



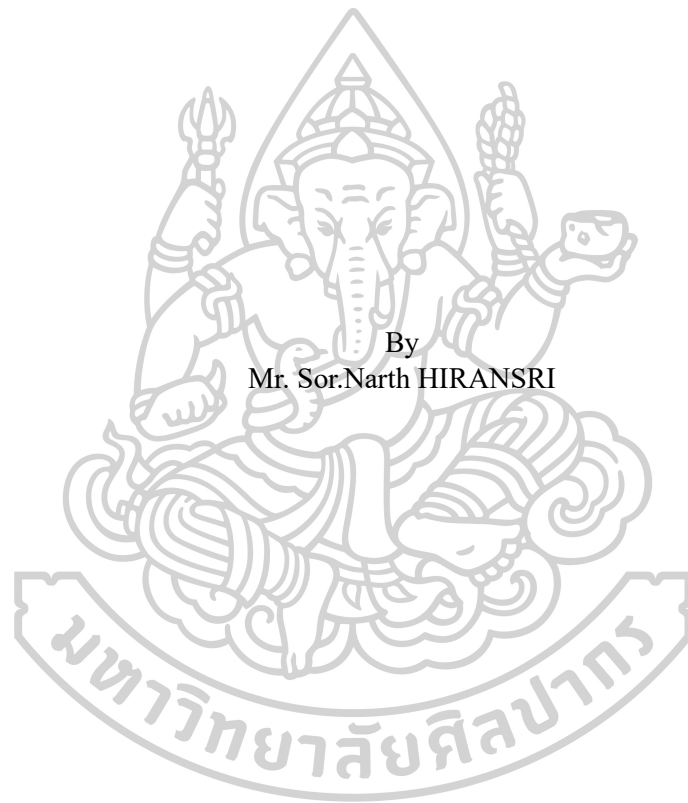
Characterizations of inversely S -invariant elements in semigroups of full transformations as magnifying elements



BY
Mr. Sor.Narth HIRANSRI

A Thesis Submitted in Partial Fulfillment of the Requirements
for Master of Science MATHEMATICS
Department of MATHEMATICS
Silpakorn University
Academic Year 2025
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Mr. Sor.Narth HIRANSRI : Characterizations of inversely S -invariant elements in semigroups of full transformations as magnifying elements Thesis Advisor : Assistant Professor Thanakorn Prinyasart, Ph.D.,

This research studies inversely S -invariant elements in the full transformation semigroup $T(X)$ and their relationship with magnifying elements. We define inversely left- S -invariant and inversely right- S -invariant elements, and establish conditions under which they become left and right magnifying, respectively. The results are applied to well-known subsemigroups of $T(X)$. We also extend the study to the Δ_γ -structure semigroup.



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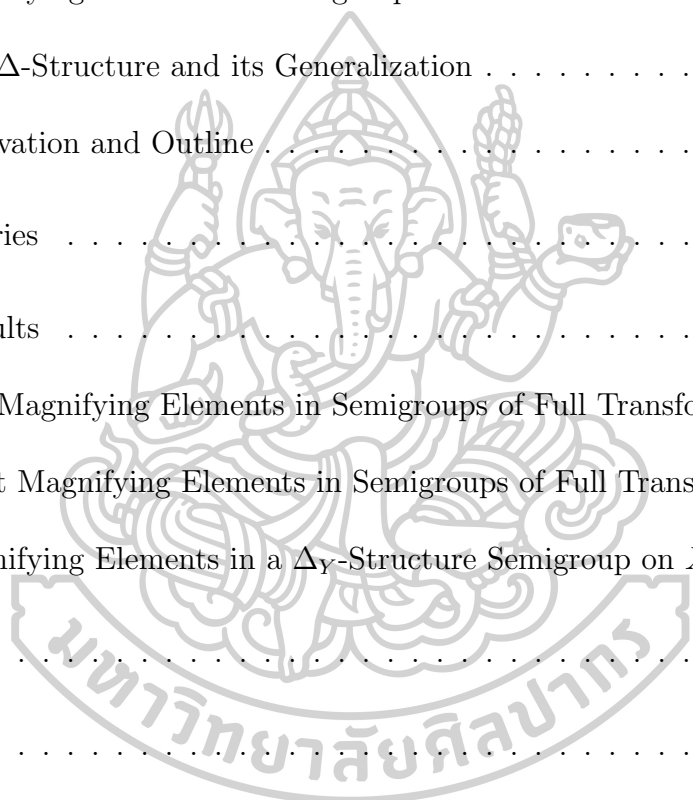
looks decent. This thesis stands as a testament to the invaluable support I have received. My deepest appreciation goes to everyone who has been part of this journey.

Sor.Narth HIRANSRI



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Chapter 1

Introduction and Literature Review

This chapter outlines the fundamental concepts, interpretations, and initial findings that facilitate readers' understanding of the research presented in the subsequent sections.

A *semigroup* (S, \cdot) is an algebraic structure made of a non-empty set S and an associative binary operation \cdot , that is,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c), \quad \text{for all } a, b, c \in S.$$

Transformation semigroups are the primary focus in this work. Consider a non-empty set X . The set $T(X)$, consisting of all transformations from X to itself, forms a semigroup under the operation of composition. The composition of two transformations α and β is defined as follows:

$$(\alpha \circ \beta)(x) = \alpha(\beta(x)), \quad \text{for all } x \in X.$$

This semigroup is called the *full transformation semigroup* on X . In general, a *transformation semigroup* is any semigroup whose elements are transformations from a set into itself, with the semigroup operation given by function composition. So, a transformation semigroup is indeed a subsemigroup of a full transformation semigroup.

Transformations with an invariant set arise when one considers a non-empty subset Y of X . We define

$$\bar{T}(X, Y) = \{\alpha \in T(X) : \alpha(Y) \subseteq Y\}.$$

This means that every element α in $\bar{T}(X, Y)$ sends all elements of Y back into Y , so the subset Y stays invariant. Since the composition of two functions with images contained in Y also maps X into Y , $\bar{T}(X, Y)$ is closed under composition and forms a subsemigroup of $T(X)$.

1.1 Magnifying Elements in Semigroups

The concept of magnifying elements was introduced by E. S. Ljapin in 1978 [1].

An element α of a semigroup S is said to be *left magnifying* if there exists a proper subset $M \subset S$ such that

$$S = \alpha M. \tag{1.1.1}$$

Analogously, α is termed a *right magnifying* element if there is a proper subset M of S such that

$$S = M\alpha. \tag{1.1.2}$$

In 1994, K. D. Magill, Jr. [2] investigated the necessary and sufficient conditions for an element to be a magnifying element within any subsemigroup of $T(X)$ which contains the identity map id_X , as well as a characterization of transformation semigroups with identity that have magnifying elements. Besides, they applied their results to the semigroup of all linear transformations of a vector space V and the semigroup of all continuous selfmaps of a topological space X .

In particular, research in recent years has focused on characterizing magnifying elements in certain subsemigroups of $\overline{T}(X, Y)$. Notable works include those by Chinram and others [3, 4, 5, 6], who examined magnifying elements in semigroups such as:

- (1) $\overline{T}(X, Y)$,
- (2) $T(X, Y) := \{\alpha \in T(X) \mid \text{ran}(\alpha) \subseteq Y\}$,
- (3) $B_T(X, Y) := \{\alpha \in T(X) \mid \alpha|_Y : Y \rightarrow Y \text{ is bijective}\}$,
- (4) $\text{Fix}_T(X, Y) := \{\alpha \in T(X) \mid \alpha|_Y = \text{id}_Y\}$.

When Y is a subspace of a vector space X , some researchers have explored magnifying elements in semigroups of linear transformations on X . For instance, in [7] and [8], they gave necessary and sufficient conditions for elements to be left and right magnifying elements in the following semigroups of linear transformations, respectively:

- (1) $\overline{L}(X, Y) := \{\alpha \in \overline{T}(X, Y) \mid \alpha \text{ is a linear transformation}\}$,
- (2) $L(X, Y) := \overline{L}(X, Y) \cap T(X, Y)$.

1.2 The Δ -Structure and its Generalization

In 1974, Magill Jr. and Subbiah [9] developed a comprehensive framework to study regularity and Green's relations in transformation semigroups through what they termed a Δ -structure. For a non-empty set X , a pair $(\mathcal{A}, \mathcal{M})$ consisting of a family \mathcal{A} of nonempty subsets of X with $X \in \mathcal{A}$ and $\mathcal{M} = \{\text{Hom}(A, B) : (A, B) \in \mathcal{A} \times \mathcal{A}\}$,

where $\text{Hom}(A, B)$ is a nonempty set of functions from A into B for all $(A, B) \in \mathcal{A} \times \mathcal{A}$, is called a Δ -structure on X if the following conditions are satisfied:

- (1) $\text{End}(X) := \text{Hom}(X, X)$ is a monoid;
- (2) $\text{ran}(\alpha) \in \mathcal{A}$ for all $\alpha \in \text{End}(X)$;
- (3) for all $B \in \mathcal{A}, \alpha \in \text{End}(X)$ and $\beta \in \text{Hom}(\text{ran}(\alpha), B), \beta\alpha \in \text{End}(X)$;
- (4) for all $\alpha, \beta \in \text{End}(X)$ and $A, B \in \mathcal{A}$ with $\alpha(B) \subseteq A$ and $\beta(A) \subseteq B$, if $(\alpha\beta)|_A = \text{id}_A$ and $(\beta\alpha)|_B = \text{id}_B$, then $\beta|_A \in \text{Hom}(A, B)$ and $\alpha|_B \in \text{Hom}(B, A)$, where id_Z denotes the identity function on Z for any nonempty set Z .

Later, Supaporn et al. [10] expanded this idea by introducing the Δ_Y -structure. They adapted the original framework to fit semigroups of transformations X preserving a subset Y of X . Let X be a non-empty set and Y be a non-empty subset of X . For a subsemigroup S of $\bar{T}(X, Y)$ and $\mathcal{M} = \{\text{Hom}(\alpha, \beta) : (\alpha, \beta) \in S \times S\}$, where $\text{Hom}(\alpha, \beta)$ is a nonempty set of functions from $\text{ran}(\alpha)$ into $\text{ran}(\beta)$, a pair (S, \mathcal{M}) is called a Δ -structure on X with Y invariant, or a Δ_Y -structure on X , if the following conditions are satisfied:

- ($\Delta 1$) for all $\alpha, \beta \in S, \rho \in \text{Hom}(\alpha, \beta)$ and $\gamma \in S$ with $\text{ran}(\gamma) = \text{ran}(\alpha)$ and $\gamma(Y) = \alpha(Y), \rho\gamma \in S$;
- ($\Delta 2$) for all $\alpha, \beta \in S$ and $\rho, \lambda \in S \cap F(X, Y)$ with $\alpha(\text{ran}(\rho)) \subseteq \text{ran}(\lambda)$ and $\beta(\text{ran}(\lambda)) \subseteq \text{ran}(\rho)$, if $(\alpha\beta)|_{\text{ran}(\lambda)} = \text{id}_{\text{ran}(\lambda)}$ and $(\beta\alpha)|_{\text{ran}(\rho)} = \text{id}_{\text{ran}(\rho)}$, then $\beta|_{\text{ran}(\lambda)} \in \text{Hom}(\lambda, \rho)$ and $\alpha|_{\text{ran}(\rho)} \in \text{Hom}(\rho, \lambda)$, where $F(X, Y) := \{\gamma \in T(X) : \text{ran}(\gamma) \cap Y = \gamma(Y)\}$;
- ($\Delta 3$) for all $\alpha, \beta \in S \cap F(X, Y)$, if $\alpha \sim \beta$, then $\text{id}_{\text{ran}(\alpha)} \in \text{Hom}(\alpha, \beta)$.

So, in this setting, the semigroup S , which is denoted by $\text{End}(X, Y)$, is known as a Δ_Y -structure semigroup on X . For any $\alpha, \beta \in \text{End}(X, Y)$, an element λ of $\text{Hom}(\alpha, \beta)$ is called a Δ_Y -isomorphism if there is $\rho \in \text{Hom}(\beta, \alpha)$ such that $\lambda\rho = \text{id}_{\text{ran}(\beta)}$ and $\rho\lambda = \text{id}_{\text{ran}(\alpha)}$.

1.3 Motivation and Outline

From the above background, we can see that the concept of magnifying elements was first introduced in 1978. Later, in 1994, magnifying elements in semigroups of linear transformations were studied. Over the past ten years, there has been increasing interest in studying magnifying elements in various subsemigroups of $\bar{T}(X, Y)$.

In this thesis, we introduce the concept of inversely left- S -invariant elements in a subsemigroup S of $T(X)$, and we establish necessary and sufficient conditions under which such an element is left magnifying. Similarly, inversely right- S -invariant elements are introduced and studied. We then apply these results to characterize left and right magnifying elements in some well-known semigroups.

- (1) $\bar{T}(X, Y)$,
- (2) $T(X, Y)$,
- (3) $B_T(X, Y)$,
- (4) $\text{Fix}_T(X, Y)$,
- (5) $\bar{F}(X, Y) := \{\alpha \in T(X, Y) \mid \text{ran}(\alpha) = \alpha(Y)\} = T(X, Y) \cap F(X, Y)$,
- (6) $\bar{L}(X, Y)$,

$$(7) L(X, Y),$$

$$(8) B_L(X, Y) := \bar{L}(X, Y) \cap B_T(X, Y),$$

$$(9) \text{Fix}_L(X, Y) := \bar{L}(X, Y) \cap \text{Fix}_T(X, Y),$$

$$(10) \bar{G}(X, Y) := \bar{L}(X, Y) \cap \bar{F}(X, Y).$$

In addition, we also apply the main results to a Δ_Y -structure semigroup on X .



Chapter 2

Preliminaries

First, we begin with results that gives a necessary and sufficient conditions for an element of a subsemigroup of $T(X)$ to be left (right) magnifying.

Lemma 2.0.1. *Let S be a subsemigroup of $T(X)$. Then the following statements hold:*

- (1) *If α is a left magnifying element of S , then there exist distinct $\gamma_1, \gamma_2 \in S$ such that $\alpha\gamma_1 = \alpha\gamma_2$.*
- (2) *If $\alpha \in S$ with $\alpha S = S$ and there exist distinct $\gamma_1, \gamma_2 \in S$ such that $\alpha\gamma_1 = \alpha\gamma_2$, then α is a left magnifying element.*

Proof. To prove (i), let α be a left magnifying element of S . Then $S = \alpha M$ for some proper subset M of S . Let $\gamma_1 \in S \setminus M$. Since $\alpha\gamma_1 \in S = \alpha M$, we get $\alpha\gamma_1 = \alpha\gamma_2$ for some $\gamma_2 \in M$. Since $\gamma_1 \notin M$ but $\gamma_2 \in M$, we conclude that $\gamma_1 \neq \gamma_2$.

To prove (ii), let $\alpha \in S$ such that $\alpha S = S$ and there are $\gamma_1, \gamma_2 \in S$ such that $\alpha\gamma_1 = \alpha\gamma_2$. Choose $M = S \setminus \{\gamma_2\}$. Since $\alpha\gamma_2 = \alpha\gamma_1 \in \alpha M$, we have $\alpha M = \alpha M \cup \{\alpha\gamma_2\} = \alpha S = S$. Therefore, α is a left magnifying element of S . \square

Lemma 2.0.2. *Let S be a subsemigroup of $T(X)$. Then the following statements hold:*

(1) If α is a right magnifying element of S , then there exist distinct $\gamma_1, \gamma_2 \in S$ such that $\gamma_1\alpha = \gamma_2\alpha$.

(2) If $\alpha \in S$ with $S\alpha = S$ and there exist distinct $\gamma_1, \gamma_2 \in S$ such that $\gamma_1\alpha = \gamma_2\alpha$, then α is a right magnifying element.

Proof. To prove (i), let α be a right magnifying element of S . Then $S = M\alpha$ for some proper subset M of S . Let $\gamma_1 \in S \setminus M$. Since $\gamma_1\alpha \in S = M\alpha$, we get $\gamma_1\alpha = \gamma_2\alpha$ for some $\gamma_2 \in M$. Since $\gamma_1 \notin M$ but $\gamma_2 \in M$, we conclude that $\gamma_1 \neq \gamma_2$.

To prove (ii), let $\alpha \in S$ such that $S\alpha = S$ and there are distinct $\gamma_1, \gamma_2 \in S$ such that $\gamma_1\alpha = \gamma_2\alpha$. Choose $M = S \setminus \{\gamma_2\}$. Since $\gamma_2\alpha = \gamma_1\alpha \in M\alpha$, we have $M\alpha = M\alpha \cup \{\gamma_2\alpha\} = S\alpha = S$. Therefore, α is a right magnifying element of S . \square

In the rest of this chapter, we focus on studying left and right inversely S -invariant. For any semigroup S and $a \in S$, we call a a *regular element* of S if there exists $b \in S$ such that $aba = a$. An element c is called an *inverse* of a if there exists $c \in S$ such that $aca = a$ and $cac = c$. It is well-known that every element α of $T(X)$ is regular, that is, $\alpha\gamma\alpha = \alpha$ for some $\gamma \in T(X)$. However, in some cases, the element γ may not satisfy $\gamma S \subseteq S$. Consider the following example.

Example 1. Let $Y = \{-1, 0, 1\}$, and define $\alpha \in \overline{T}(\mathbb{Z}, Y)$ by $\alpha(x) = |x| - 1$ for all $x \in \mathbb{Z}$. If $\gamma \in T(\mathbb{Z})$ such that $\alpha\gamma\alpha = \alpha$, then $\gamma(1) \in \{-2, 2\}$, so $\gamma \notin \overline{T}(\mathbb{Z}, Y)$, which implies that $\gamma\overline{T}(\mathbb{Z}, Y) \not\subseteq \overline{T}(\mathbb{Z}, Y)$.

Based on this, we now introduce the following definition of left and right inversely S -invariant elements.

Definition 1. Let S be a subsemigroup of $T(X)$ and $\alpha \in S$. We say that α is *inversely left- S -invariant* in S if there exists $\gamma \in T(X)$ such that $\alpha\gamma\alpha = \alpha$ and $\gamma S \subseteq S$. Similarly, α is *inversely right- S -invariant* in S if there exists $\gamma \in T(X)$ such that $\alpha\gamma\alpha = \alpha$ and $S\gamma \subseteq S$.

Since α is inversely left- S -invariant, there exists $\gamma \in T(X)$ such that $\alpha\gamma\alpha = \alpha$ and $\gamma S \subseteq S$. Then, $\gamma\alpha\gamma \in T(X)$ is an inverse of α because $\alpha(\gamma\alpha\gamma)\alpha = (\alpha\gamma\alpha)\gamma\alpha = \alpha\gamma\alpha = \alpha$ and $(\gamma\alpha\gamma)\alpha(\gamma\alpha\gamma) = \gamma(\alpha\gamma\alpha)\gamma\alpha\gamma = \gamma(\alpha\gamma\alpha)\gamma = \gamma\alpha\gamma$. Besides, $(\gamma\alpha\gamma)S \subseteq S$. Consequently, we obtain the following results about inversely left- S -invariant elements.

Remark 1. (1) α is inversely left- S -invariant if and only if there exists an inverse γ of α in $T(X)$ such that $\gamma S \subseteq S$.
 (2) If α is a regular element of S , then α is inversely left- S -invariant.

In the same way, the basic results for inversely right- S -invariant elements are presented.

Remark 2. (1) α is inversely right- S -invariant if and only if there exists an inverse γ of α in $T(X)$ such that $S\gamma \subseteq S$.
 (2) If α is a regular element of S , then α is inversely right- S -invariant.

Chapter 3

Main Results

This chapter presents the main results in two parts: first, a characterization of left magnifying elements in subsemigroups of $T(X, Y)$, and second, a characterization of right magnifying elements in subsemigroups of $T(X, Y)$.

3.1 Left Magnifying Elements in Semigroups of Full Transformations

Theorem 3.1.1. *Let S be a subsemigroup of $T(X)$, and let $\alpha \in S$ be inversely left- S -invariant. Then α is a left magnifying element of S if and only if the following two statements hold:*

- (1) $\bigcup_{\beta \in S} \text{ran}(\beta) \subseteq \text{ran}(\alpha)$.
- (2) There exist distinct $\gamma_1, \gamma_2 \in S$ such that $\alpha\gamma_1 = \alpha\gamma_2$.

Proof. Assume that α is a left magnifying element of S . By Lemma 2.0.1 (1), we have that (2) holds. To show (1), let $\beta \in S$. Since $S = \alpha S$, we have $\beta = \alpha\gamma$ for some $\gamma \in S$. Therefore, $\text{ran}(\beta) \subseteq \text{ran}(\alpha)$. Hence, (1) holds.

Conversely, assume that (1) and (2) hold. By Lemma 2.0.1(2), it is enough to show that $\alpha S = S$. Let $\beta \in S$ be arbitrary. Since α is inversely left- S -invariant, we get $\alpha\gamma\alpha = \alpha$ for some $\gamma \in T(X)$ such that $\gamma S \subseteq S$. By (1), we

have $\text{ran}(\beta) \subseteq \text{ran}(\alpha)$. Therefore, $\alpha\gamma\beta = \beta$. Since $\gamma\beta \in \gamma S \subseteq S$, we obtain that $\beta = \alpha(\gamma\beta) \in \alpha S$. Hence, $S = \alpha S$. \square

Next, we apply the main result to the semigroup S , where S is one of the following: $\bar{T}(X, Y), T(X, Y), B_T(X, Y), \text{Fix}_T(X, Y), \bar{F}(X, Y), \bar{L}(X, Y), L(X, Y), B_L(X, Y), \text{Fix}_L(X, Y)$, or $\bar{G}(X, Y)$. We begin by showing that every left magnifying element in these semigroups is also inversely left- S -invariant.

Lemma 3.1.2. *Let S be a subsemigroup of $\bar{T}(X, Y)$. If α is a left magnifying element of S , then $\beta(Y) \subseteq \alpha(Y)$ for all $\beta \in S$.*

Proof. Let α be a left magnifying element of S and $\beta \in S$. Then $\beta = \alpha\gamma$ for some $\gamma \in S$. Let $y \in Y$. Since $\gamma(y) \in \gamma(Y) \subseteq Y$, we get $\beta(y) = \alpha(\gamma(y)) \in \alpha(Y)$. \square

Theorem 3.1.3. *Let S be $\bar{T}(X, Y), T(X, Y), B_T(X, Y), \text{Fix}_T(X, Y), \bar{F}(X, Y), \bar{L}(X, Y), L(X, Y), B_L(X, Y), \text{Fix}_L(X, Y)$ or $\bar{G}(X, Y)$. If α is a left magnifying element of S , then $\alpha(Y) = Y$.*

Proof. Let $y \in Y$. First, assume that S is $\bar{T}(X, Y), T(X, Y), B_T(X, Y), \text{Fix}_T(X, Y)$ or $\bar{F}(X, Y)$. Define $\beta \in S$ by

$$\beta(x) := \begin{cases} x, & \text{if } x \in Y, \\ y, & \text{otherwise.} \end{cases}$$

We can see that $\beta(Y) = Y$. By Lemma 3.1.2, $Y = \beta(Y) \subseteq \alpha(Y)$.

Next, assume that S is $\bar{L}(X, Y), L(X, Y), B_L(X, Y), \text{Fix}_L(X, Y)$ or $\bar{G}(X, Y)$. Let B be a basis for Y and extend B to a basis \tilde{B} for X . Let $\beta : X \rightarrow X$

be the linear transformation such that

$$\beta(x) := \begin{cases} x, & \text{if } x \in B, \\ 0 & \text{if } x \in \tilde{B} \setminus B. \end{cases}$$

Then $\beta \in S$ and $\beta(Y) = Y$. By Lemma 3.1.2, $Y = \beta(Y) \subseteq \alpha(Y)$. \square

Theorem 3.1.4. *Let S be $\overline{T}(X, Y)$, $T(X, Y)$, $B_T(X, Y)$, $Fix_T(X, Y)$, $\overline{F}(X, Y)$, $\overline{L}(X, Y)$, $L(X, Y)$, $B_L(X, Y)$, $Fix_L(X, Y)$ or $\overline{G}(X, Y)$. For any $\alpha \in S$, if $\text{ran}(\alpha) \cap Y = \alpha(Y)$, then α is regular in S .*

Proof. Let $\alpha \in S$ such that $\text{ran}(\alpha) \cap Y = \alpha(Y)$. First, assume that S is $\overline{T}(X, Y)$, $T(X, Y)$, $B_T(X, Y)$, $Fix_T(X, Y)$ or $\overline{F}(X, Y)$. Since $Y \neq \emptyset$, there exists $y \in Y$. Since $\text{ran}(\alpha) \cap Y = \alpha(Y)$, for each $b \in \text{ran}(\alpha) \cap Y$, we can fix an element a_b in $Y \cap \alpha^{-1}(b)$. For each $b \in \text{ran}(\alpha) \setminus Y$, we also fix an element a_b in $\alpha^{-1}(b)$. Define $\gamma : X \rightarrow X$ by

$$\gamma(x) := \begin{cases} a_x, & \text{if } x \in \text{ran}(\alpha), \\ a_{\alpha(y)}, & \text{otherwise.} \end{cases}$$

It can be shown that $\gamma \in S$ and $\alpha\gamma\alpha = \alpha$.

Next, assume that S is $\overline{L}(X, Y)$, $L(X, Y)$, $B_L(X, Y)$, $Fix_L(X, Y)$ or $\overline{G}(X, Y)$. Let B be a basis for $\text{ran}(\alpha) \cap Y$, and extend B to a basis for $\text{ran}(\alpha)$ and Y , say B_α and B_Y , respectively. Then $B_\alpha \cup B_Y$ is a basis for $Y + \text{ran}(\alpha)$. we extend $B_\alpha \cup B_Y$ to a basis \tilde{B} for X . Since $\text{ran}(\alpha) \cap Y = \alpha(Y)$, for each $b \in B$, we can fix an element a_b in $Y \cap \alpha^{-1}(b)$. For each $b \in B_\alpha \setminus B$, we also fix an element

a_b in $\alpha^{-1}(b)$. Let $\gamma : X \rightarrow X$ be the linear transformation such that

$$\gamma(x) := \begin{cases} a_x, & \text{if } x \in B_\alpha, \\ 0, & \text{if } x \in \tilde{B} \setminus B_\alpha. \end{cases}$$

It can be shown that $\gamma \in S$ and $\alpha\gamma\alpha = \alpha$. □

For any $\alpha \in T(X)$, if $\alpha(Y) = Y$, then $\text{ran}(\alpha) \cap Y = \text{ran}(\alpha) \cap \alpha(Y) = \alpha(Y)$. Then, by Theorem 3.1.3 and Theorem 3.1.4, we obtain the following corollary.

Corollary 3.1.5. *Let S be $\overline{T}(X, Y)$, $T(X, Y)$, $B_T(X, Y)$, $Fix_T(X, Y)$, $\overline{F}(X, Y)$, $\overline{L}(X, Y)$, $L(X, Y)$, $B_L(X, Y)$, $Fix_L(X, Y)$ or $\overline{G}(X, Y)$. If α is a left magnifying element of S , then α is regular, which also implies that α is inversely left- S -invariant.*

Now we can state the necessary and sufficient conditions for an element in these semigroups to be a left magnifying element.

Theorem 3.1.6. *Let S be $T(X, Y)$, $\overline{F}(X, Y)$, $L(X, Y)$ or $\overline{G}(X, Y)$. Then, for any $\alpha \in S$, α is a left magnifying element of S if and only if $\alpha(Y) = Y$, and $\alpha|_Y$ is not injective.*

Proof. Assume that α is a left magnifying element of S . By Theorem 3.1.3, we have that $\alpha(Y) = Y$. By Corollary 3.1.5, we have that α is an inversely left- S -invariant element in S . Since $\text{ran}(\gamma) \subseteq Y$ for all $\gamma \in S$, by Theorem 3.1.1(2), we can conclude that $\alpha|_Y$ is not injective.

Coversely, assume that $\alpha(Y) = Y$ and $\alpha|_Y$ is not injective. Then $\text{ran}(\alpha) \cap Y = \text{ran}(\alpha) \cap \alpha(Y) = \alpha(Y)$. By Theorem 3.1.4, we have that α is

an inversely left- S -invariant element in S . Notice that condition (1) in Theorem 3.1.1 holds since $\text{ran}(\beta) \subseteq Y = \alpha(Y) \subseteq \text{ran}(\alpha)$ for all $\beta \in S$. To show that condition (2) also holds, notice that since $\alpha|_Y$ is not injective, there exist distinct $y_1, y_2 \in Y$ such that $\alpha(y_1) = \alpha(y_2)$. For $i \in \{1, 2\}$, if S is $T(X, Y)$ or $\overline{F}(X, Y)$, define $\gamma_i \in S$ by

$$\gamma_i(x) := y_i$$

for all $x \in X$. If S is $L(X, Y)$ or $\overline{G}(X, Y)$, let B be a basis for X , and for $i \in \{1, 2\}$, let $\gamma_i \in S$ such that $\gamma_i(x) = y_i$ for all $x \in B$. Thus, $\alpha\gamma_1 = \alpha\gamma_2$ but $\gamma_1 \neq \gamma_2$. By Theorem 3.1.1, α is a left magnifying element of S . \square

Theorem 3.1.7. *Let S be $\overline{T}(X, Y)$, $B_T(X, Y)$, $Fix_T(X, Y)$, $\overline{L}(X, Y)$, $B_L(X, Y)$ or $Fix_L(X, Y)$. Then, for any $\alpha \in S$, α is a left magnifying element of S if and only if $\text{ran}(\alpha) = X$, $\alpha(Y) = Y$, and α is not injective.*

Proof. Assume that α is a left magnifying element of S . By Theorem 3.1.3, we have that $\alpha(Y) = Y$. By Corollary 3.1.5, we have that α is an inversely left- S -invariant element in S . Since $\text{id}_X \in S$, by Theorem 3.1.1(1), $X = \text{ran}(\text{id}_X) \subseteq \text{ran}(\alpha)$. By Theorem 3.1.1(2), we also have that α is not injective.

Conversely, assume that $\text{ran}(\alpha) = X$, $\alpha(Y) = Y$, and α is not injective. Then, $\text{ran}(\alpha) \cap Y = \text{ran}(\alpha) \cap \alpha(Y) = \alpha(Y)$. By Theorem 3.1.4, we have that α is an inversely left- S -invariant element in S . Notice that condition (1) in Theorem 3.1.1 clearly holds. To show that condition (2) also holds, notice that since α is not injective, there exist distinct $x_1, x_2 \in X$ such that $\alpha(x_1) = \alpha(x_2)$. First, assume that S is $\overline{T}(X, Y)$, $B_T(X, Y)$ or $Fix_T(X, Y)$. If $Y \subsetneq X$, then for $i \in \{1, 2\}$, we

define $\gamma_i \in S$ by

$$\gamma_i(x) := \begin{cases} x_i, & \text{if } x \in X \setminus Y, \\ x, & \text{if } x \in Y. \end{cases}$$

If $Y = X$, then $S = \overline{T}(X, Y)$ since α is not injective, so for $i \in \{1, 2\}$, we define

$$\gamma_i(x) := x_i$$

for all $x \in X$. Then $\alpha\gamma_1 = \alpha\gamma_2$ but $\gamma_1 \neq \gamma_2$. By Theorem 3.1.1, α is a left magnifying element of S . Next, assume that S is $\overline{L}(X, Y)$, $B_L(X, Y)$ or $Fix_L(X, Y)$.

Let B be a basis for Y , and extend B to a basis \tilde{B} for X . If $B \subsetneq \tilde{B}$, then for $i \in \{1, 2\}$, let $\gamma_i \in S$ such that

$$\gamma_i(x) := \begin{cases} x_i, & \text{if } x \in \tilde{B} \setminus B, \\ x, & \text{if } x \in B. \end{cases}$$

If $B = \tilde{B}$, then $S = \overline{L}(X, Y)$ since α is not injective, so for $i \in \{1, 2\}$, we let $\gamma_i \in S$ such that

$$\gamma_i(x) := x_i$$

for all $x \in \tilde{B}$. Then $\alpha\gamma_1 = \alpha\gamma_2$ but $\gamma_1 \neq \gamma_2$. By Theorem 3.1.1, α is a left magnifying element of S . □

3.2 Right Magnifying Elements in Semigroups of Full Transformations

Theorem 3.2.1. *Let S be a subsemigroup of $T(X)$, and let $\alpha \in S$ be inversely right- S -invariant. Then α is a right magnifying element of S if and only if the*

following two statements hold:

(1) $\alpha^{-1}(\alpha(x)) \subseteq \bigcap_{\beta \in S} \beta^{-1}(\beta(x))$ for all $x \in X$.

(2) There exist distinct $\gamma_1, \gamma_2 \in S$ such that $\gamma_1\alpha = \gamma_2\alpha$.

Proof. Assume that α is a right magnifying element of S . By Lemma 2.0.2(1), we have that (2) holds. To show (1), let $x \in X$, $\beta \in S$, and $a \in \alpha^{-1}(\alpha(x))$. Since $S = S\alpha$, we get $\beta = \gamma\alpha$ for some $\gamma \in S$. Since $\alpha(a) = \alpha(x)$, we have that $\beta(a) = \gamma\alpha(a) = \gamma\alpha(x) = \beta(x)$, which implies that $a \in \beta^{-1}(\beta(x))$. Hence, (1) holds.

Conversely, assume that (1) and (2) hold. By Lemma 2.0.2(2), it is enough to show that $S\alpha = S$. Let $\beta \in S$ be arbitrary. Since α is inversely right- S -invariant, we get $\alpha\gamma\alpha = \alpha$ for some $\gamma \in T(X)$ such that $S\gamma \subseteq S$. By (1), we have that $\beta\gamma\alpha = \beta$. Since $\beta\gamma \in S\gamma \subseteq S$, we obtain that $\beta = (\beta\gamma)\alpha \in S\alpha$. Hence, $S = S\alpha$. \square

Next, we apply the main result to the semigroup S , where S is one of the following: $\bar{T}(X, Y), T(X, Y), B_T(X, Y), Fix_T(X, Y), \bar{F}(X, Y), \bar{L}(X, Y), L(X, Y), B_L(X, Y), Fix_L(X, Y)$, or $\bar{G}(X, Y)$. First, we prove that in these semigroups, every right magnifying element is inversely right- S -invariant.

Lemma 3.2.2. *Let S be a subsemigroup of $\bar{T}(X, Y)$. If α is a right magnifying element of S , then $\alpha^{-1}(Y) \subseteq \beta^{-1}(Y)$ for all $\beta \in S$.*

Proof. Let α be a right magnifying element of S and $\beta \in S$. Then $\beta = \gamma\alpha$ for some $\gamma \in S$. Thus, $\alpha^{-1}(Y) \subseteq \alpha^{-1}(\gamma^{-1}(Y)) = \beta^{-1}(Y)$. \square

Theorem 3.2.3. *Let S be $\bar{T}(X, Y)$, $B_T(X, Y)$, $Fix_T(X, Y)$, $\bar{F}(X, Y)$, $\bar{L}(X, Y)$, $B_L(X, Y)$, $Fix_L(X, Y)$ or $\bar{G}(X, Y)$. If α is a right magnifying element of S , then $\text{ran}(\alpha) \cap Y = \alpha(Y)$.*

Proof. This theorem is trivial when S is $B_T(X, Y)$, $Fix_T(X, Y)$, $\bar{F}(X, Y)$, $B_L(X, Y)$, $Fix_L(X, Y)$ or $\bar{G}(X, Y)$ since for every $\alpha \in S$, $\text{ran}(\alpha) \cap Y = \alpha(Y)$. Now, assume that S is $\bar{T}(X, Y)$ or $\bar{L}(X, Y)$, and let α be a right magnifying element of S . Since $\text{id}_X \in S$, by Lemma 3.2.2, we have that $\alpha^{-1}(Y) \subseteq \text{id}_X^{-1}(Y) = Y$. Thus, $\text{ran}(\alpha) \cap Y = \alpha(\alpha^{-1}(Y)) \subseteq \alpha(Y)$. Moreover, since $\alpha(Y) \subseteq Y$, we conclude that $\text{ran}(\alpha) \cap Y = \alpha(Y)$. \square

Theorem 3.2.4. *Let S be $\bar{T}(X, Y)$, $T(X, Y)$, $B_T(X, Y)$, $Fix_T(X, Y)$, $\bar{F}(X, Y)$, $\bar{L}(X, Y)$, $L(X, Y)$, $B_L(X, Y)$, $Fix_L(X, Y)$ or $\bar{G}(X, Y)$. If α is a right magnifying element of S , then α is inversely right- S -invariant.*

Proof. By Theorem 3.1.4 and Theorem 3.2.3, this theorem holds when S is $\bar{T}(X, Y)$, $B_T(X, Y)$, $Fix_T(X, Y)$, $\bar{F}(X, Y)$, $\bar{L}(X, Y)$, $B_L(X, Y)$, $Fix_L(X, Y)$ or $\bar{G}(X, Y)$. Now, let α be a right magnifying element of S , and assume that S is $T(X, Y)$. Since $Y \neq \emptyset$, there exists $y \in Y$. For each $b \in \text{ran}(\alpha)$, fix an element a_b in $\alpha^{-1}(b)$.

Define $\gamma : X \rightarrow X$ by

$$\gamma(x) := \begin{cases} a_x, & \text{if } x \in \text{ran}(\alpha), \\ y, & \text{otherwise.} \end{cases}$$

It can be shown that $\alpha\gamma\alpha = \alpha$. Since $T(X, Y)$ is a right ideal of $T(X)$, it is also obvious that $T(X, Y)\gamma \subseteq T(X, Y)$.

Next, assume that S is $L(X, Y)$. Let B be a basis for $\text{ran}(\alpha)$, and extend

B to a basis for X , say \tilde{B} . For each $b \in \text{ran}(\alpha)$, fix an element a_b in $\alpha^{-1}(b)$. Let $\gamma : X \rightarrow X$ be the linear transformation such that

$$\gamma(x) = \begin{cases} a_x, & \text{if } x \in B, \\ 0, & \text{if } x \in \tilde{B} \setminus B. \end{cases}$$

It can be shown that $\alpha\gamma\alpha = \alpha$. Since $L(X, Y)$ is a right ideal of $L(X)$, it is obvious that $L(X, Y)\gamma \subseteq L(X, Y)$. \square

For convenience, we will use the following two lemmas.

Lemma 3.2.5. *Let S be $T(X, Y)$, $B_T(X, Y)$, $Fix_T(X, Y)$, $\overline{F}(X, Y)$, $L(X, Y)$, $B_L(X, Y)$, $Fix_L(X, Y)$ or $\overline{G}(X, Y)$. If $\alpha \in S$ is injective, then α is inversely right- S -invariant.*

Proof. Let $\alpha \in S$ be injective. First, assume that S is $T(X, Y)$, $B_T(X, Y)$, $Fix_T(X, Y)$ or $\overline{F}(X, Y)$. Since $Y \neq \emptyset$, there exists $y \in Y$. For each $b \in \text{ran}(\alpha)$, let a_b be the unique element in $\alpha^{-1}(b)$. Define $\gamma : X \rightarrow X$ by

$$\gamma(x) := \begin{cases} a_x, & \text{if } x \in \text{ran}(\alpha), \\ y, & \text{otherwise.} \end{cases}$$

It can be shown that $\alpha\gamma\alpha = \alpha$ and $S\gamma \subseteq S$.

Next, assume that S is $L(X, Y)$, $B_L(X, Y)$, $Fix_L(X, Y)$ or $\overline{G}(X, Y)$.

Let B be a basis for $\text{ran}(\alpha)$, and extend B to a basis for X , say \tilde{B} . For each $b \in \text{ran}(\alpha)$, let a_b be the unique element in $\alpha^{-1}(b)$. Let $\gamma : X \rightarrow X$ be the linear

transformation such that

$$\gamma(x) = \begin{cases} a_x, & \text{if } x \in B, \\ 0, & \text{if } x \in \tilde{B} \setminus B. \end{cases}$$

It can be shown that $\alpha\gamma\alpha = \alpha$ and $S\gamma \subseteq S$. \square

Lemma 3.2.6. *Let S be $\bar{T}(X, Y)$, $T(X, Y)$, $B_T(X, Y)$, $Fix_T(X, Y)$, $\bar{F}(X, Y)$, $\bar{L}(X, Y)$, $L(X, Y)$, $B_L(X, Y)$, $Fix_L(X, Y)$ or $\bar{G}(X, Y)$. For any $\alpha \in S$, if α is injective and $\text{ran}(\alpha) \neq X$, then there are $\gamma_1, \gamma_2 \in S$ such that $\gamma_1\alpha = \gamma_2\alpha$ but $\gamma_1 \neq \gamma_2$.*

Proof. Let $\alpha \in S$ such that α is injective and $\text{ran}(\alpha) \neq X$. First, assume that S is $\bar{T}(X, Y)$, $T(X, Y)$, $B_T(X, Y)$, $Fix_T(X, Y)$ or $\bar{F}(X, Y)$. Let $a \in X \setminus \text{ran}(\alpha)$ and $b \in Y \cap \text{ran}(\alpha)$. Let $\gamma_1 := \alpha \in S$ and define $\gamma_2 \in S$ by

$$\gamma_2(x) := \begin{cases} \alpha(b), & \text{if } x = a, \\ \alpha(x), & \text{otherwise.} \end{cases}$$

Then, $\gamma_1\alpha = \gamma_2\alpha$. Since α is injective and $a \neq b$, we get that $\gamma_1(a) = \alpha(a) \neq \alpha(b) = \gamma_2(a)$.

Next, assume that S is $\bar{L}(X, Y)$, $L(X, Y)$, $B_L(X, Y)$, $Fix_L(X, Y)$ or $\bar{G}(X, Y)$. Let B be a basis for $Y \cap \text{ran}(\alpha)$. Then extend B to bases for Y and $\text{ran}(\alpha)$, say B_Y and B_α , respectively. Then $B_Y \cup B_\alpha$ is a basis for $Y + \text{ran}(\alpha)$. We extend $B_Y \cup B_\alpha$ to a basis \tilde{B} for X . Let $\gamma_1 := \alpha \in S$ and let $\gamma_2 \in S$ such that

$$\gamma_2(x) := \begin{cases} 0, & \text{if } x \in \tilde{B} \setminus B_\alpha, \\ \alpha(x), & \text{if } x \in B_\alpha. \end{cases}$$

Then, $\gamma_1\alpha = \gamma_2\alpha$. Since α is injective and $B_\alpha \subsetneq \tilde{B}$, we get that $\gamma_1(\tilde{B} \setminus B_\alpha) = \alpha(\tilde{B} \setminus B_\alpha) \neq \{0\} = \gamma_2(\tilde{B} \setminus B_\alpha)$, so $\gamma_1 \neq \gamma_2$. \square

Now we can give the necessary and sufficient conditions for an element in these semigroups to be a right magnifying element.

Theorem 3.2.7. *Let S be $T(X, Y)$, $B_T(X, Y)$, $Fix_T(X, Y)$, $\overline{F}(X, Y)$, $L(X, Y)$, $B_L(X, Y)$, $Fix_L(X, Y)$ or $\overline{G}(X, Y)$. For any $\alpha \in S$, α is a right magnifying element of S if and only if α is injective and $\text{ran}(\alpha) \neq X$.*

Proof. Assume that α is a right magnifying element of S . By Theorem 3.2.4, we have that α is an inversely right- S -invariant element in S . By Theorem 3.2.1(2), we have that $\text{ran}(\alpha) \neq X$. Next, we will show that α is injective. First, assume that S is $B_T(X, Y)$, $Fix_T(X, Y)$, $B_L(X, Y)$ or $Fix_L(X, Y)$. Then $\text{id}_X \in S$. By Theorem 3.2.1(1), $\alpha^{-1}(\alpha(x)) \subseteq \text{id}_X^{-1}(\text{id}_X(x)) = \{x\}$ for all $x \in X$. Thus, α is injective. Now, assume that S is $T(X, Y)$, $\overline{F}(X, Y)$, $L(X, Y)$ or $\overline{G}(X, Y)$. If $|Y| = 1$, then $|S| = 1$, so S does not contain any right magnifying element, which contradicts to the assumption that α is a right magnifying element of S . Thus, $|Y| > 1$. Then, for any distinct elements $x_1, x_2 \in X$, there is $\beta \in S$ such that $\beta(x_1)$ and $\beta(x_2)$ are distinct elements in Y . It follows that $\alpha^{-1}(\alpha(x)) = \{x\}$ for all $x \in X$ by Theorem 3.2.1(1). Hence, α is injective.

Conversely, assume that α is injective and $\text{ran}(\alpha) \neq X$. By Lemma 3.2.5, α is inversely right- S -invariant. Since α is injective, condition (1) in Theorem 3.2.1 holds. By Lemma 3.2.6, condition (2) in Theorem 3.2.1 holds. Hence, α is a right magnifying element of S . \square

Theorem 3.2.8. *Let S be $\overline{T}(X, Y)$ or $\overline{L}(X, Y)$. For any $\alpha \in S$, α is a right magnifying element of S if and only if α is injective, $\text{ran}(\alpha) \neq X$, and $\text{ran}(\alpha) \cap Y = \alpha(Y)$.*

Proof. Assume that α is a right magnifying element of S . By Theorem 3.2.3, we have that $\text{ran}(\alpha) \cap Y = \alpha(Y)$. By Theorem 3.2.4, α is inversely right- S -invariant. By Theorem 3.2.1(2), we have that $\text{ran}(\alpha) \neq X$. Since $\text{id}_X \in S$, we have that $\alpha^{-1}(\alpha(x)) \subseteq \text{id}_X^{-1}(\text{id}_X(x)) = \{x\}$ for all $x \in X$ by Theorem 3.2.1(1). Hence, α is injective.

Conversely, assume that α is injective, $\text{ran}(\alpha) \neq X$ and $\text{ran}(\alpha) \cap Y = \alpha(Y)$. Since $\text{ran}(\alpha) \cap Y = \alpha(Y)$, by Theorem 3.1.4, we get that α is inversely right- S -invariant. Since α is injective, condition (1) in Theorem 3.2.1 holds. By Lemma 3.2.6, condition (2) in Theorem 3.2.1 holds. Hence, α is a right magnifying element of S . \square

3.3 Magnifying Elements in a Δ_Y -Structure Semigroup on X

In this section, we apply the main results to the Δ_Y -structure semigroup on X , denoted by $\text{End}(X, Y)$. To derive a criterion—formulated in terms of Δ_Y -isomorphisms—for an element $\alpha \in \text{End}(X, Y)$ to be left and right magnifying, we first require that α be regular. The necessary and sufficient conditions are presented in the following two theorems.

Theorem 3.3.1. *Let α be a regular element in $\text{End}(X, Y)$ with $\alpha\gamma\alpha = \alpha$ for some $\gamma \in \text{End}(X, Y)$. Assume that there exists $\beta \in \text{End}(X, Y)$ such that $\beta \text{End}(X, Y) =$*

$\text{End}(X, Y)$. Then, α is a left magnifying element of $\text{End}(X, Y)$ if and only if the following statements hold:

1. $(\alpha\gamma)|_{\text{ran}(\beta)}$ is a Δ_Y -isomorphism in $\text{Hom}(\beta, \alpha)$.
2. There exist distinct $\gamma_1, \gamma_2 \in \text{End}(X, Y)$ such that $\alpha\gamma_1 = \alpha\gamma_2$.

Proof. Assume that α is a left magnifying element of $\text{End}(X, Y)$. Since α is regular, it is also inversely left- $\text{End}(X, Y)$ -invariant. By Theorem 3.1.1, condition (2) holds and $\text{ran}(\beta) \subseteq \text{ran}(\alpha)$. Since $\alpha\gamma\alpha = \alpha$ and $\text{ran}(\beta) \subseteq \text{ran}(\alpha)$, we have that $\alpha\gamma\beta = \beta$. Since $\alpha \in \text{End}(X, Y) = \beta \text{End}(X, Y)$, we have that $\alpha = \beta\delta$ for some $\delta \in \text{End}(X, Y)$ and $\text{ran}(\alpha) \subseteq \text{ran}(\beta)$. Then $\beta\delta\gamma\beta = \alpha\gamma\beta = \beta$. Thus, β is regular, which implies that $\beta \in F(X, Y)$. Since $\text{ran}(\alpha) = \text{ran}(\beta)$ and $\alpha\gamma\beta = \beta$, $(\alpha\gamma)|_{\text{ran}(\beta)}$ is a Δ_Y -isomorphism in $\text{Hom}(\beta, \alpha)$ by ($\Delta 3$).

Conversely, assume that (1) and (2) hold. By (1), there exists $\delta \in \text{Hom}(\alpha, \beta)$ such that $\beta = \delta\alpha\gamma\beta$. Since $\alpha \in \text{End}(X, Y) = \beta \text{End}(X, Y)$, we get that $\text{ran}(\alpha) \subseteq \text{ran}(\beta)$. Then $\alpha = \delta\alpha\gamma\alpha = \delta\alpha$. Thus, $\alpha\gamma\beta = \delta\alpha\gamma\beta = \beta$. Therefore, $\text{ran}(\beta) \subseteq \text{ran}(\alpha)$. For any $\epsilon \in \text{End}(X, Y)$, since $\epsilon \in \text{End}(X, Y) = \beta \text{End}(X, Y)$, we have that $\text{ran}(\epsilon) \subseteq \text{ran}(\beta) \subseteq \text{ran}(\alpha)$. By Theorem 3.1.1, α is a left magnifying element of $\text{End}(X, Y)$. □

Theorem 3.3.2. Let α be a regular element in $\text{End}(X, Y)$ with $\alpha\gamma\alpha = \alpha$ for some $\gamma \in \text{End}(X, Y)$. Assume that there exists $\beta \in \text{End}(X, Y)$ such that $\text{End}(X, Y)\beta = \text{End}(X, Y)$. Then, α is a right magnifying element of $\text{End}(X, Y)$ if and only if the following statements hold:

1. $(\beta\gamma)|_{\text{ran}(\alpha)}$ is a Δ_Y -isomorphism in $\text{Hom}(\alpha, \beta)$.

2. There exist distinct $\gamma_1, \gamma_2 \in \text{End}(X, Y)$ such that $\gamma_1\alpha = \gamma_2\alpha$.

Proof. Assume that α is a right magnifying element of $\text{End}(X, Y)$. Since α is regular, it is also inversely right- $\text{End}(X, Y)$ -invariant. By Theorem 3.2.1, condition (2) holds. Since $\alpha \in \text{End}(X, Y) = \text{End}(X, Y)\beta$, we get that $\alpha = \delta\beta$ for some $\delta \in \text{End}(X, Y)$. Since $\alpha\gamma\alpha = \alpha$, we have that $\beta\gamma\alpha = \beta$ by Theorem 3.2.1(1). Thus, $\beta\gamma\delta\beta = \beta\gamma\alpha = \beta$, which implies that β is regular. Therefore, $\alpha, \beta \in F(X, Y)$. Since $(\delta(\beta\gamma))\alpha = \delta\beta = \alpha$ and $((\beta\gamma)\delta)\beta = \beta\gamma\alpha = \beta$, we have that $(\beta\gamma)|_{\text{ran}(\alpha)}$ is a Δ_Y -isomorphism in $\text{Hom}(\alpha, \beta)$ by $(\Delta 2)$.

Conversely, assume that (1) and (2) hold. By (1), there exists $\delta \in \text{Hom}(\beta, \alpha)$ such that $\beta\gamma\delta\beta = \beta$. Since $\alpha\gamma\alpha = \alpha$ and $\text{ran}(\delta) \subseteq \text{ran}(\alpha)$, we have that $\alpha\gamma\delta = \delta$. Since $\alpha \in \text{End}(X, Y) = \text{End}(X, Y)\beta$, we get that $\alpha = \epsilon\beta$ for some $\epsilon \in \text{End}(X, Y)$. Then

$$\beta\gamma\alpha = \beta\gamma\epsilon\beta = \beta\gamma\epsilon\beta\gamma\delta\beta = \beta\gamma\alpha\gamma\delta\beta = \beta\gamma\delta\beta = \beta.$$

Notice that for any $\zeta \in \text{End}(X, Y)$, since $\text{End}(X, Y) = \text{End}(X, Y)\beta$, there exists $\eta \in \text{End}(X, Y)$ such that $\zeta = \eta\beta$, which implies that $\zeta = \eta\beta = \eta\beta\gamma\alpha$, so $\alpha^{-1}(\alpha(x)) \subseteq \zeta^{-1}(\zeta(x))$ for all $x \in X$. By Theorem 3.2.1, α is a right magnifying element of $\text{End}(X, Y)$. \square

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