[S]$ is the subgraph of G whose vertex set is S and whose edge set consists of all the edges in $E(G)$ that have both endpoints in S. That is, for any two vertices $u, v \in S$, u and v are adjacent in $G[S]$ if and only if they are adjacent in G. In this thesis, we only consider simple graphs i.e., graphs with no multiple edges and loops.

1.2 Domination in graphs

A subset S of the vertex set of a graph G is a *dominating set* if every vertex in $V(G) \setminus S$ is adjacent to at least one vertex of S. The *domina-*

tion number of G, denoted by $\gamma(G)$, is the number of vertices in a minimum dominating set of G.

Claude Berge [2] introduced the concept of the domination number of a graph in 1962, where he referred to it as the "co-efficient of external stability". The term "dominating set" and "domination number" were originally coined by Ore [16] in 1962. In 1977, Cockayne and Hedetniemi [5] wrote a report summarizing what was known at that time about dominating sets in graphs. In that paper, they introduced the notation $\gamma(G)$ for the domination number of a graph, which later became the standard notation used.

Domination on graphs is the focus of many studies. Many researchers are interested in this field because it has many practical applications. For example, in the context of surveillance cameras [14], a dominating set can be utilized to determine the minimum number of cameras required to cover a certain area. This area can be represented by a graph, with each vertex within the graph indicating a potential camera location, and each edge representing the visibility between those positions. By finding a minimum dominating set, the number of necessary cameras can be minimized, thus providing full coverage while reducing costs in a surveillance camera system. Then domination can be used to solve this kind of resource allocation problem. Refer to [9, 10, 11, 12, 13] for additional information on domination.

Many variations of domination arise from imposing an additional condition on the dominating set. Here, we are interested in connected domination, k-domination, and the combination of these two.

1.3 Connected domination in graphs

A *connected dominating set* of a connected graph G is a dominating set S of G such that $G[S]$ is connected. The *connected domination number* of G, denoted by $\gamma_c(G)$, is the minimum cardinality of a connected dominating set of G. Any connected dominating set of cardinality $\gamma_c(G)$ is called a γ_c*-set* of G. Since connected dominating sets are dominating sets, $\gamma(G) \leqslant \gamma_c(G)$ for any connected graph G.

The concept of connected domination in graphs was introduced by Sampathkumar and Walikar [18] in 1979. Various researchers have conducted extensive analyses to determine upper and lower bounds for the connected domination number in arbitrary graphs. For instance, Sampathkumar and Walikar [18] have derived many bounds. Kleitman and West [15] looked into connected graphs that have spanning trees with many leaves in 1991. The connected domination number of a tree is the number of vertices that are not leaf within the tree. Thus, determining the minimum connected dominating set D is equivalent to find a spanning tree of G with maximum number of leaves. Kleitman and West [15] provided various bounds for the connected domination number $\gamma_c(G)$. Subsequently, Caro, West, and Yuster [3] presented an enhanced upper bound, surpassing the results of Kleitman and West, which is asymptotically sharp.

Example 1.1. Consider the graph C_6 in Figure 1.1.

Let $S = \{a, b, c, d\}$. Since $C_6[S]$ is connected and every vertex in

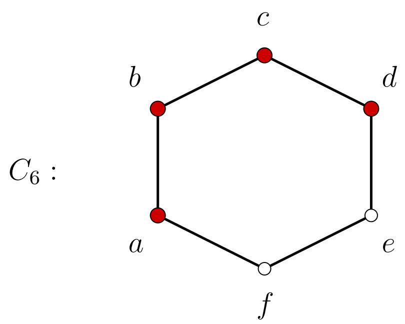

Figure 1.1: Graph C_6 and a connected dominating set of C_6 (the set of red vertices)

the set $V(C_6) \setminus S = \{e, f\}$ is adjacent to some vertex in S, it implies that S is a connected dominating set of C_6, so $\gamma_c(C_6) \leqslant 4$. Notice that any set of three vertices whose induced subgraph forms a connected subgraph of C_6 is not a connected dominating set of C_6. Thus, $\gamma_c(C_6) \geqslant 4$. Hence, $\gamma_c(C_6) = 4$ and S is a minimum connected dominating set of C_6.

1.4 k-domination in graphs

A *k-dominating set* of a graph G is a subset S of the vertex set $V(G)$ such that every vertex in $V(G) \setminus S$ has at least k neighbors in S. The *k-domination number* of G, denoted by $\gamma_k(G)$, is the minimum cardinality of a k-dominating set of G. Any k-dominating set of cardinality $\gamma_k(G)$ is called a *γ_k-set* of G.

The k-domination in graphs was introduced by Fink and Jacobson [6] in 1985. In addition, several researchers have studied the bounds of k-domination in graphs. In 1985, Cockayne, Gamble, and Shepherd [4] proved that for a graph G with a minimum degree δ and an integer k satisfying $\delta \geqslant k$, the k-domination number $\gamma_k(G) \leqslant \dfrac{k|V(G)|}{|V(G)|+1}$. Rautenbach and Volkmann [17] later relaxed the minimum degree requirement significantly and introduced an upper bound on the k-domination number $\gamma_k(G)$ in 2007. Hansberg and Volkmann [8] presented two new upper bounds for the k-domination number $\gamma_k(G)$ for a graph G, which are better than the previous bounds in 2009.

Example 1.2. Consider the graph C_6 in Figure 1.2.

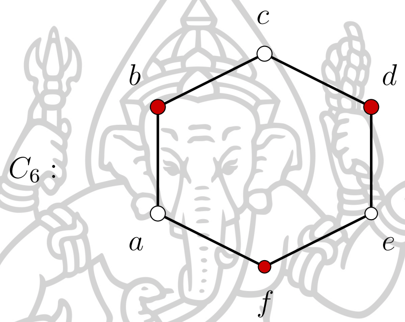

Figure 1.2: Graph C_6 and a 2-dominating set of C_6 (the set of red vertices)

Let $S = \{b, d, f\}$. Since every vertex in the set $V(C_6) \setminus S = \{a, c, e\}$ has 2 neighbors in S, it follows that S is a 2-dominating set of C_6, so $\gamma_2(C_6) \leqslant 3$. Note that any 2-dominating set in a graph with at least two vertices must contain at least two members. From Figure 1.2, we observe that all dominating sets of size 2 in C_6 are $\{a, d\}$, $\{b, e\}$, and $\{c, f\}$. However, these sets are not 2-dominating sets. Therefore, C_6 has no 2-dominating sets of size 2. It follows that a 2-dominating set in C_6 must contain at least three members. Thus, $\gamma_2(C_6) \geqslant 3$. Consequently, $\gamma_2(C_6) = 3$, and S is a minimum 2-dominating set of C_6.

1.5 Connected k-domination in graph

A *connected k-dominating set* of a graph G is a subset S of the vertex set $V(G)$ such that every vertex in $V(G) \setminus S$ has at least k neighbors in S and the subgraph $G[S]$ is connected. The *connected k-domination number* of G, denoted by $\gamma_k^c(G)$, is the minimum cardinality of a connected k-dominating set of G. Any connected k-dominating set of cardinality $\gamma_k^c(G)$ is called a *γ_k^c-set* of G.

In 2009, Volkmann [19] introduced the connected k-dominating in graphs. In the paper, he characterized connected graphs G with $\gamma_k^c(G) = |V(G)|$. For $\delta(G) \geqslant k \geqslant 2$, he also characterized connected graphs G with $\gamma_k^c(G) = |V(G)| - 1$. Moreover, he presented various bounds of $\gamma_k^c(G)$ and proposed some open problems.

The bound $\gamma_k(G) \geqslant \gamma(G)+k-2$ for any graph G with $\delta(G) \geqslant k \geqslant 2$ was given by Fink and Jacobson in [6]. In 2010, Hansberg [7] presented a bound similar to Fink and Jacobson for the connected case, that is $\gamma_k^c(G) \geqslant \gamma_c(G) + k - 2$ where $\delta(G) \geqslant k \geqslant 2$. Moreover, they established various sharp bounds on the connected k-domination number and the k-domination number. For $k = 2$, Volkmann [19] established the sharp bound $\gamma_2^c(G) \geqslant \gamma_c(G)+1$. This implies that $\gamma_2^c(G) \geqslant \gamma(G) + 1$.

In 2013, Karima and Mustapha [1] gave some properties of connected graphs G with $\gamma_k^c(G) = |V(G)| - 2$. Then they provided a complete characterization of connected cubic graphs G with $\gamma_2^c(G) = |V(G)| - 2$ and connected 4-regular claw-free graphs G with $\gamma_3^c(G) = |V(G)| - 2$.

Example 1.3. Consider the graph C_6 in Figure 1.3.

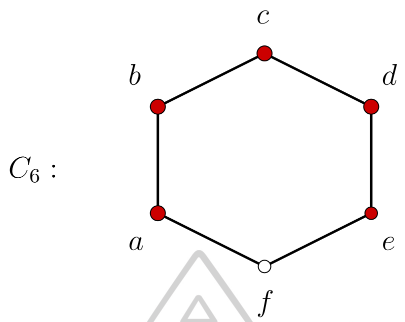

Figure 1.3: Graph C_6 and a connected 2-dominating set of C_6 (the set of red vertices)

Let $S = \{a, b, c, d, e\}$. Since $G[S]$ is connected and f has 2 neighbors in S, it implies that S is a connected 2-dominating set of C_6, so $\gamma_2^c(C_6) \leqslant 5$. Notice that any set S' of four vertices whose induced subgraph forms a connected subgraph of C_6 is not a connected 2-dominating set of C_6 because each of the remaining vertices has only one neighbor in S'. Thus, $\gamma_2^c(C_6) \geqslant 5$. Hence, $\gamma_2^c(C_6) = 5$ and S is a minimum connected 2-dominating set of C_6.

In this thesis, we study two of the open problems posted by Volkmann [19] in 2009. In particular, we study graphs with the smallest possible connected 2-domination numbers with respect to domination numbers and connected domination numbers. In Chapter 2, we recall necessary definitions and relevant results. In Chapter 3, we provide a characterization of the connected graphs G with $\gamma(G) = 1$ and $\gamma_2^c(G) = 2$. Moreover, we present a necessary condition of the connected graphs G with $\gamma_2^c(G) = \gamma(G) + 1$ and a necessary condition of the connected graphs G with $\gamma_2^c(G) = \gamma_c(G) + 1$, when $\gamma_2^c(G) \geqslant 3$.

Lastly, we present a graph construction that takes in any connected graph with k vertices and gives a graph G with $\gamma_2^c(G) = k$, $\gamma_c(G) = k-1$ and $\gamma(G) \in \{k-1, k-2\}$.

CHAPTER 2

PRELIMINARIES

In this chapter, we introduce fundamental definitions and relevant known results. Regarding graph theory in general, we follow the notations used in West's book [20].

Definition 2.1. The *open neighborhood* $N_G(v)$ of vertex v in a graph G is the set of vertices adjacent to v, and the *closed neighborhood* of v is $N_G[v] := N_G(v) \cup \{v\}$. For $X \subseteq V(G)$, its *open neighborhood* is the set $N_G(X) := \bigcup_{v \in X} N_G(v)$, and its *closed neighborhood* is the set $N_G[X] := N_G(X) \cup X$.

Definition 2.2. The *degree* of a vertex v in a graph G, written as $deg_G(v)$, is the number of edges that are connected to v. A vertex v of G is said to be an *isolated vertex* if $deg_G(v) = 0$. A vertex v of G is said to be *a leaf* or *a pendant* if $deg_G(v) = 1$. The vertex that is adjacent to a pendant is its *support vertex*.

Definition 2.3. The *order* of a graph G is the number of vertices in G.

Definition 2.4. A *universal vertex* in a graph G is a vertex that is adjacent to all other vertices of G.

Definition 2.5. An *independent set* in a graph is a set of pairwise nonadjacent vertices.

Definition 2.6. A graph G is *connected* if each pair of vertices in G belongs to a path; otherwise, G is *disconnected*.

Definition 2.7. A *path* P_n is a graph whose vertices can be listed in the order $v_1, v_2, \ldots, v_n$ such that v_i and v_{i+1} are adjacent where $i = 1, 2, \ldots, n-1$. A *cycle* C_n is a connected graph of order n such that every vertex has degree 2.

Definition 2.8. An *isomorphism* from a simple graph G to a simple graph H is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. We say G is *isomorphic* to H, written $G \cong H$, if there is an isomorphism from G to H.

Definition 2.9. A *complete graph* is a graph whose vertices are pairwise adjacent. A complete graph with n vertices is denoted by K_n.

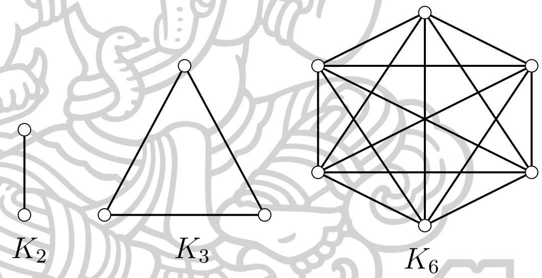

Figure 2.1: Graphs K_2, K_3, and K_6, respectively

Definition 2.10. A graph with no cycles is *acyclic*. A *tree* is a connected acyclic graph. A *spanning subgraph* of a graph G is a subgraph of G with vertex set $V(G)$. A *spanning tree* is a spanning subgraph that is a tree.

Lemma 2.11. *[20] Every tree with at least two vertices has at least two leaves.*

Lemma 2.12. *[20] Every connected graph contains a spanning tree.*

Definition 2.13. The graph obtained by taking the union of disjoint graphs G and H is the *disjoint union* or *sum* of G and H, written $G + H$.

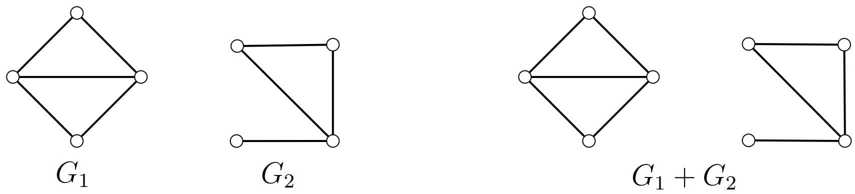

Figure 2.2: Graphs G_1, G_2, and $G_1 + G_2$, respectively

Definition 2.14. The *join* of disjoint graphs G and H, written $G \vee H$, is the graph obtained from the disjoint union $G + H$ by adding the edges $\{xy \mid x \in V(G), y \in V(H)\}$.

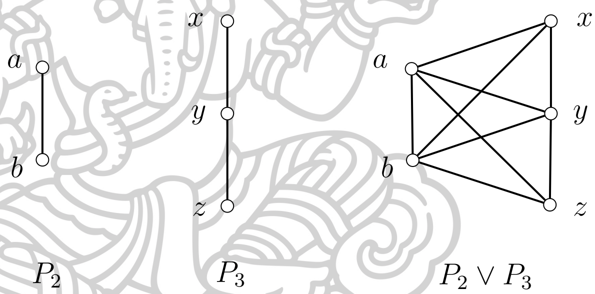

Figure 2.3: Graphs P_2, P_3, and $P_2 \vee P_3$, respectively

Definition 2.15. A *star graph* S_n of order n is a tree consisting of one vertex adjacent to all other $n - 1$ vertices.

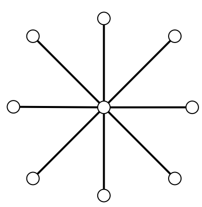

Figure 2.4: Graph S_9

CHAPTER 3

MAIN RESULTS

In this chapter, we find a necessary condition for a connected graph G to have $\gamma_2^c(G) = \gamma(G)+1$ and a necessary condition for a connected graph G to have $\gamma_2^c(G) = \gamma_c(G)+1$. First we provide a characterization of the connected graphs G with $\gamma(G) = \gamma_c(G) = 1$ and $\gamma_2^c(G) = 2$.

3.1 Graphs G with $\gamma_2^c(G) = 2$ and $\gamma(G) = \gamma_c(G) = 1$

Observation 3.1. Let G be a connected graph with $\gamma_2^c(G) = 2$. Let D be a γ_2^c-set of G. Then each vertex in D is a universal vertex. In particular, $\gamma(G) = \gamma_c(G) = 1$.

Theorem 3.2. *Let G be a connected graph of order at least 2. Then the following are equivalent.*

(i) $\gamma_2^c(G) = 2$,

(ii) $G \cong K_2 \vee H$ *for some graph* H.

Proof. $(i) \Rightarrow (ii)$ Assume that $\gamma_2^c(G) = 2$. Let $\{x, y\}$ be a γ_2^c-set of G. Then x and y are universal vertices of G. Hence, $G = G[\{x, y\}] \vee G[V(G) \setminus \{x, y\}]$. Observe that $G[\{x, y\}] \cong K_2$.

$(ii) \Rightarrow (i)$ Assume that $G \cong K_2 \vee H$ for some graph H. Then the vertex set of K_2 is a γ_2^c-set of G. Thus, $\gamma_2^c(G) \leqslant 2$. Clearly, for graphs of order at least 2, a connected 2-dominating set have at least two vertices. Hence, $\gamma_2^c(G) \geqslant 2$. Therefore, $\gamma_2^c(G) = 2$. □

From now on, we only consider connected graphs whose connected 2-domination numbers are at least 3.

3.2 Necessary condition for graphs G with $\gamma_2^c(G) = \gamma(G) + 1$

We begin by showing the existence of vertices x and y in a γ_2^c-set D of a graph G such that $x, y \in N_G(D \setminus \{x, y\})$, which shows that the coming necessary conditions are not null.

Lemma 3.3. *Let G be a connected graph with $\gamma_2^c(G) \geqslant 3$. Let D be a γ_2^c-set of G. Then there exist distinct vertices $x, y \in D$ such that $x, y \in N_G(D \setminus \{x, y\})$. Moreover, x, y can be chosen so that $G[D \setminus \{x, y\}]$ is connected.*

Proof. Since $G[D]$ is connected, by Lemma 2.12, there exists a spanning tree T of $G[D]$. Since T is a tree of order greater than 2, it has at least two leaves. Let x and y be two distinct leaves in T. Then $x, y \in N_G(D \setminus \{x, y\})$ and $G[D \setminus \{x, y\}]$ is connected. □

From Figure 3.1, we observe that G has a connected 2-domination number of at least 3. Let $D = \{t, u, x, y\}$. Then D is a connected 2-dominating set of G. Note that $G[D]$ is a connected graph. Moreover, upon considering a spanning tree T of $G[D]$, we notice that there exist distinct vertices x and y in D such that both x and y are neighbors of $D \setminus \{x, y\}$ in G. Notice that

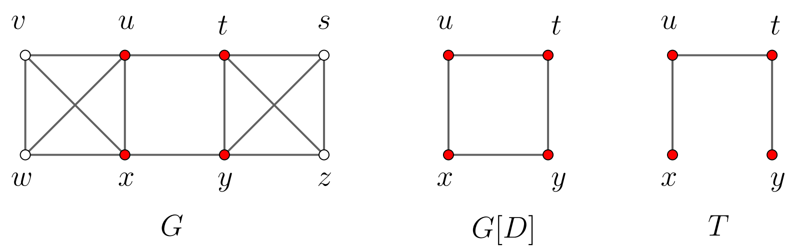

Figure 3.1: Graphs G, $G[D]$, and a spanning tree T of $G[D]$

$G[D \setminus \{x, y\}] = G[\{t, u\}]$ and $G[\{t, u\}] \cong P_2$. It implies that $G[\{t, u\}]$ forms a connected graph.

The following result provides a necessary condition of a connected graph G with $\gamma_2^c(G) = \gamma(G) + 1$.

Theorem 3.4. *Let G be a connected graph with $\gamma_2^c(G) \geqslant 3$ and $\gamma_2^c(G) = \gamma(G) + 1$. Let D be a γ_2^c-set of G. Then $N_G(x) \cap N_G(y) \nsubseteq N_G(D \setminus \{x, y\})$ for every pair of distinct vertices x and y in D such that $x, y \in N_G(D \setminus \{x, y\})$.*

Proof. Let x and y be two distinct vertices in D such that $x, y \in N_G(D \setminus \{x, y\})$. Suppose that $N_G(x) \cap N_G(y) \subseteq N_G(D \setminus \{x, y\})$. So, the vertices in $N_G(x) \cap N_G(y)$ are dominated by $D \setminus \{x, y\}$. Since $x, y \in N_G(D \setminus \{x, y\})$, the vertices x and y are also dominated by $D \setminus \{x, y\}$. Let v be a vertex of G not in $D \cup (N_G(x) \cap N_G(y))$. Then v is adjacent to at least one vertex in $D \setminus \{x, y\}$. Therefore, $D \setminus \{x, y\}$ is a dominating set of G of size $|D| - 2 = \gamma(G) - 1$, a contradiction. Consequently, $N_G(x) \cap N_G(y) \nsubseteq N_G(D \setminus \{x, y\})$. $\square$

3.3 Necessary condition for graphs G with $\gamma_2^c(G) = \gamma_c(G) + 1$

Similarly, we obtain a necessary condition of a connected graph G with $\gamma_2^c(G) = \gamma_c(G) + 1$.

Theorem 3.5. *Let G be a connected graph with $\gamma_2^c(G) \geqslant 3$ and $\gamma_2^c(G) = \gamma_c(G) + 1$. Let D be a γ_2^c-set of G. Then $N_G(x) \cap N_G(y) \nsubseteq N_G(D \setminus \{x, y\})$ for every pair of distinct vertices x and y in D such that $x, y \in N_G(D \setminus \{x, y\})$ and $G[D \setminus \{x, y\}]$ is connected.*

Proof. Let x and y be two distinct vertices in D such that $x, y \in N_G(D \setminus \{x, y\})$ and $G[D \setminus \{x, y\}]$ is connected. Suppose that $N_G(x) \cap N_G(y) \subseteq N_G(D \setminus \{x, y\})$. So the vertices in $N_G(x) \cap N_G(y)$ are dominated by $D \setminus \{x, y\}$. Since $x, y \in N_G(D \setminus \{x, y\})$, the vertices x and y are also dominated by $D \setminus \{x, y\}$. Let v be a vertex of G not in $D \cup (N_G(x) \cap N_G(y))$. Then v is adjacent to at least one vertex in $D \setminus \{x, y\}$. Since $G[D \setminus \{x, y\}]$ is connected, it follows that $D \setminus \{x, y\}$ is a connected dominating set of size $|D| - 2 = \gamma_c(G) - 1$, a contradiction. Consequently, $N_G(x) \cap N_G(y) \nsubseteq N_G(D \setminus \{x, y\})$. □

Graph $K_{2,4}$ in Figure 3.2 is an example of a connected graph where $\gamma_2^c(K_{2,4}) = 3$ and $\gamma(K_{2,4}) = \gamma_c(K_{2,4}) = 2$, satisfying the condition $\gamma_2^c(K_{2,4}) = \gamma(K_{2,4}) + 1$ and $\gamma_2^c(K_{2,4}) = \gamma_c(K_{2,4}) + 1$. Let $D = \{x_1, x_2, y_2\}$. Then D is a γ_2^c-set of $K_{2,4}$. If we consider each pair of distinct vertices in D except for the pair of x_1 and x_2, we observe that there exists one vertex among them that

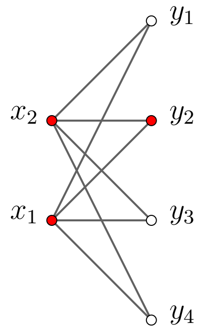

Figure 3.2: Graph $K_{2,4}$

is not a neighbor of the remaining vertices in D. Next, we consider a pair of distinct vertices x_1 and x_2 in D. We observe that both x_1 and x_2 are adjacent to y_2 and y_2 is the only common neighbor of x_1 and x_2 in D. Notice that $D \setminus \{x_1, x_2\} = \{y_2\}$ and its induced subgraph is a connected graph. Clearly, $N_{K_{2,4}}(x_1) \cap N_{K_{2,4}}(x_2) = \{y_1, y_2, y_3, y_4\}$ and $N_{K_{2,4}}(D \setminus \{x_1, x_2\}) = N_{K_{2,4}}(\{y_2\}) = \{x_1, x_2\}$. It implies that $N_{K_{2,4}}(x) \cap N_{K_{2,4}}(y) \nsubseteq N_{K_{2,4}}(D \setminus \{x, y\})$.

After obtaining the necessary conditions, we discover that graphs with such conditions have no universal vertices, as shown in the following propositions.

Proposition 3.6. *Let G be a connected graph with $\gamma_2^c(G) \geqslant 3$. For every γ_2^c-set D of G, assume that $N_G(x) \cap N_G(y) \nsubseteq N_G(D \setminus \{x, y\})$ for every pair of distinct vertices x and y in D such that $x, y \in N_G(D \setminus \{x, y\})$. Then G has no universal vertices.*

Proof. Let x and y be two distinct vertices in a γ_2^c-set D of G such that $x, y \in N_G(D \setminus \{x, y\})$. So, $N_G(x) \cap N_G(y) \nsubseteq N_G(D \setminus \{x, y\})$. Suppose that G has a universal vertex u. There are two possibilities.

- ▷ **Case 1**: $u \in D$. Since $N_G(x) \cap N_G(y) \nsubseteq N_G(D \setminus \{x, y\})$, there is a vertex z such that $z \in N_G(x) \cap N_G(y)$, but $z \notin N_G(D \setminus \{x, y\})$. Suppose that $u \in D \setminus \{x, y\}$. Since u is a universal vertex, it is adjacent to z. So, $z \in N_G(D \setminus \{x, y\})$, which is a contradiction. Thus, $u \in \{x, y\}$. Without loss of generality, we assume that $u = x$. Then x is adjacent to all vertices in $D \setminus \{x, y\}$. Since $|D| \geqslant 3$, we have $D \setminus \{x, y\} \neq \phi$. Let w be a vertex in $D \setminus \{x, y\}$. Since x is a universal vertex, the vertices $w, y \in N_G[x] \subseteq N_G(D \setminus \{w, y\})$. By the assumption, $N_G(w) \cap N_G(y) \nsubseteq N_G(D \setminus \{w, y\})$. However, $N_G(w) \cap N_G(y) \subseteq N_G[x] \subseteq N_G(D \setminus \{w, y\})$, a contradiction. Therefore, this case cannot happen.

- ▷ **Case 2**: $u \notin D$. Then u is adjacent to every vertex in D. Since $|D| \geqslant 3$, the set $D \setminus \{x, y\} \neq \phi$. Let w be a neighbor of x in $D \setminus \{x, y\}$. Let $D' = (D \setminus \{w\}) \cup \{u\}$. Since u is a universal vertex, the set D' is a connected 2-dominating set of G. Since $|D'| = |D|$, the set D' is also a γ_2^c-set of G. However, $u \in D'$. Just as in Case 1, this cannot happen.

From both cases, we conclude that G has no universal vertices. □

Proposition 3.7. *Let G be a connected graph with $\gamma_2^c(G) \geqslant 3$. For every γ_2^c-set D of G, assume that $N_G(x) \cap N_G(y) \nsubseteq N_G(D \setminus \{x, y\})$ for every pair of distinct vertices x and y in D such that $x, y \in N_G(D \setminus \{x, y\})$ and $G[D \setminus \{x, y\}]$ is connected. Then G has no universal vertices.*

Proof. Similar to the proof of Proposition 3.6. □

3.4 Construction of graphs G with $\gamma_2^c(G) = \gamma_c(G) + 1$

In this section, we use the necessary condition to construct an infinite family of graphs G that satisfy $\gamma_2^c(G) = \gamma_c(G) + 1$. Note that the condition $N_G(x) \cap N_G(y) \nsubseteq N_G(D \setminus \{x, y\})$ in Theorems 3.4 and 3.5 implies that $N_G(x) \cap N_G(y)$ must contain a vertex outside of $N_G(D \setminus \{x, y\})$.

Definition 3.8. For a connected graph H of order at least 3, we let $g(H)$ be the connected graph obtained from H by adding new vertices in the following way. For every pair of distinct vertices x and y in $V(H)$ such that $x, y \in N_H(V(H) \setminus \{x, y\})$, we add one new vertex and join it to x and y.

Observation 3.9. For any connected graph H, its vertex set $V(H)$ is a connected 2-dominating set of $g(H)$.

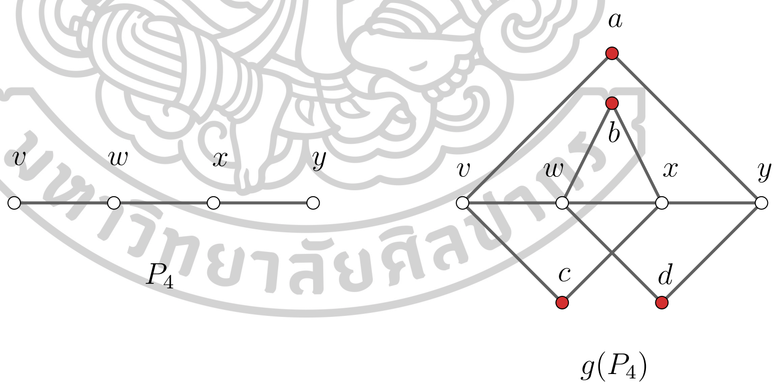

Figure 3.3: Graphs P_4 and $g(P_4)$

For example, let $H = P_4$. The connected graph $g(P_4)$ is obtained from P_4 by adding the red vertices, as illustrated in Figure 3.3. Note that $v \notin N_H(V(H) \setminus \{v, w\})$, so no new vertex was created for the pair v, w. In this case, we say v *and* w *do not create a new vertex in* G. Similarly, x and y do not create a new vertex in G. We also note that each new vertex has degree 2.

Next, we discuss some properties of graphs $g(H)$.

Lemma 3.10. *Let* H *be a connected graph of order* k *where* $k \geqslant 3$ *and let* $G = g(H)$. *The vertices* x *and* y *in* H *do not create a new vertex in* G *if and only if* x *and* y *are adjacent and one of the two vertices has degree* 1 *in* H.

Proof. We will prove the forward direction by the contrapositive method. Assume that x and y are not adjacent or both x and y have degree at least 2 in H. Since H is a connected graph, it implies that $x, y \in N_H(V(H) \setminus \{x, y\})$. By construction, x and y create a new vertex in G.

Conversely, assume that x and y are adjacent and one of the two vertices has degree 1 in H. Without loss of generality, let $deg_H(x) = 1$. Then $x \notin N_H(V(H) \setminus \{x, y\})$. It follows that x and y do not create a new vertex in G. □

Lemma 3.11. *Let* H *be a connected graph of order* k *where* $k \geqslant 3$ *and let* $G = g(H)$. *Then among any three vertices of* $V(H)$, *there exist two vertices that create a new vertex in* G.

Proof. Let $x, y, z \in V(H)$. Suppose there are no pairs of vertices among x, y and z that create a new vertex in G. By Lemma 3.10 and since x and y do not create a new vertex in G, the vertices x and y are adjacent and one of the two vertices has degree 1 in H, say y. Similarly, since x and z do not create a new vertex in G, the vertices x and z are adjacent and z has degree 1 in H.

Note that y and z are not adjacent in H. By Lemma 3.10, the vertices y and z create a new vertex in G, a contradiction. Hence, there exist two vertices among x, y and z that create a new vertex in G. □

Lemma 3.12. *Let H be a connected graph of order k where $k \geqslant 3$ and let l be the number of pendants in H. Then $|V(g(H))| = k + \binom{k}{2} - l$.*

Proof. Let $G = g(H)$. If every pair of vertices in H creates a new vertex in G, then the number of new vertices in G is $\binom{k}{2}$. By Lemma 3.10, the number of new vertices in G is $\binom{k}{2} - l$. By Definition 3.8, $|V(G)| = |V(H)| + \binom{k}{2} - l$. □

We proceed to find the connected 2-domination numbers of the graphs $g(H)$. We begin by proving two useful lemmas.

Lemma 3.13. *Let H be a connected graph of order k where $k \geqslant 3$. Let D be a connected 2-dominating set of $g(H)$. If $V(H) \setminus D$ contains a vertex u that does not create new vertices with any vertices in $D \cap V(H)$, then $D \cap V(H)$ is an independent set and u is adjacent to every vertex in $D \cap V(H)$.*

Proof. Assume that $V(H) \setminus D$ contains a vertex u that does not create new vertices with any vertices in $D \cap V(H)$. By Lemma 3.10, each vertex in $D \cap V(H)$ is adjacent to the vertex u. If $|D \cap V(H)| = 1$, then we are done. Otherwise, we have $deg_H(u) \geqslant 2$ so each vertex in $D \cap V(H)$ has degree 1 in H. Hence, $D \cap V(H)$ is an independent set. □

Lemma 3.14. *Let H be a connected graph of order 3 and let $G = g(H)$. Suppose that D is a connected 2-dominating set of G of size 2 such that $D \nsubseteq$*

$V(H)$. If there exist two vertices in $V(H) \setminus D$ that do not create a new vertex in G, then $|D \cap V(H)| = 1$.

Proof. Let $V(H) = \{x, y, z\}$. Assume that $x, y \in V(H) \setminus D$ and they do not create a new vertex in G. By Lemma 3.10, x and y are adjacent and one of the two has degree 1 in H, say y. Then y and z create a new vertex v in G. Next, we will show that $v \in D$. Suppose that $v \notin D$. Since D is a 2-dominating set and v is only adjacent to z and y, we have $y, z \in D$. This is a contradiction to $y \in V(H) \setminus D$. It follows that $v \in D$. Suppose that $D \cap V(H) = \phi$. Since $|D| = 2$, there exists a vertex $w \in D \setminus \{v\}$. Since $N_G(v) = \{y, z\}$, the vertex w is not adjacent to v. This is a contradiction to $G[D]$ being a connected graph. Hence, $|D \cap V(H)| = 1$. □

Theorem 3.15. *Let H be a connected graph of order k where $k \geqslant 3$ and let $G = g(H)$. Then $V(H)$ is a γ_2^c-set of G. In particular, $\gamma_2^c(G) = k$.*

Proof. By construction, $V(H)$ is a connected 2-dominating set of G of size k. Suppose that there exists a connected 2-dominating set D of G of size $k - 1 \geqslant 2$. Suppose that $D \subseteq V(H)$. Let u be the single vertex in $V(H) \setminus D$. If u does not create new vertices with any vertices in D, then by Lemma 3.13, the set D is independent. This contradicts $G[D]$ being a connected graph. Consequently, u creates a new vertex $v \in G \setminus H$ with some vertex w in D. Since $u \notin D$ and $N_G(v) = \{u, w\}$, it follows that D is not a 2-dominating set of G, a contradiction. Hence, $D \nsubseteq V(H)$. Then there is at least one vertex in D that does not belong to $V(H)$. So, $|D \cap V(H)| \leqslant k - 2$. It implies that there exist at least two vertices x and y in $V(H) \setminus D$. There are two possibilities.

▷ **Case 1**: x and y create a new vertex z in G. Suppose that $z \in D$. Since $N_G(z) = \{x, y\}$, the graph $G[D]$ is disconnected, a contradiction. Thus, $z \notin D$. Then the new vertex z is not dominated by D. This is a contradiction to D being a 2-dominating set of G.

▷ **Case 2**: x and y do not create a new vertex in G. By Lemma 3.10, the two vertices are adjacent and one of the two has degree 1 in H, say y. Note that $|V(H) \setminus \{x, y\}| = |V(H)| - 2 = k - 2$. Let $V(H) \setminus \{x, y\} = \{u_1, u_2, \ldots, u_{k-2}\}$. Since H is a connected graph and y is adjacent to x in $V(H) \setminus D$, for each $i \in \{1, ..., k-2\}$, we have that $u_i, y \in N_H(V(H) \setminus \{u_i, y\})$ so u_i and y create a new vertex v_i in G. Let $S = \{v_1, v_2, \ldots, v_{k-2}\}$. Next, we will show that $S \subseteq D$. Suppose that there exists an $i \in \{1, ..., k-2\}$ such that $v_i \notin D$. Since D is a 2-dominating set and $N_G(v_i) = \{u_i, y\}$, the vertices u_i and y are in D. This is a contradiction to $y \in V(H) \setminus D$. It implies that $v_i \in D$ for all $i \in \{1, ..., k-2\}$. So, $S \subseteq D$.

If $k = 3$, then $|S| = 1$ and $|D| = 2$. Thus, $S = \{v_1\}$. By Lemma 3.14, $|D \cap V(H)| = 1$. Since $V(H) = \{x, y, u_1\}$ and $x, y \notin D$, we have $D \cap V(H) = \{u_1\}$. Since $S \subseteq D$, the vertex v_1 belongs to $D \setminus V(H)$. Thus, $D = \{u_1, v_1\}$. Since y is a pendant with x as its support, y is not adjacent to u_1. It follows that D is not a 2-dominating set of G, a contradiction. Thus, $k \neq 3$.

Now suppose $k \geqslant 4$ so there exist at least 2 vertices in S. By construction, S is an independent set. Since each vertex v_i in S is created by joining it to y and $u_i \in V(H) \setminus \{x, y\}$, the vertices in S have only one common neighbor, namely y. But y is not in D. Since $S \subseteq D$ and $|D \setminus S| = 1$, the induced subgraph $G[D]$ is disconnected, a contradiction.

We conclude from the above two cases that a connected 2-dominating set of G has at least k members. Therefore, $V(H)$ is a γ_2^c-set of G and $\gamma_2^c(G) = k$. □

Let H be a path P_3 and let $G = g(H)$. Then G is isomorphic to a cycle C_4. Moreover, any three vertices in C_4 form a γ_2^c-set of C_4. Hence, any three vertices in G form a γ_2^c-set of G. It implies that the γ_2^c-set of G is not unique.

The next following result shows that $V(H)$ is the unique γ_2^c-set of G, where H is a connected graph of order $k \geqslant 3$ not isomorphic to a path P_3 and $G = g(H)$.

Theorem 3.16. *Let H be a connected graph of order $k \geqslant 3$ not isomorphic to a path on 3 vertices and let $G = g(H)$. Then $V(H)$ is the unique γ_2^c-set of G.*

Proof. By Theorem 3.15, we have that $V(H)$ is a γ_2^c-set of G. If $k = 3$, then H is a cycle on 3 vertices and it is easy to check that $V(H)$ is the only γ_2^c-set of G. It remains to consider $k \geqslant 4$. Suppose that there exists a γ_2^c-set D of G such that $D \neq V(H)$. So, $|D| = |V(H)|$ and $|V(H) \setminus D| = |D \setminus V(H)|$. Consider the following 3 cases.

- ▷ **Case 1**: $|V(H) \setminus D| = |D \setminus V(H)| = 1$. Let u be the unique vertex in $V(H) \setminus D$. Suppose that u does not create new vertices with any vertices in $D \cap V(H)$. By Lemma 3.13, the set $D \cap V(H)$ is independent and u is adjacent to every vertex in $D \cap V(H)$. Since $D \cap V(H)$ is an independent set of size at least 3 and the unique vertex in $D \setminus V(H)$ has degree 2, the graph $G[D]$ is disconnected, a contradiction. Therefore, u creates new vertices with some vertices in $D \cap V(H)$. Suppose u creates exactly one new vertex. Let a be the vertex in $D \cap V(H)$ that creates the new vertex with u. Since $k \geqslant 4$ and $|V(H) \setminus D| = 1$, we have $|(D \cap V(H)) \setminus \{a\}| \geqslant 2$. By Lemma 3.10, every vertex in $(D \cap V(H)) \setminus \{a\}$ is adjacent to u and has degree 1 in H. Then a is not adjacent to any vertex in $(D \cap V(H)) \setminus \{a\}$. Thus, $N_H(a) \subseteq \{u\}$. By this and Lemma 3.10, the vertices u and a are not adjacent. Therefore, a is not adjacent to any vertices in $V(H) \setminus \{a\}$. Consequently, H is disconnected, a contradiction. Thus, u creates at least two new vertices with some vertices in $D \cap V(H)$. Since $|D \setminus V(H)| = 1$, at least one of the new vertices above is not in D and is not 2-dominated by D, a contradiction.

▷ **Case 2**: $|V(H) \setminus D| = |D \setminus V(H)| = 2$. Let $V(H) \setminus D = \{x, y\}$. Suppose that x and y create a new vertex z in G. Suppose that $z \in D$. Since $deg_G(z) = 2$, the graph $G[D]$ is disconnected, a contradiction. So, $z \notin D$. Thus, D is not a dominating set of G, a contradiction. Therefore, x and y do not create a new vertex in G. By Lemma 3.10, the vertices x and y are adjacent and one of the two has degree 1 in H, say y.

Now, suppose x does not create new vertices with any vertices in $D \cap V(H)$. By Lemma 3.13, the set $D \cap V(H)$ is independent and x is adjacent to every vertex in $D \cap V(H)$. Since $D \cap V(H)$ is an independent set of size at least 2, the graph H is a star with at least 3 pendants. By Lemma 3.11, there exist at least $|D \cap V(H)|$ new vertices in G that are created by joining them to y and $D \cap V(H)$. If $|D \cap V(H)| > 2$, then at least one of the new vertices above is not in D and so it is not 2-dominated by D, a contradiction. Thus, $|D \cap V(H)| = 2$ and H is a star of order 4. By Lemma 3.12, the number of new vertices in $g(H)$ is three. Suppose that two new vertices in $g(H)$ that are created by joining them to y and $D \cap V(H)$ belong to $D \setminus V(H)$. Since both of the two new vertices have degree two and $D \cap V(H)$ is an independent set, the graph $G[D]$ is disconnected, a contradiction. Hence, at least one of the two new vertices in $g(H)$ that is created by joining them to y and $D \cap V(H)$ does not belong to D, and so it is not 2-dominated by D, a contradiction. Therefore, x creates new vertices with some vertices in $D \cap V(H)$.

Since y is a pendant with x as its support, by Lemma 3.14 the vertex y creates a new vertex with each vertex in $D \cap V(H)$. It follows that there exist at least $|D \cap V(H)| + 1 \geqslant 3$ new vertices in G that are adjacent to x or y. Since $|D \setminus V(H)| = 2$, at least one of the new vertices above is not in D and is not 2-dominated by D, a contradiction.

▷ **Case 3**: $|V(H) \setminus D| \geqslant 3$. Let $x, y, z \in V(H) \setminus D$. By Lemma 3.11, there exist two vertices in $\{x, y, z\}$ that create a new vertex in G. Without loss of generality, let x and y create a new vertex v in G. Suppose that $v \in D$. Since $deg_G(v) = 2$, the graph $G[D]$ is disconnected, a contradiction. So,

$v \notin D$. Thus, D is not a dominating set of G, a contradiction.

From the above three cases, we conclude that $V(H)$ is the unique γ_2^c-set of G. □

Now, we find the connected domination numbers of the graphs $g(H)$ and show how they are related to the connected 2-domination numbers.

Theorem 3.17. *Let H be a connected graph of order k where $k \geqslant 3$ and let $G = g(H)$. Then $\gamma_c(G) = k - 1$.*

Proof. Let S be a subset of $V(H)$ such that $|S| = k - 1$ and $G[S]$ is connected. Since $V(H)$ is a 2-dominating set of G, the set S is a connected dominating set of G. Thus, $\gamma_c(G) \leqslant k - 1$. Suppose that there exists a connected dominating set D of G of size $k - 2$. Suppose that $D \subseteq V(H)$. Then there exist $u, v \in V(H) \setminus D$. We consider the vertices u and v in $V(H) \setminus D$ in two cases.

- ▷ **Case 1**: u and v create a new vertex in G. Then the new vertex is not dominated by D. This is a contradiction to D being a dominating set.
- ▷ **Case 2**: u and v do not create a new vertex in G. By Lemma 3.10, u and v are adjacent and one of the two has degree 1 in $V(H)$, say v. Then v is not dominated by D, a contradiction.

From the two cases, we conclude that $D \nsubseteq V(H)$. Then at least one vertex in D does not belong to $V(H)$. So, $|D \cap V(H)| \leqslant k - 3$. It implies that there exist at least 3 vertices in $V(H) \setminus D$. Let $x, y, z \in V(H) \setminus D$. By Lemma 3.11, there exist two vertices in $V(H) \setminus D$ that create a new vertex in

G. Without loss of generality, let x and y create a new vertex t in G. Suppose that $t \in D$. Since $N_G(t) = \{x, y\}$, we have that $t \notin N_G(D)$, a contradiction. So, $t \notin D$. It follows that the new vertex t in G is not dominated by D, a contradiction. Hence, a connected dominating set of G has at least $k-1$ members. Therefore, $\gamma_c(G) = k - 1$. □

Corollary 3.18. *Let H be a connected graph of order k where $k \geqslant 3$ and let $G = g(H)$. Then $\gamma_2^c(G) = \gamma_c(G) + 1$.*

3.5 $\gamma_2^c(g(H)) = \gamma(g(H)) + 1$ or $\gamma_2^c(g(H)) = \gamma(g(H)) + 2$

In this section, we show that for any connected graph H of order at least 3, the graphs $g(H)$ satisfies either $\gamma_2^c(g(H)) = \gamma(g(H)) + 1$ or $\gamma_2^c(g(H)) = \gamma(g(H)) + 2$.

Firstly, let's begin by determining the domination number of $g(H)$ for any connected graph H of order 3.

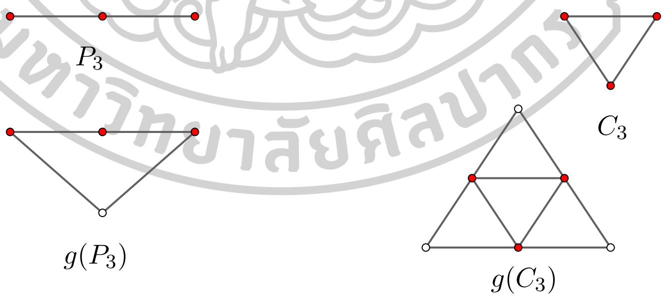

Figure 3.4: Graphs P_3, $g(P_3)$, C_3 and $g(C_3)$

Theorem 3.19. *Let H be a connected graph of order 3 and let $G = g(H)$. Then $\gamma(G) = 2$.*

Proof. Since H is a connected graph of order 3, it follows that H is either a path P_3 or a cycle C_3 of order 3. Since $g(P_3)$ is a cycle of order 4, it implies that $\gamma(g(P_3)) = 2$. From the graph $g(C_3)$ in Figure 3.4, it is easy to see that $\gamma(g(C_3)) = 2$. □

Next, we determine the domination number of $g(H)$ for any connected graph H of order at least 4.

Lemma 3.20. *Let H be a connected graph of order k where $k \geqslant 4$ and let $G = g(H)$. Then $\gamma(G) \geqslant k - 2$.*

Proof. Let $V(H) = \{v_1, v_2, v_3, \ldots, v_k\}$. Let $X = V(G) \setminus V(H)$. Then X consists of the new vertices. Suppose there exists $D \subsetneq V(G)$ such that $|D| = k-3$ and D dominates X. If D contains a new vertex x in X, then x was created by some vertices u and v in H. Since $N_G[x] \cap X \subseteq N_G[u] \cap X$, we can use the vertex u in H to dominate new vertices in X instead of the vertex x. Hence, it is sufficient to consider that the vertices in D are from $V(H)$. Without loss of generality, let $D = \{v_1, v_2, v_3, \ldots, v_{k-3}\}$. We divide the argument into two cases according to the number of pendants in $\{v_{k-2}, v_{k-1}, v_k\}$.

- ▷ **Case 1**: $\{v_{k-2}, v_{k-1}, v_k\}$ contains at most one pendant. Without loss of generality, assume v_{k-1} and v_k are not pendants. By Lemma 3.10, v_{k-1} and v_k create a new vertex in G which is not dominated by D, a contradiction.
- ▷ **Case 2**: $\{v_{k-2}, v_{k-1}, v_k\}$ contains at least two pendants. Without loss of generality, assume v_{k-1} and v_k are the two pendants. By Lemma 3.10,

v_{k-1} and v_k create a new vertex in G which is not dominated by D, a contradiction.

We conclude from the above two cases that at least $k-2$ vertices are required to dominate X. Thus, $\gamma(G) \geqslant k-2$. □

Theorem 3.21. *Let H be a connected graph of order k where $k \geqslant 4$ and let $G = g(H)$. If H contains two pendants that share a support vertex in H, then $\gamma(G) = k-2$.*

Proof. Let $V(H) = \{v_1, v_2, v_3, \ldots, v_k\}$. Assume that H contains 2 pendants that share a support vertex in H. For $i \neq j$, when v_i and v_j create a new vertex in G, we let v_{ij} denote the new vertex. Since $|V(H)| = k \geqslant 4$, no two pendants are adjacent. Without loss of generality, let v_{k-1} and v_k be two pendants of H with the common support vertex v_{k-2}. Let $D = \{v_1, v_2, v_3, \ldots, v_{k-3}\} \cup \{v_{k-1,k}\}$. By Lemma 3.10, v_{k-2} does not create a new vertex with either v_{k-1} or v_k. Since H is connected, the vertex v_{k-2} is adjacent to some vertex in $\{v_1, v_2, \ldots, v_{k-3}\}$. By construction, all vertices in G except v_{k-1}, v_k and $v_{k-1,k}$ are dominated by $\{v_1, v_2, v_3, \ldots, v_{k-3}\}$ but v_{k-1}, v_k and $v_{k-1,k}$ are dominated by $v_{k-1,k}$. Hence, D dominates all vertices in G. Since $|D| = k-2$, we have that $\gamma(G) \leqslant k-2$. By Lemma 3.20, we have $\gamma(G) = k-2$. □

Theorem 3.22. *Let H be a connected graph of order k such that $k \geqslant 4$ and no two pendants share a support vertex. Let $G = g(H)$. Then $\gamma(G) = k-1$.*

Proof. Let $V(H) = \{v_1, v_2, v_3, \ldots, v_k\}$. Let $X = V(G) \setminus V(H)$. For $i \neq j$, when v_i and v_j create a new vertex in G, we let v_{ij} denote the new vertex.

Suppose there exists $D \subseteq V(G)$ such that $|D| = k-2$ and D dominates X. Similar to the proof of Theorem 3.19, we can assume that $D \subseteq V(H)$ and let $D = \{v_1, v_2, v_3, \ldots, v_{k-2}\}$. Let α be the number of vertices in X that are dominated by D. Let l be the number of pendants in H. By Lemma 3.12, we have $\alpha = |X| = \binom{k}{2} - l$. We will also compute α by counting the number of additional vertices that are dominated by each v_i for $1 \leqslant i \leqslant k-2$. By Lemma 3.10, for each $v \in D$, if v is a pendant or a support of a pendant, then v is adjacent to $k-2$ vertices in X; otherwise, v is adjacent to $k-1$ vertices in X.

First, suppose both v_{k-1} and v_k are not pendants in H. Then all l pendants are in D so $\alpha = (k-1) + (k-2) + \cdots + 2 - l = \binom{k}{2} - 1 - l$. Thus, $\alpha < |X|$, a contradiction.

Suppose both v_{k-1} and v_k are pendants in H. Then the support vertices of v_{k-1} and v_k are distinct and are in D. It implies that $\alpha = (k-1) + (k-2) + \cdots + 2 - l = \binom{k}{2} - 1 - l$. Thus, $\alpha < |X|$, a contradiction.

Therefore, exactly one vertex in $\{v_{k-1}, v_k\}$ is a pendant in H. Then D contains $l-1$ pendants. Without loss of generality, let v_k be a pendant. First, suppose that the support vertex of v_k is in D. It follows that $\alpha = (k-1) + (k-2) + \cdots + 2 - l = \binom{k}{2} - 1 - l$. Thus, $\alpha < |X|$, a contradiction. Thus, the support vertex of v_k is not in D, i.e. v_{k-1} is the support vertex of v_k. Then $\alpha = (k-1) + (k-2) + \cdots + 2 - (l-1) = \binom{k}{2} - l$. It follows that we need at least $k-2$ vertices to dominate every vertex in X. Each vertex v_i in D dominates at least 2 additional vertices $v_{i,k-1}$ and v_{ik}. Each vertex v_{ij} in X can only dominate one vertex (itself) in X. So, to use exactly $k-2$ vertices to dominate X, we cannot use any vertex from X. Since the pendant v_k and its support vertex v_{k-1} are not in D, the vertex v_k is not dominated by D. Thus, we must use one more vertex to dominate v_k. Then a dominating set of G has at least $k-1$ members. So, $\gamma(G) \geqslant k-1$.

Let $D' = \{v_1, v_2, v_3, \ldots, v_{k-1}\}$. Clearly, D' dominate all vertices in G. Since $|D'| = k-1$, we have that $\gamma(G) \leqslant k-1$. Therefore, $\gamma(G) = k-1$. □

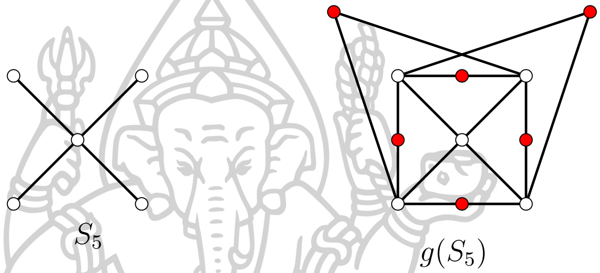

Figure 3.5: Graphs S_5 and $g(S_5)$

Remark 3.23. *Theorem 3.15 and 3.21 imply that our necessary condition for graphs G with $\gamma_2^c(G) = \gamma(G) + 1$ is not a sufficient condition.*

Lastly, we apply Theorems 3.15, 3.17, 3.19, 3.21, and 3.22 to stars, paths and cycles.

Corollary 3.24. *For $k \geqslant 4$, let $G = g(S_k)$. Then $\gamma_2^c(G) = k$, $\gamma_c(G) = k-1$ and $\gamma(G) = k-2$.*

Corollary 3.25. *For $k \geqslant 3$, let $G = g(P_k)$. Then $\gamma_2^c(G) = k$, $\gamma_c(G) = k-1$ and $\gamma(G) = k-1$.*

Corollary 3.26. *For $k \geqslant 3$, let $G = g(C_k)$. Then $\gamma_2^c(G) = k$, $\gamma_c(G) = k-1$ and $\gamma(G) = k-1$.*

CHAPTER 4

CONCLUSIONS

In this chapter, we present a summary resulting from this thesis.

This thesis determines some necessary conditions for graphs with extremal connected 2-dominating sets in Chapter 3. Firstly, we examine graphs that have the smallest possible connected 2-domination numbers with respect to domination numbers and connected domination numbers. In Section 3.1, the connected graphs G with $\gamma(G) = 1$ and $\gamma_2^c(G) = 2$ are characterized.

Particularly, in Section 3.2, for the connected graphs G with $\gamma_2^c(G) \geqslant 3$ and $\gamma_2^c(G) = \gamma(G) + 1$, the necessary condition of the connected graphs G with $\gamma_2^c(G) = \gamma(G) + 1$ is presented. Similarly, in Section 3.3, the necessary condition of the connected graphs G with $\gamma_2^c(G) = \gamma_c(G) + 1$ is also presented.

In Section 3.4, we utilize the necessary condition to construct an infinite family of graphs G that satisfy $\gamma_2^c(G) = \gamma_c(G) + 1$. Additionally, for a connected graph H of order $k \geqslant 3$, let $G = g(H)$. Theorems 3.15 and 3.16 state that $\gamma_2^c(G) = k$. Moreover, $V(H)$ is the unique γ_2^c-set of G if H is not a path on 3 vertices. Subsequently, we show that our graph construction that takes any connected graph with k vertices and gives a graph G with $\gamma_2^c(G) = k$, $\gamma_c(G) = k - 1$.

Finally, in Section 3.5, we show that $\gamma(G) = k - 1$ if H does not contain 2 pendants that share a support vertex; otherwise $\gamma(G) = k - 2$.

Furthermore, by Theorems 3.15 and 3.21, our necessary condition for graphs G with $\gamma_2^c(G) = \gamma(G) + 1$ is not a sufficient condition.

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